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Are classes of deterministic integrands for fractional Brownian motion on an interval complete?

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Let B_H be a fractional Brownian motion with self-similarity parameter $H \in (0, 1)$ and $a > 0$ be a fixed real number. Consider the integral $\int_0^a f(u) \mathrm{d}B_H(u)$, where f belongs to a class of non-random integrands $\Lambda_{H,a}$. The integral will then be defined in the $L^2(\Omega)$ sense. One would like $\Lambda_{H,a}$ to be a complete inner-product space. This corresponds to a desirable situation because then there is an isometry between $\Lambda_{H,a}$ and the closure of the span generated by $B_H(u)$, $0 \leqslant u \leqslant a$. We show in this work that, when $H \in (\frac{1}{2}, 1)$, the classes of integrands $\Lambda_{H,a}$ one usually considers are not complete inner-product spaces even though they are often assumed in the literature to be complete. Thus, they are isometric not to $\overline{\mathrm{sp}}\{B_H(u), 0 \leqslant u \leqslant a\}$ but only to a proper subspace. Consequently, there are (random) elements in that closure which cannot be represented by functions f in $\Lambda_{H,a}$. We also show, in contrast to the case $H \in (\frac{1}{2}, 1)$ that there is a class of integrands for fractional Brownian motion B_H with $H \in (0, \frac{1}{2})$ on an interval $[0, a]$ which is a complete inner-product space.

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1. Introduction

A fractional Brownian motion (FBM) $B_H = \{B_H(u)\}_{u \in \mathbb{R}}$ with self-similarity parameter $H \in (0, 1)$ is a zero-mean Gaussian process which has stationary increments and is self-similar with index H, that is, for any $c > 0$,

$$\{B_H(cu)\}_{u \in \mathbb{R}} \stackrel{d}{=} \{c^H B_H(u)\}_{u \in \mathbb{R}},$$

where $\stackrel{d}{=}$ stands for equality in the sense of finite-dimensional distributions. For notational simplicity we will use another parametrization of FBM. Let

$$\kappa = H - \tfrac{1}{2},$$

so that the range $H \in (0, 1)$ now corresponds to the range $\kappa \in (-\frac{1}{2}, \frac{1}{2})$. We will denote the FBM $B_H = \{B_H(u)\}_{u \in \mathbb{R}}$ in terms of the parameter κ as $B^\kappa = \{B^\kappa(u)\}_{u \in \mathbb{R}}$. We will also assume that the FBM B^κ is standard, i.e. $\mathrm{E}(B^\kappa(1))^2 = 1$. By using stationarity of the

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increments and the self-similarity with index $\kappa + \frac{1}{2}$ of a standard FBM B^κ, it is easy to see that the covariance function of B^κ is given by

$$\Gamma^\kappa(u, v) = \mathrm{E}B^\kappa(u)B^\kappa(v) = \tfrac{1}{2}\{|u|^{2\kappa+1} + |v|^{2\kappa+1} - |u-v|^{2\kappa+1}\}, \qquad u, v \in \mathbb{R}. \tag{1.1}$$

When $\kappa = 0$, the FBM $B^\kappa = B^0$ is the usual Brownian motion which has independent increments. If $\kappa \neq 0$, the increments of the FBM B^κ are no longer independent: they are positively correlated if $\kappa \in (0, \frac{1}{2})$, and negatively correlated if $\kappa \in (-\frac{1}{2}, 0)$. Moreover, when $\kappa \in (0, \frac{1}{2})$, the dependence at large time lags is so strong that the series $\sum_{k=1}^\infty \Gamma^\kappa(1, k)$ diverges. In this case, one says that the FBM B^κ exhibits long-range dependence. (For more information on FBM see, for example, Samorodnitsky and Taqqu 1994).

In this work we deal with questions related to the $L^2(\Omega)$-integration of *deterministic* functions with respect to the FBM B^κ when $\kappa \in (-\frac{1}{2}, \frac{1}{2})$. To define such integrals, one typically starts with an inner-product space $(\mathcal{C}, (\cdot, \cdot)_{\mathcal{C}})$ of functions on a region of integration R (say, $R = \mathbb{R}$ or $[0, a]$ with $a > 0$) such that $(1_{[0,s)}, 1_{[0,t)})_{\mathcal{C}} = \mathrm{E}B^\kappa(s)B^\kappa(t)$ for all $s, t \in R$. Let $\overline{\mathrm{sp}}_R(B^\kappa)$ be the closure in $L^2(\Omega)$ of all posible linear combinations of the increments of FBM on R. If the map $1_{[0,t)} \mapsto B^\kappa(t)$ extends to the isometry between this class of functions $\mathscr{C}$ and the space $\overline{\mathrm{sp}}_R(B^\kappa)$, then the resulting isometry map is called the integral in the $L^2(\Omega)$ sense with respect to FBM of functions from $\mathcal{C}$.

