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Brownian motion in self-similar domains

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For $T \neq 1$, the domain G is T-homogeneous if $TG = G$. If $0 \notin G$, then necessarily $0 \in \partial G$. It is known that for some $p > 0$, the Martin kernel K at infinity satisfies $K(Tx) = T^p K(x)$ for all $x \in G$. We show that in dimension $d \geqslant 2$, if G is also Lipschitz, then the exit time τ_G of Brownian motion from G satisfies $P_x(\tau_G > t) \approx K(x)t^{-p/2}$ as $t \to \infty$. An analogous result holds for conditioned Brownian motion, but this time the decay power is $1 - p - d/2$. In two dimensions, we can relax the Lipschitz condition at 0 at the expense of making the rest of the boundary C^2.

Keywords: Brownian motion; lifetime; Martin kernel; self-similar sets; T-homogeneous domains

1. Introduction

Let B_t be Brownian motion in $\mathbb{R}^d$, $d \geqslant 2$. For a domain (i.e., an open connected set) $G \subseteq \mathbb{R}^d$, denote the first exit time of B_t from G by

$$\tau_G = \inf\{t > 0 : B_t \notin G\}.$$

Following the usual convention, we use P_x and E_x to denote probability and expectation, respectively, associated with $B_0 = x$.

If G is smooth and bounded, then it is well known that

$$\log P_x(\tau_G > t) \sim -\lambda_G t \qquad \text{as } t \to \infty,$$

where λ_G is the first Dirichlet eigenvalue of half the Laplacian in G, and we write

$$f \sim g \qquad \text{as } t \to \infty$$

to mean

$$\frac{f}{g} \to 1 \qquad \text{as } t \to \infty.$$

In addition, if G is simply connected for $d = 2$ or strongly regular for $d \geqslant 3$, then for the inner radius of G defined by

$$\mathrm{Inr}(G) = \sup\{d(x, G^c) : x \in G\},$$

we have

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$$\frac{C_1}{\mathrm{Inr}(G)} \leqslant \lambda_G \leqslant \frac{C_2}{\mathrm{Inr}(G)}$$

for some positive dimension-dependent constants C_1 and C_2 which are independent of G (see Davies 1989, p. 30, Theorem 1.5.8).

From these considerations, it is clear that if G contains arbitrarily large balls, then the tail $P_x(\tau_G > t)$ cannot have exponential decay as $t \to \infty$. Recently several authors have studied such domains with known expansion rate as a traveller moves out to ∞ from within.

For example, given $a > 0$ and $p > 1$, consider the 'parabolic-type' domain

$$G = G_{a,p} = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : y > a|x|^p\}.$$

In this case

$$\log P_x(\tau_G > t) \sim -Kt^{(p-1)/(p+1)} \qquad \text{as } t \to \infty. \tag{1.1}$$

For $p = 2$ and $d = 2$, the $\liminf/\limsup$ behaviour was determined in Bañuelos *et al.* (2001). This was extended to $p > 1$ and $d \geqslant 2$ by Li (2003). Lifshits and Shi (2002) found the exact limiting behaviour and explicitly described K.

Convexity and symmetry played a great role in the derivation of (1.1). In DeBlassie and Smits (2005) we showed that such symmetry is irrelevant and proved analogous results for a new class of domains – the *twisted domains*. Such a domain G is generated as follows. Given a curve $C \subseteq \mathbb{R}^2$, as a traveller moves out along the curve, the boundary curves of G are obtained by moving out $\pm g(r)$ units along the unit normal to C when the traveller is r units from the origin. The function g is known as the *growth radius*. If, for some $\gamma > 0$ and $0 < p < 1$, the growth radius is $g(r) = \gamma r^p$ for large r, then it was shown that

$$\log P_x(\tau_G > t) \sim -Kt^{(1-p)/(1+p)} \qquad \text{as } t \to \infty,$$

and K was explicitly determined. Convexity played no role in the analysis, essentially being replaced by geometric arguments from conformal mapping.

In this paper we take a different point of view for a domain G containing arbitrarily large balls: The decay rate of $P_x(\tau_G > t)$ as $t \to \infty$ is intimately connected with the growth of the Martin kernel $K^\infty(x)$ of G with pole at ∞. Here are two motivating examples.

Example 1.1. Suppose $K \subseteq \mathbb{R}^2$ is compact and non-polar. Let $G = K^c$. Such a domain is called an *exterior domain*. It is known that the Martin boundary of G at ∞ is a singleton and the corresponding normalized Martin kernel $K^\infty(x)$ satisfies

$$K^\infty(x) \sim \log|x| \qquad \text{as } x \to \infty.$$

This is proved by Collet *et al.* (2000); they also show that

$$P_x(\tau_G > t) \sim 2K^\infty(x)(\log t)^{-1} \qquad \text{as } t \to \infty.$$

Example 1.2. An *open cone* in $\mathbb{R}^d$ is a domain of the form

$$G = \{r\theta : r > 0, \theta \in \Omega\},$$

where Ω is a domain in S^{d-1}, the unit sphere in $\mathbb{R}^d$. It is known that the Martin boundary of G at ∞ is a singleton (Pinsky 1995, p. 391, Theorem 6.4). Moreover, it is easy to check directly that

$$K^\infty(x) = |x|^{a_1} m_1\left(\frac{x}{|x|}\right),$$

where

$$a_1 = \sqrt{\lambda_1 + \left(\frac{d}{2} - 1\right)^2} - \left(\frac{d}{2} - 1\right)$$

and $m_1 \geqslant 0$ is the first normalized eigenfunction of the Dirichlet Laplace–Beltrami operator $\Delta_{S^{d-1}}$ on Ω with corresponding eigenvalue λ_1:

$$\begin{aligned} \Delta_{S^{d-1}} m_1 &= -\lambda_1 m_1 \qquad \text{on } \Omega, \\ m_1|_{\partial\Omega} &= 0, \\ \int_\Omega m_1^2 \, \mathrm{d}\sigma &= 1 \end{aligned}$$

(here $\mathrm{d}\sigma$ is $(d-1)$-dimensional volume measure on S^{d-1}). In Bañuelos and Smits (1997) it was shown that

$$P_x(\tau_G > t) \sim cK^\infty(x)t^{-a_1/2} \qquad \text{as } t \to \infty, \tag{1.2}$$

and c was explicitly identified (see also DeBlassie 1987; 1988).

The hypotheses on G were very general: Ω must be regular for the Laplace–Beltrami operator $\Delta_{S^{d-1}}$. When G is Lipschitz, (1.2) was extended in Bañuelos and Smits (1997) to the case of conditioned Brownian motion. Recall that if $h > 0$ is harmonic in G, then Brownian motion conditioned by h (also known as the h-process or the h-path) is the process corresponding to the semigroup

$$f \mapsto h(x)^{-1}\mathrm{E}_x[h(B_t)f(B_t)I_{\tau_G > t}].$$

This process was introduced in Doob (1957); see also Doob (1984). When h is the normalized Martin kernel K^y of G with pole at $y \in \partial G$, we use the notation P_x^y for the probability associated with the corresponding h-process starting at x. It is known that as $t \to \tau_G$, the h-process converges to y. For this reason it is called Brownian motion conditioned to converge to $y \in \partial G$. The Lipschitz nature of G ensures that the Martin boundary less ∞ coincides with the Euclidean boundary. The extension of (1.2) is

$$P_x^y(\tau_G > t) \sim H(x, y)t^{1-a_1-d/2} \qquad \text{as } t \to \infty. \tag{1.3}$$

The relations (1.2)–(1.3) required the scale invariance of the cone G (i.e., $\gamma G = G$ for all $\gamma > 0$) and the boundary Harnack principle in truncations of G. Our main results extend (1.2)–(1.3) to more general domains that scale only for a discrete set of dilations. Given a positive $T \neq 1$, a domain $G \subseteq \mathbb{R}^d$ is *T-homogeneous* if $TG = G$. Thus G is invariant under the discrete set of dilations $\{T^n : n \in \mathbb{Z}\}$. Clearly any cone is T-homogeneous for all $T > 0$.

Remark 1.1. (a) We will always assume $0 \notin G$, for otherwise $G = \mathbb{R}^d$. Notice that a T-homogeneous domain is unbounded and contains arbitrarily large balls.

(b) Since $0 \notin G$, we must have $0 \in \partial G$.

(c) If G is T-homogeneous then it is $(1/T)$-homogeneous. Thus it is no loss to assume $T > 1$.

(d) Since G is connected, $G \cap \{|x| = 1\} \neq \emptyset$, so after a rotation we can always assume that $(1, 0, \ldots, 0) \in G$.

In what follows we will say that G is a *Lipschitz domain* if each point $x \in \partial G$ has a neighbourhood N for which $N \cap \partial G$ is a rotated graph of some Lipschitz function from $\mathbb{R}^{d-1}$ to $\mathbb{R}$. Assuming $0 \in \partial G$, if we require this condition only for $x \in \partial G \backslash \{0\}$, then we say that $\partial G \backslash \{0\}$ is Lipschitz. The corresponding definition holds when we say that $\partial G \backslash \{0\}$ is C^2.

The key to our theorems is the following recent result of Azarin *et al.* (2005).

Theorem 1.1. *Suppose G is T-homogeneous and Lipschitz. Then the Martin boundary of G is $\partial G \cup \{\infty\}$, where ∂G is the Euclidean boundary. For $y \in \partial G \cup \{\infty\}$, denote the corresponding Martin kernel by $K^y(\cdot)$, normalized by $K^y(1, 0, \ldots, 0) = 1$. Then there exists $p(G) > 0$ such that for all $x \in G$,*

$$K^{\infty}(Tx) = T^{p(G)} K^{\infty}(x),$$

$$K^0(Tx) = T^{2-d-p(G)} K^0(x).$$

When $d = 2$, this result is an amalgamation of several results in Azarin *et al.* (2005). It is not difficult to check that the arguments in the two-dimensional case can be modified to carry through to higher dimensions.

Remark 1.2. By Section 5 in Azarin *et al.* (2005), for dimension $d = 2$, Theorem 1.1 continues to hold if instead of assuming that G is Lipschitz, we assume that the boundary Harnack principle holds on $G \cap \{|z| = 1\}$. For instance, this is true if $G \cap \{|z| = 1\}$ consists of finitely many arcs and in a neighbourhood of each of these arcs the boundary of G is a (possibly rotated) graph of a Lipschitz function.

In what follows we will write

$$f \approx g \qquad \text{as } t \to \infty$$

to mean, for some positive constants C_1 and C_2,

$$C_1 \leqslant \frac{f}{g} \leqslant C_2 \qquad \text{as } t \to \infty.$$

We can now state our main results.

Theorem 1.2. *Let $G \subseteq \mathbb{R}^d$ be T-homogeneous and Lipschitz. Then for $x \in G$, with $p(G)$ from Theorem 1.1,*

$$P_x(\tau_G > t) \approx K^\infty(x) t^{-p(G)/2} \qquad \text{as } t \to \infty, \tag{1.4}$$

and for $y \in \partial G$,

$$P_x^y(\tau_G > t) \approx K^\infty(x) K^y(x)^{-1} t^{1-p(G)-d/2} \qquad \text{as } t \to \infty. \tag{1.5}$$

In two dimensions, we can relax the Lipschitz requirement at $0 \in \partial G$ at the expense of more regularity away from 0. This allows for some interesting examples not covered by Theorem 1.2. See Section 4 for some examples.

If $\hat{\mathbb{C}}$ is the Riemann sphere, a *Jordan curve* is a continuous mapping $\gamma : [a, b] \to \hat{\mathbb{C}}$ with $\gamma(a) = \gamma(b)$. The values $a = -\infty$ and $b = \infty$ are allowed.

Theorem 1.3. *Let $G \subseteq \mathbb{R}^2$ be a T-homogeneous domain such that ∂G is given by a Jordan curve. If $\partial G \backslash \{0\}$ is C^2, then for $z \in G$ and $y \in \partial G$,*

$$P_z(\tau_G > t) \approx K^\infty(x) t^{-p(G)/2} \qquad \text{as } t \to \infty, \tag{1.6}$$

and

$$P_z^y(\tau_G > t) \approx K^\infty(x) K^y(x)^{-1} t^{-p(G)} \qquad \text{as } t \to \infty. \tag{1.7}$$

This paper is organized as follows. In Section 2 we prove Theorem 1.2 using the parabolic boundary Harnack principle. In Section 3 we prove Theorem 1.3 using conformal maps. Finally, in Section 4 we present some two-dimensional examples.

2. Proof of Theorem 1.2

Throughout this section $G \subseteq \mathbb{R}^d$ will be a Lipschitz T-homogeneous domain. We will need the following result of Davis and Zhang (1994). It is important in the proof of (1.5) for $y = 0$. Even though their result was given for the case of a cone, it is not hard to verify that their proof works for T-homogeneous domains too.

Lemma 2.1. *Let $G \subseteq \mathbb{R}^d$ be Lipschitz and T-homogeneous. Then for $p > 0$,*

$$\mathrm{E}_x^0(\tau_G^p) < \infty \Leftrightarrow p < p(G) + \frac{d}{2} - 1.$$

We will also need the following consequence of the parabolic boundary Harnack principle.

Theorem 2.2. *Let $u(t, x)$ and $v(t, x)$ be positive solutions of the heat equation $\frac{1}{2}\Delta u - \partial u / \partial t = 0$ in $\{(t, x) : \frac{1}{2} < t < 2, x \in G, |x| < T\}$ vanishing continuously on $\{(t, x) : \frac{1}{2} < t < 2, x \in \partial G, |x| < T\}$. Then for some constant C, if $x \in G$ with $|x| \leqslant 1$ then*

$$u(1, x) \leqslant Cv(1, x).$$

Proof. Given $r > 0$ and $x \in \mathbb{R}^d$, we will write

$$B_r(x) = \{y : |x - y| < r\}.$$

By the parabolic boundary Harnack principle (Kenig and Pipher 1989, Lemma 2.6) applied to the Lipschitz cylinder

$$\left\{(t, x) : \frac{1}{2} < t < 2, x \in G, |x| < T\right\},$$

for each $x \in \partial G$ with $|x| \leqslant 1$ there exist $\delta = \delta(x)$ and $A = A(x)$ such that for all $y \in G \cap B_\delta(x)$,

$$u(1, y) \leqslant Av(1, y).$$

Then since $\partial G \cap \{|x| \leqslant 1\}$ is compact and since $u(1, y)/v(1, y)$ is bounded on compact subsets of $G \cap \{|x| < T\}$, the desired conclusion holds. □

One application below of Theorem 2.2 requires the next result.

Lemma 2.3. *The Martin kernel $K^0(x)$ of G with pole at 0 satisfies*

$$\lim_{\substack{x\to 0\\ x\in G}} \mathrm{E}_x[I_{\tau_G > t} K^0(B_t)] = 0$$

for all $t > 0$.

Proof. By Lemma 2.1, for $q < p(G) + d/2 - 1$

$$C_1 := \sup_{1/T \leqslant |x| \leqslant 1} \mathrm{E}_x^0(\tau_G^q) < \infty. \tag{2.1}$$

Since $p(G) > 0$, we can choose q satisfying

$$\frac{1}{2}[p(G) + d - 2] < q < p(G) + \frac{d}{2} - 1. \tag{2.2}$$

By scaling and the homogeneity property of K^0,

$$P_x^0(\tau_G > t) = P_{T^n x}^0(T^{-2n}\tau_G > t). \tag{2.3}$$

Hence for $|x|$ small, choose $n \geqslant 0$ such that

$$T^{-n-1} < |x| \leqslant T^{-n}.$$

Then by (2.1) and (2.3),

$$\begin{aligned}
\mathrm{E}_x[I_{\tau_G>t}K^0(B_t)] &= K^0(x)P^0_x(\tau_G > t) \\
&= K^0(x)P^0_{T^n x}(T^{-2n}\tau_G > t) \\
&\leqslant K^0(x)C_1 T^{-2nq}t^{-q} \\
&\leqslant K^0(x)C_1 t^{-q}(T|x|)^{2q}. \qquad (2.4)
\end{aligned}$$

By the homogeneity property of $K^0(x)$ and the fact that $2 - d - p(G) < 0$,

$$\begin{aligned}
K^0(x) &= K^0(T^n x)T^{-n(2-p(G)-d)} \\
&\leqslant \left[\sup_{1/T\leqslant|y|\leqslant 1} K^0(y)\right] T^{-n(2-p(G)-d)} \\
&\leqslant C_2|x|^{2-d-p(G)},
\end{aligned}$$

where C_2 is independent of x. Combined with (2.4), we have for some positive constant C independent of x,

$$\begin{aligned}
\mathrm{E}_x[I_{\tau_G>t}K^0(B_t)] &\leqslant C|x|^{2q-(d+p(G)-2)} \\
&\to 0 \qquad \text{as } x \to 0,
\end{aligned}$$

by (2.2). □

We now prove Theorem 1.2. For the rest of this section, fix $x \in G$ and consider any $n \in \mathbb{Z}$ so large that

$$|x| \leqslant T^n. \qquad (2.5)$$

Proof of (1.4). By harmonicity of K^∞, T-homogeneity of G and scaling,

$$\begin{aligned}
\frac{P_x(\tau_G > T^{2n})}{K^\infty(x)} &= \frac{P_x(\tau_G > T^{2n})}{E_x[K^\infty(B_{T^{2n}})I_{\tau_G>T^{2n}}]} \\
&= \frac{P_{xT^{-n}}(\tau_G > 1)}{E_{xT^{-n}}[K^\infty(T^n B_1)I_{\tau_G>1}]} \\
&= \frac{P_{xT^{-n}}(\tau_G > 1)}{T^{np(G)}E_{xT^{-n}}[K^\infty(B_1)I_{\tau_G>1}]}. \qquad (2.6)
\end{aligned}$$

Apply Theorem 2.2 to the functions

$$\begin{aligned}
u(t, w) &= P_w(\tau_G > t), \\
v(t, w) &= \mathrm{E}_w[K^\infty(B_t)I_{\tau_G>t}],
\end{aligned}$$

and use (2.5) to obtain

$$C_1 \leqslant \frac{P_{xT^{-n}}(\tau_G > 1)}{E_{xT^{-n}}[K^\infty(B_1)I_{\tau_G>1}]} \leqslant C_2,$$

where C_1 and C_2 are independent of n. Using this in (2.6) yields

$$C_1 T^{-np(G)} \leqslant \frac{P_x(\tau_G > T^{2n})}{K^\infty(x)} \leqslant C_2 T^{-np(G)}. \tag{2.7}$$

Next, given large $t > 0$, choose $m \in \mathbb{N}$ such that $T^{2m-2} \leqslant t < T^{2m}$. Then

$$P_x(\tau_G > T^{2m}) \leqslant P_x(\tau_G > t) \leqslant P_x(\tau_G > T^{2m-2}).$$

By (2.7),

$$C_1 T^{-mp(G)} \leqslant \frac{P_x(\tau_G > t)}{K^\infty(x)} \leqslant C_2 T^{-(m-1)p(G)}.$$

After multiplying by $t^{p(G)/2}$ we obtain

$$C_1 \Big(\frac{t}{T^{2m}}\Big)^{p(G)/2} \leqslant \frac{t^{p(G)/2} P_x(\tau_G > t)}{K^\infty(x)} \leqslant C_2 \Big(\frac{t}{T^{2m-2}}\Big)^{p(G)/2}.$$

By choice of m, this yields

$$C_1 T^{-p(G)} \leqslant \frac{t^{p(G)/2} P_x(\tau_G < t)}{K^\infty(x)} \leqslant C_2 T^{-p(G)},$$

from which (1.4) follows. □

Proof of (1.5) for $y = 0$. With x and n as above (see (2.5)),

$$\frac{P_x^0(\tau_G > T^{2n})}{P_x(\tau_G > T^{2n})} = \frac{K^0(x)^{-1} \mathrm{E}_x[K^0(B_{T^{2n}})I_{\tau_G > T^{2n}}]}{P_x(\tau_G > T^{2n})}.$$

By the homogeneity property of K^0, thanks to Lemma 2.3, we can argue exactly as above to obtain the following analogue of (2.7):

$$C_1 T^{n(2-p(G)-d)} K_0(x)^{-1} \leqslant \frac{P_x^0(\tau_G > T^{2n})}{P_x(\tau_G > T^{2n})} \leqslant C_2 T^{n(2-p(G)-d)} K_0(x)^{-1}.$$

Combined with (2.7),

$$C_1^2 T^{n(2-2p(G)-d)} \frac{K^\infty(x)}{K^0(x)} \leqslant P_x^0(\tau_G > T^{2n}) \leqslant C_2^2 T^{n(2-2p(G)-d)} \frac{K^\infty(x)}{K^0(x)}.$$

As above, this yields

$$P_x^0(\tau_G > t) \approx t^{1-p(G)-d/2} \frac{K^\infty(x)}{K^0(x)} \qquad \text{as } t \to \infty,$$

as desired. □

Proof of (1.5) for y ≠ 0. In this case, the proof is much more difficult. We need the next two results for the lower bound.

Lemma 2.4. *For $r = 2|y|$, we have*

$$K^0(w) \approx K^y(w), \qquad w \in G \cap \{|w| \geqslant 2r\}.$$

Proof. This is an immediate consequence of Jerison and Kenig (1982, Theorem 5.20). They consider bounded Lipschitz domains, but the argument depends only on the fact that the local Lipschitz characteristics at each boundary point can be chosen independent of the boundary point. Even though G is unbounded, by T-homogeneity, this independence also holds in this case. □

Lemma 2.5. *Let $r = 2|y|$. Given n_0 large, there exists positive C_1 such that for all $n \geqslant n_0$,*

$$C_1 \leqslant \frac{\mathrm{E}_x[K^0(B_{T^{2n}})I_{\tau_G > T^{2n}} I_{|B(T^{2n})| \geqslant 2r}]}{\mathrm{E}_x[K^0(B_{T^{2n}})I_{\tau_G > T^{2n}}]}.$$

Proof. By scaling and the homogeneity property of K^0, the fraction in question is equal to

$$\frac{\mathrm{E}_{xT^{-n}}[K^0(B_1)I_{\tau_G > 1} I_{|B_1| \geqslant 2rT^{-n}}]}{\mathrm{E}_{xT^{-n}}[K^0(B_1)T_{\tau_G > 1}]},$$

which in turn is bounded below by

$$\frac{\mathrm{E}_{xT^{-n}}[K^0(B_1)I_{\tau_G > 1} I_{|B_1| \geqslant 2rT^{-n_0}}]}{\mathrm{E}_{xT^{-n}}[K^0(B_1)I_{\tau_G > 1}]}. \tag{2.8}$$

Hence by Theorem 2.2 and (2.5), for n_0 large, there is a positive constant C such that (2.8) is bounded below by C for $n \geqslant n_0$. □

We can now prove the lower bound in (1.5) for $y \neq 0$. Given n_0 large, we have for $r = 2|y|$ and $n \geqslant n_0$,

$$\begin{aligned} P_x^y(\tau_G > T^{2n}) &= K^y(x)^{-1}\mathrm{E}_x[K^y(B_{T^{2n}})I_{\tau_G > T^{2n}}] \\ &\geqslant K^y(x)^{-1}\mathrm{E}_x[K^y(B_{T^{2n}})I_{\tau_G > T^{2n}} I_{|B(T^{2n})| \geqslant 2r}] \\ &\geqslant CK^y(x)^{-1}\mathrm{E}_x[K^0(B_{T^{2n}})I_{\tau_G > T^{2n}} I_{|B(T^{2n})| \geqslant 2r}] \qquad \text{(by Lemma 2.4)} \\ &\geqslant CK^y(x)^{-1}\mathrm{E}_x[K^0(B_{T^{2n}})I_{\tau_G > T^{2n}}] \qquad \text{(by Lemma 2.5)} \\ &= CK^y(x)^{-1}K^0(x)P_x^0(\tau_G > T^{2n}) \\ &\geqslant CK^y(x)^{-1}K^\infty(x)T^{n(2-2p(G)-d)}, \end{aligned}$$

by the case $y = 0$ of (1.5) proved above. This easily leads to the desired lower bound.

As for the upper bound, we will use the next result.

Lemma 2.6. *Let* $r = 2[|y| \vee |x|]$ *and set*

$$D = G \cap B_{2r}(0).$$

Then for some positive constants C *and* λ,

$$\mathrm{E}_x[K^y(B_t)I_{\tau_D \geqslant t}] \leqslant CK^y(x)\mathrm{e}^{-\lambda t}, \qquad t > 0.$$

Proof. Since $h(x) = K^y(x)$ is positive and harmonic in D,

$$\mathrm{E}_x[K^y(B_t)I_{\tau_D \geqslant t}] = h(x)P_x^h(\tau_D \geqslant t). \tag{2.9}$$

Since D is bounded, the first Dirichlet eigenvalue λ_D of $\frac{1}{2}\Delta$ exists and is positive. Since $\log P_x^h(\tau_D \geqslant t) \sim -\lambda_D t$ as $t \to \infty$ (see Kenig and Pipher 1989), using $\lambda = \lambda_D/2$ and (2.9) we obtain

$$\mathrm{E}_x[K^y(B_t)I_{\tau_D \geqslant t}] \leqslant Ch(x)\mathrm{e}^{-\lambda t}\mathrm{E}_x^h[e^{\lambda \tau_D}] = CK^y(x)\mathrm{e}^{-\lambda t},$$

as claimed. $\square$

We now prove the upper bound in (1.5). First note that by the strong Markov property, with D from Lemma 2.6,

$$\begin{aligned} &\mathrm{E}_x[K^y(B_{T^{2n}})I_{\tau_G > T^{2n}}I_{\tau_D < T^{2n}}] \\ &\quad = \mathrm{E}_x[I_{\tau_D < \tau_G}I_{\tau_D < T^{2n}}\mathrm{E}^{\omega}_{B(\tau_D)}[I_{\tau_G > T^{2n} - \tau_D(w)}K^y(B(T^{2n} - \tau_D(w)))]], \end{aligned} \tag{2.10}$$

where, in an abuse of notation, we have written $\mathrm{E}^{\omega}_{B(\tau_D)}$ to emphasize that any ω within the conditional expectation is to be regarded as constant. In particular, since K^y is harmonic on G,

$$\mathrm{E}^{\omega}_{B(\tau_D)}[I_{\tau_G > T^{2n} - \tau_D(w)}K^y(B(T^{2n} - \tau_D(w)))] = K^y(B(\tau_D)).$$

Using this in (2.10), we obtain

$$\mathrm{E}_x[K^y(B_{T^{2n}})I_{\tau_G > T^{2n}}I_{\tau_D < T^{2n}}] = \mathrm{E}_x[I_{\tau_D < \tau_G}I_{\tau_D < T^{2n}}K^y(B(\tau_D))]. \tag{2.11}$$

By the boundary Harnack principle, for some positive C_1 and C_2,

$$C_1 \leqslant \frac{K^y(w)}{K^0(w)} \leqslant C_2, \qquad w \in G \cap \{|w| = 2r\}.$$

Then since $\tau_D < \tau_G$ implies $|B(\tau_D)| = 2r$, (2.11) becomes

$$\mathrm{E}_x[K^y(B_{T^{2n}})I_{\tau_G > T^{2n}}I_{\tau_D < T^{2n}}] \leqslant C_2\mathrm{E}_x[I_{\tau_D < \tau_G}I_{\tau_D < T^{2n}}K^0(B(\tau_D))].$$

Note that (2.11) holds for K^y replaced by K^0, since we only have used the fact that K^y is positive and harmonic on G. Hence

$$
\begin{aligned}
\mathrm{E}_x[K^y(B_{T^{2n}})I_{\tau_G>T^{2n}}I_{\tau_D<T^{2n}}] &\leqslant C_2\mathrm{E}_x[K^0(B_{T^{2n}})I_{\tau_G>T^{2n}}I_{\tau_D<T^{2n}}] \\
&\leqslant C_2\mathrm{E}_x[K^0(B_{T^{2n}})I_{\tau_G>T^{2n}}] \\
&= C_2K^0(x)P_x^0(\tau_G > T^{2n}) \\
&\leqslant C_2K^\infty(x)T^{n(2-2p(G)-d)}, \qquad (2.12)
\end{aligned}
$$

by the case $y = 0$ of (5). To finish, note that by Lemma 2.6 and (2.12),

$$
\begin{aligned}
P_x^y(\tau_G > T^{2n}) &= K^y(x)^{-1}\mathrm{E}_x[K^y(B_{T^{2n}})I_{\tau_G>T^{2n}}] \\
&\leqslant K^y(x)^{-1}\mathrm{E}_x[K^y(B_{T^{2n}})I_{\tau_D\geqslant T^{2n}}] \\
&\quad + K^y(x)^{-1}\mathrm{E}_x[K^y(B_{T^{2n}})I_{\tau_G>T^{2n}}I_{\tau_D<T^{2n}}] \\
&\leqslant C\mathrm{e}^{-\lambda T^{2n}} + C_2K^y(x)^{-1}K^\infty(x)T^{n(2-2p(G)-d)} \\
&\leqslant CK^y(x)^{-1}K^\infty(x)T^{n(2-2p(G)-d)}.
\end{aligned}
$$

This leads to the desired upper bound. The proof of Theorem 1.2 is complete. □

3. Proof of Theorem 1.3

Let

$$H = \{z = x + \mathrm{i}y : x > 0\}$$

be the right half-space. By our hypothesis on G, there exists a continuous bijection $F : \overline{G} \to \overline{H}$ such that $F : G \to H$ is conformal with $F(0) = 0$. In particular, $\operatorname{Re} F$ is positive and harmonic on G and vanishes on ∂G. By Remark 1.2, there exists a positive constant c such that for $K = K^\infty$, $\operatorname{Re} F = cK$. By replacing F by $c^{-1}F$, there is no loss in assuming $c = 1$. Thus we can write

$$F = K + \mathrm{i}V, \qquad (3.1)$$

where V is the harmonic conjugate of K in G. We now derive an important property of F. Recall from Theorem 1.1, with $p = p(G)$,

$$K(Tz) = T^pK(z), \qquad z \in G. \qquad (3.2)$$

Lemma 3.1. *The function $F : \overline{G} \to \overline{H}$ satisfies*

$$F(Tz) = T^pF(z), \qquad z \in G.$$

Proof. It suffices to show

$$V(Tz) = T^pV(z), \qquad z \in G.$$

By (3.2),

$$\begin{aligned} K_x(Tz) &= T^{p-1}K_x(z),\\ K_y(Tz) &= T^{p-1}K_y(z) \end{aligned} \tag{3.3}$$

where $z = x + \mathrm{i}y \in G$ and $K_x = \partial K/\partial x$, $K_y = \partial K/\partial y$.

Consider any $z_1 \in G$. Then

$$V(z) = \int_{z_1}^{z} -K_t(r, t)\mathrm{d}r + K_r(r, t)\mathrm{d}t + V(z_1),$$

where the line integral is along any path from z_1 to z in G. Let $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to \mathbb{R}^2$ be a smooth path in G connecting z_1 to z. Then $T\gamma$ is a smooth path in G connecting Tz_1 to Tz and so

$$\begin{aligned} V(Tz) &= V(Tz_1) + \int_{Tz_1}^{Tz} -K_t(r, t)\mathrm{d}r + K_r(r, t)\mathrm{d}t \\ &= V(Tz_1) + \int_0^1 [-K_t(T\gamma(s))T\gamma_1'(s) + K_r(T\gamma(s))T\gamma_2'(s)]\mathrm{d}s \\ &= V(Tz_1) + \int_0^1 [-T^{p-1}K_t(\gamma(s))T\gamma_1'(s) + T^{p-1}K_r(\gamma(s))T\gamma_2'(s)]\mathrm{d}s \qquad \text{(by (3.3))} \\ &= V(Tz_1) + T^p\int_0^1 [-K_t(\gamma(s))\gamma_1'(s) + K_r(\gamma(s))\gamma_2'(s)]\mathrm{d}s \\ &= V(Tz_1) + T^p[V(z) - V(z_1)] \\ &= T^pV(z) + [V(Tz_1) - T^pV(z_1)]. \end{aligned}$$

Let $z_1 \to 0$, use continuity of V and the fact that $V(0) = 0$ to obtain

$$V(Tz) = T^pV(z),$$

as desired. □

For $\xi > 0$, define

$$D_\xi = D(\xi) = (0, \infty) \times \left(-\frac{\xi}{2}, \frac{\xi}{2}\right),$$

$$L = \frac{1}{2}\left[\frac{\partial^2}{\partial u_1^2} + \frac{1}{u_1}\frac{\partial}{\partial u_1} + \frac{1}{u_1^2}\frac{\partial^2}{\partial u_2^2}\right], \qquad u = (u_1, u_2) \in D_\xi. \tag{3.4}$$

For $u = (u_1, u_2) \in \overline{D}_\xi$ and $t > 0$ set

$$q_\xi(u, t) = \sum_{j=1}^{\infty} B_j\left(\frac{u_1^2}{2t}\right)^{a_j/2} {}_1F_1\left(\frac{a_j}{2}, a_j + 1, -\frac{u_1^2}{2t}\right)\sqrt{\frac{2}{\xi}}\cos(a_ju_2), \tag{3.5}$$

where

$$a_j = \frac{2j-1}{\xi}\pi,$$

$$B_j = \frac{\Gamma\big(a_j/2+1\big)}{\Gamma(a_j+1)} \int_{-\xi/2}^{\xi/2} \sqrt{\frac{2}{\xi}} \cos(a_j\theta)\mathrm{d}\theta$$

and

$$ {}_1F_1(a,\, b;\, z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \cdots \qquad (b > 0)$$

is the confluent hypergeometric function.

The following two results can be found in Bañuelos and Smits (1997).

Theorem 3.2. *The convergence in* (3.5) *is uniform for* $(u,\, t) = (u_1,\, u_2,\, t) \in [0,\, K] \times [-\xi/2,\, \xi/2] \times [T,\, \infty)$ *for all positive K and T. Moreover,*

$$\lim_{u\to v} q_\xi(u,\, t) = 0, \qquad v \in (0,\, \infty) \times \left\{\pm\frac{\xi}{2}\right\} \text{ and } t > 0,$$

$$\lim_{t\to 0} q_\xi(u,\, t) = 1, \qquad u \in D_\xi,$$

$$Lq_\xi = \frac{\partial}{\partial t} q_\xi \qquad \text{on } D_\xi \times (0,\, \infty).$$

Corollary 3.3. *As* $t \to \infty$,

$$q_\xi(u,\, t) \sim B_1 \left(\frac{u_1^2}{2t}\right)^{a_1/2} \sqrt{\frac{2}{\xi}} \cos(a_1 u_2),$$

and as $u_1 \to 0$,

$$q_\xi(u,\, t) \sim B_1 \left(\frac{u_1^2}{2t}\right)^{a_1/2} \sqrt{\frac{2}{\xi}} \cos(a_1 u_2).$$

Remark 3.1. (a) Thus if U is the diffusion in D_ξ associated with the operator L and

$$\tau_{D(\xi)}(u) = \inf\{t > 0 : U_t \notin D(\xi)\},$$

then

$$P_u(\tau_{D(\xi)}(u) > t) = q_\xi(u,\, t).$$

(b) Bañuelos and Smits (1997) proved this for $\xi < 2\pi$ because they were interested in the distribution of the exit time of two-dimensional Brownian motion in a wedge

$$W_\xi = \left\{ r\mathrm{e}^{\mathrm{i}\theta} : r > 0,\, -\frac{\xi}{2} < \theta < \frac{\xi}{2} \right\}.$$

However, the proof works for arbitrary $\xi > 0$, although we lose the interpretation of q_ξ as the distribution of the exit time of Brownian motion from a wedge.

Our next idea is to obtain a map J from G into a region $\tilde{G}$ such that Brownian motion is converted into a diffusion in $\tilde{G}$ governed by an operator of the form $a(z)\Delta$, where $0 < \inf a \leqslant \sup a < \infty$. Then because of the next lemma and Corollary 3.3, we will be able to prove Theorem 1.3.

Lemma 3.4. *Suppose U and $\tilde{U}$ are the diffusions in D_ξ associated with the operators L and $a(u)L$, respectively, where L is from* (3.4) *and $0 < \inf_{D(\xi)} a \leqslant \sup_{D(\xi)} a < \infty$. Then for some constants c_1 and c_2, for $u \in D_\xi$ and $t > 0$,*

$$P_u(\tau_{D(\xi)}(U) > c_1 t) \leqslant P_u(\tau_{D(\xi)}(\tilde{U}) > t) \leqslant P_u(\tau_{D(\xi)}(U) > c_2 t).$$

Proof. Since the first component of U is a two-dimensional Bessel process starting away from 0, it never hits 0. Then standard time-change arguments yield the desired conclusion. □

The question is how to define J. In the case $p > \frac{1}{4}$, it turns out $J = F^{1/p} : G \to W_{\pi/p}$ will do the trick. This motivates the following definition. Let $J : G \to D(\pi/p)$ be given by

$$J(z) = \left(|F(z)|^{1/p}, \frac{1}{p} \arctan \frac{V(z)}{K(z)} \right).$$

Since $F(G) = H = \left\{ r\mathrm{e}^{\mathrm{i}\theta} : r > 0, -\pi/2 < \theta < \pi/2 \right\}$, we see that J is onto. Notice that

$$\lim_{z \to \partial G \setminus \{0\}} J(z) = \left(0, \pm \frac{\pi}{2p} \right),$$

where the plus sign is chosen if $\lim_{z \to \partial G \setminus \{0\}} V(z) > 0$ and the negative sign is chosen if the limit is negative. Thus J has a continuous extension from $\overline{G} \setminus \{0\}$ onto $(0, \infty) \times [-\pi/2p, \pi/2p]$. Moreover, J is one-to-one: if $J(z_1) = J(z_2)$ then

$$|F(z_1)| = |F(z_2)| \quad \text{and} \quad \arctan \frac{V(z_1)}{K(z_1)} = \arctan \frac{V(z_2)}{K(z_2)}.$$

Now $K \geqslant 0$, so either $K(z_1) > 0$ or $K(z_1) = 0$. When $K(z_1) > 0$ we have $F(z_1) = F(z_2)$, which implies $z_1 = z_2$, since F is one-to-one. When $K(z_1) = 0$ we have $V(z_1) \neq 0$ and

$$\arctan \frac{V(z_1)}{K(z_1)} = \frac{\pi}{2} \operatorname{sgn}(V(z_1)).$$

Hence $V(z_1)V(z_2) > 0$ and since $|V(z_1)| = |F(z_1)| = |F(z_2)| = |V(z_2)|$, we obtain $V(z_1) = V(z_2)$. Thus $F(z_1) = F(z_2)$ again, and so $z_1 = z_2$.

In what follows, let

$$\tilde{U}_t = J(B_t), \tag{3.6}$$

$$\tilde{L} = aL, \tag{3.7}$$

where L is from (3.4) with $\xi = \pi/p$ and

$$a(u) = \frac{1}{p^2}|F' \circ J^{-1}(u)|^2 |F \circ J^{-1}(u)|^{\frac{2}{p}-2}. \tag{3.8}$$

Before proving the next two lemmas, we show how to obtain (1.6) in Theorem 1.3.

Lemma 3.5. *The differential operator associated with $\tilde{U}$ in $D_{\pi/p}$ is $\tilde{L}$. Equivalently, $\frac{1}{2}\Delta(h \circ J) = (\tilde{L}h) \circ J$ for smooth $h : D_{\pi/p} \to \mathbb{R}$.*

Lemma 3.6. *The function $a(u)$ from* (3.8) *satisfies*

$$0 < \inf_{D(\pi/p)} a \leqslant \sup_{D(\pi/p)} a < \infty.$$

Lemma 3.7. *Up to positive constant multiples, for $u = J(z)$,*

$$K^{\infty}(z) = u_1^p \cos(pu_2)$$

and

$$K^0(z) = u_1^{-p} \cos(pu_2).$$

Proof of (1.6). Let U be the diffusion in $D_{\pi/p}$ associated with L from (3.4), where $\xi = \pi/p$. Then for $u = J(z)$, by Lemmas 3.4–3.6,

$$\begin{aligned} P_z(\tau_G > t) &= P_u(\tau_{D(\pi/p)}(\tilde{U}) > t) \\ &\leqslant P_u(\tau_{D(\pi/p)}(U) > c_2 t). \end{aligned} \tag{3.9}$$

By Remark 3.1(a), Corollary 3.3 and the fact that, for $\xi = \pi/p$,

$$a_1 = \frac{\pi}{\xi} = p,$$

we have

$$P_u(\tau_{D(\pi/p)}(U) > t) \sim B_1 u_1^p \cos(pu_2)(2t)^{-p/2} \qquad \text{as } t \to \infty.$$

Hence by (3.9) and Lemma 3.7, for some constant $C > 0$,

$$P_z(\tau_G > t) \leqslant CK^{\infty}(x)t^{-p/2} \qquad \text{as } t \to \infty.$$

A similar argument yields a similar lower bound and the proof of (1.6) in Theorem 1.3 is complete. □

Proof of Lemma 3.5. The quickest way to see this is to observe that locally, $J(z)$ gives the polar coordinates of the appropriate branch $M(z)$ of $F(z)^{1/p}$. More precisely, if $u_1 = |z|$ and $R(z) = (u_1, u_2) = u$ gives the polar coordinates of z, then

$$J(z) = R \circ M(z).$$

It is well known that

$$\frac{1}{2}\Delta(h \circ R) = (Lh) \circ R$$

and, since M is conformal,

$$\Delta(f \circ M) = |M'|^2 (\Delta f) \circ M.$$

Hence

$$\begin{aligned}
\frac{1}{2}\Delta(h \circ J) &= \frac{1}{2}\Delta(h \circ R \circ M) \\
&= \frac{1}{2}|M'|^2 (\Delta(h \circ R)) \circ M \\
&= |M'|^2 (Lh) \circ R \circ M \\
&= \frac{1}{p^2}|F'|^2 |F|^{2/p-2} (Lh) \circ J \\
&= (a \circ J)(Lh) \circ J \\
&= (\tilde{L}h) \circ J,
\end{aligned}$$

as desired. Alternatively, one can use the Cauchy–Riemann equations $K_x = V_y$ and $K_y = -V_x$ to compute $\mathrm{d}\tilde{U}_t$ and then $\mathrm{d}g(\tilde{U}_t)$ for smooth g. One can read off $\tilde{L}$ from the latter. □

Proof of Lemma 3.6. Since J is one-to-one, it suffices to show that

$$0 < \inf_G [|F'|^2 |F|^{2/p-2}] \leqslant \sup_G [|F'|^2 |F|^{2/p-2}] < \infty.$$

Given $z \in G$, choose $n \in \mathbb{Z}$ such that

$$T^{n-1} < |z| \leqslant T^n.$$

Then by Lemma 3.1,

$$\begin{aligned}
|F'(z)F(z)|^{1/p-1} &= \left|F'\left(T^n \frac{z}{T^n}\right)\right| \left|F\left(T^n \frac{z}{T^n}\right)\right|^{1/p-1} \\
&= \left|F'\left(\frac{z}{T^n}\right)\right| \left|F\left(\frac{z}{T^n}\right)\right|^{1/p-1}.
\end{aligned}$$

Since $1/T < |z|/T^n \leqslant 1$, by continuity of F,

$$0 < \inf_{z \in G} \left|F\left(\frac{z}{T^n}\right)\right|^{1/p-1} \leqslant \sup_{z \in G} \left|F\left(\frac{z}{T^n}\right)\right|^{1/p-1} < \infty.$$

Thus it suffices to show that

$$0 < \inf\left\{|F'(z)| : z \in G, \frac{1}{T} \leqslant |z| \leqslant 1\right\} \leqslant \sup\left\{|F'(z)| : z \in G, \frac{1}{T} \leqslant |z| \leqslant 1\right\} < \infty.$$

Since F is conformal on G, $|F'(z)| > 0$ away from ∂G, hence we need only consider z near ∂G with $1/T \leqslant |z| \leqslant 1$. Notice also that $|F'(z)| = |\nabla K(z)|$, so we need only study K. Now K is harmonic on G and vanishes continuously on ∂G. Since ∂G is C^2 away from 0, by the elliptic regularity theorem (Gilbarg and Trudinger 1983, p. 111, Lemma 6.18) we have $K \in C^2(\overline{G}\backslash\{0\})$. In particular, $|\nabla K|$ is bounded on $\overline{G} \cap \{1/T \leqslant |z| \leqslant 1\}$. Furthermore, by the Hopf maximum principle (Protter and Weinberger 1984, p. 65, Theorem 7), $|\nabla K| > 0$ on ∂G. Then by continuity of ∇K on $\overline{G}\backslash\{0\}$, $|F'| = |\nabla K|$ is positive for z near ∂G with $1/T \leqslant |z| \leqslant 1$, as desired. □

Proof of Lemma 3.7. By Lemma 3.5 the Martin boundaries of G and $D_{\pi/p}$ corresponding to Δ and $\tilde{L}$, respectively, are in one-to-one correspondence. In particular, by Remark 1.2 the normalized Martin kernels $\mathcal{K}^\infty$ and $\mathcal{K}^0$ in $D_{\pi/p}$ corresponding to $\tilde{L}$ with poles at ∞ and 0, respectively, are well defined. Moreover, up to positive constant multiples,

$$K^\infty(x) = \mathcal{K}^\infty(J(x))$$

and

$$K^0(x) = \mathcal{K}^0(J(x)).$$

It is easy to check directly that

$$\mathcal{K}^\infty(u) = u_1^p \cos(pu_2) \quad \text{and} \quad \mathcal{K}^0(u) = u_1^{-p} \cos(pu_2),$$

and the lemma follows. □

Proof of (1.7). Let U be the process in $D_{\pi/p}$ corresponding to L from (3.4) with $\xi = \pi/p$ and let $\tilde{U}$ be the process in $D_{\pi/p}$ corresponding to $\tilde{L}$ from (3.7). Then the Martin boundaries of $D_{\pi/p}$ corresponding to L and $\tilde{L}$ coincide. For $y \in \partial G \cup \{\infty\}$ and $v = J(y)$, if $\mathcal{K}^v$ is the Martin kernel in $D_{\pi/p}$ corresponding to L (or $\tilde{L}$) with pole at v, then by Lemma 3.5,

$$K^y(x) = \mathcal{K}^v \circ J(x). \tag{3.10}$$

Write U^v and $\tilde{U}^v$ to denote the $\mathcal{K}^v$-processes in $D_{\pi/p}$ and denote the corresponding differential operators by L^v and $\tilde{L}^v$.

First we show that, for $v = J(y)$ and $u = J(z)$,

$$P_z^y(\tau_G > t) = P_u^v(\tau_{D(\pi/p)}(\tilde{U}) > t). \tag{3.11}$$

Indeed, if we denote the K^y-process in G by B^y then it suffices to show that the differential operator associated with $J(B^y)$ is $\tilde{L}^v$. Equivalently, we must show that, for smooth $h : D_{\pi/p} \to \mathbb{R}$,

$$\frac{1}{2K^y}\Delta(K^y(h \circ J)) = (\tilde{L}^v h) \circ J.$$

But by (3.10) and Lemma 3.5,

$$\begin{aligned}\frac{1}{2K^y}\Delta(K^y(h\circ J)) &= \frac{1}{2\mathcal{K}^v\circ J}\Delta((\mathcal{K}^v\circ J)(h\circ J))\\ &= \frac{1}{\mathcal{K}^v\circ J}(\tilde{L}(\mathcal{K}^v h))\circ J\\ &= (\tilde{L}^v h)\circ J,\end{aligned}$$

as desired.

In Bañuelos and Smits (1997) it was shown that

$$P_u^v(\tau_{D(\pi/p)}(U) > t) \approx \mathcal{K}^\infty(u)\mathcal{K}^v(u)^{-1}t^{-p} \qquad \text{as } t\to\infty. \tag{3.12}$$

Notice that for $v\neq 0$ the $\mathcal{K}^v$-process never hits 0, whereas for $v=0$ the lifetime of the $\mathcal{K}^0$-process is the first hitting time of 0. Since $\tilde{L}=aL$, we have $\tilde{L}^v = aL^v$ and by Lemma 3.6 the following analogue of Lemma 3.4 holds: for some positive c_1 and c_2,

$$\begin{aligned}P_u^v(\tau_{D(\pi/p)}(U) > c_1 t) &\leqslant P_u^v(\tau_{D(\pi/p)}(\tilde{U}) > t)\\ &\leqslant P_u^v(\tau_{D(\pi/p)}(U) > c_2 t).\end{aligned}$$

Then by (3.11) and (3.10), (1.7) of Theorem 1.3 follows. □

4. Examples

Example 4.1. Suppose $f:\mathbb{R}^{d-1}\to\mathbb{R}$ is continuous with $f(0)=0$. If, for some $T>1$,

$$f(Tx) = Tf(x), \qquad x\in\mathbb{R}^{d-1}, \tag{4.1}$$

then it is easy to check that

$$G = \{(x,y) : y > f(x)\} \tag{4.2}$$

is T-homogeneous.

If f is locally Lipschitz, then Theorem 1.2 applies. If $d=2$ and $f\in C^2(\mathbb{R}\backslash\{0\})$ then Theorem 1.3 applies. It is not hard to check that the sufficient condition (4.1) for G of the form (4.2) to be T-homogeneous is also necessary.

Example 4.2. Suppose $f_1, f_2 : (0,\infty)\to\mathbb{R}$ are continuous with $0 < f_2(r) - f_1(r) < 2\pi$ for all $r>0$. If for some $T>0$ there exists an integer n such that

$$f_i(Tr) = f_i(r) + 2n\pi, \qquad r>0,\ i=1,2, \tag{4.3}$$

then it is easy to check

$$G = \{re^{i\theta} : r>0,\ f_1(r) < \theta < f_2(r)\} \tag{4.4}$$

is T-homogeneous.

If f_1 and f_2 are in $C^2(0,\infty)$, then Theorem 1.3 applies. Notice the condition $0 < f_2 - f_1 < 2\pi$ is natural to prevent crossing of the boundary curves. Also, the sufficient condition (4.3) is necessary for G to be of the form (4.4).

Example 4.3. Given $m > 0$ and $0 < b < 2\pi$, let

$$f_1(r) = m \log r,$$

$$f_2(r) = m \log r + b.$$

Since $\log(e^{2\pi/m} r) = 2\pi/m + \log r$, we have

$$f_i(e^{2\pi/m} r) = f_i(r) + 2\pi.$$

By Example 4.2,

$$G = \{re^{i\theta} : r > 0,\ m \log r < \theta < m \log r + b\}$$

is T-homogeneous for $T = e^{2\pi/m}$ and Theorem 1.3 applies. We can explicitly identify the power $p(G)$.

To this end, observe that G is the image of the strip

$$S = \left\{(x, y) : 0 < y < \frac{b}{\sqrt{1 + m^2}}\right\}$$

under the conformal mapping F given by a counterclockwise rotation through an angle of $\theta_0 = \tan^{-1} m$ followed by e^z. The Martin boundary of S less the Euclidean boundary consists of two points we denote by $\pm\infty$. The Martin kernel with pole at ∞ is a constant multiple of

$$f(x, y) = e^{\pi x\sqrt{m^2+1}/b} \sin \frac{\pi\sqrt{m^2 + 1}}{b} y.$$

Note that $F(-\infty) = 0$ and $F(\infty) = \infty$. Hence by conformal invariance of harmonic functions, $f \circ F^{-1}$ must be a multiple of the Martin kernel K^∞ of G with pole at ∞. Computing the composition explicitly, using $\cos \theta_0 = 1/\sqrt{m^2 + 1}$ and $\sin \theta_0 = m/\sqrt{m^2 + 1}$, we obtain

$$K^\infty(re^{i\theta}) = C \exp\Big(\frac{\pi}{b}(\log r + m\theta)\Big) \sin \frac{\pi}{b}(\theta - m \log r).$$

If $x = re^{i\theta(x)} \in G$ with $m \log r < \theta(x) < m \log r + b$, then by T-homogeneity, $\theta(Tx) = \theta(x) + 2\pi$. Therefore $Tx = Tre^{i(\theta(x)+2\pi)}$ and we can write

$$\begin{aligned} K^\infty(Tx) &= C \exp\Big(\frac{\pi}{b}(\log \mathrm{Tr} + m(\theta + 2\pi))\Big) \sin \frac{\pi}{b}(\theta + 2\pi - m \log \mathrm{Tr}) \\ &= T^{\pi b^{-1}(m^2+1)} K^\infty(x). \end{aligned}$$

Hence $p(G) = \pi b^{-1}(m^2 + 1)$.

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