The extension step is usually taken for granted (i.e. not proved). So, for example, when $\kappa \in (0, \frac{1}{2})$ and the region of integration R is $[0, a]$ with $a > 0$, Carmona *et al.* (1999) and Kleptsyna *et al.* (1999b) defined the class of integrands for FBM

$$\Lambda_a^\kappa = \left\{ f : [0, a] \mapsto \mathbb{R} \text{ such that } \int_0^a [s^{-\kappa}(I_{a-}^\kappa u^\kappa f(u))(s)]^2 \,\mathrm{d}s < \infty \right\}, \tag{1.2}$$

where I_{a-}^κ is a fractional integral operator defined in Section 2 below (see (2.1)). Another class of integrands, considered by Duncan *et al.* (2000), Kleptsyna *et al.* (1999a) and Norros *et al.* (1999), is given by

$$|\Lambda|_a^\kappa = \left\{ f : [0, a] \mapsto \mathbb{R} \text{ such that } \int_0^a\int_0^a |f(u)\|\ f(v)\ \|u-v|^{2\kappa-1} \,\mathrm{d}u\,\mathrm{d}v < \infty \right\}. \tag{1.3}$$

The classes of integrands Λ_a^κ and $|\Lambda|_a^\kappa$ are assumed to be inner-product spaces with the inner products

$$(f, g)_{\Lambda_a^\kappa} = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin\pi\kappa} \int_0^a s^{-2\kappa}(I_{a-}^\kappa u^\kappa f(u))(s)(I_{a-}^\kappa u^\kappa g(u))(s)\mathrm{d}s \tag{1.4}$$

($\Gamma(p) = \int_0^\infty \mathrm{e}^{-v} v^{p-1}\,\mathrm{d}v$, $p > 0$, is the gamma function) and

$$(f, g)_{|\Lambda|_a^\kappa} = \kappa(2\kappa+1)\int_0^a\int_0^a f(u)g(v)|u-v|^{2\kappa-1}\,\mathrm{d}u\,\mathrm{d}v, \tag{1.5}$$

respectively. (For details on the construction of classes of integrands Λ_a^κ and $|\Lambda|_a^\kappa$, see Section 4 below.) All the authors claimed that both Λ_a^κ and $|\Lambda|_a^\kappa$ are isometric to the Gaussian space $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$. Since $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ is a complete inner-product space, both Λ_a^κ and $|\Lambda|_a^\kappa$ necessarily have to be complete inner-product spaces as well. We will show in this work that, when $\kappa \in (0, \frac{1}{2})$, *neither the space of functions* Λ_a^κ *nor the space of functions* $|\Lambda|_a^\kappa$ *is a complete* *inner-product space*. Thus, they cannot be isometric to $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ itself: in fact, they are isometric only to proper linear subspaces of $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$. Consequently, there are (random) elements in $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ which cannot be represented by functions f belonging to either Λ_a^κ or $|\Lambda|_a^\kappa$.

One area of applications of integrals with respect to FBM, where completeness of classes of integrands is relevant, involves prediction problems. Consider, for example, the problem of predicting the value of an FBM at some future time $t > 0$ given its past from time 0 to time a (with $a < t$), or in mathematical terms, of computing the conditional expectation $X = \mathrm{E}(B^\kappa(t)|B^\kappa(s),\ s \in [0, a])$. It is well known that $X \in \overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$. One would expect that $X = \int_0^a f\,\mathrm{d}B^\kappa$. But, when $\kappa > 0$, in view of the above-mentioned incompleteness results, there may be no f belonging to Λ_a^κ or $|\Lambda|_a^\kappa$ such that $X = \int_0^a f\,\mathrm{d}B^\kappa$. In fact, such an f exists as is shown in Section 7 below, where the prediction problem for FBM is discussed.

Although $a > 0$ is assumed to be a real number, our results are also valid in the case $a = \infty$. (The space of integrands $|\Lambda|_a^\kappa$ with $a = \infty$ is considered in Duncan *et al*. 2000 and Norros *et al*. 1999). As shown in Section 6 below, when $\kappa \in (0, \frac{1}{2})$ incompleteness of classes of integrands in the case $a = \infty$ can in principle be deduced from that of classes of integrands in the case $a > 0$.

The results described above are in the same spirit as those of Pipiras and Taqqu (2000), where the integration is over $\mathbb{R}$. The $\mathbb{R}$ set-up, however, is quite different from that of $[0, a]$ considered here. When $\kappa \in (0, \frac{1}{2})$, an $\mathbb{R}$-analogue of the space of integrands Λ_a^κ is the inner-product space

$$\Lambda^\kappa = \left\{ f : \mathbb{R} \mapsto \mathbb{R} \text{ such that } \int_{\mathbb{R}} [(I_-^\kappa f)(s)]^2\,\mathrm{d}s < \infty \right\} \tag{1.6}$$

with the inner product

$$(f, g)_{\Lambda^\kappa} = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2} \int_{\mathbb{R}} (I_-^\kappa f)(s)(I_-^\kappa g)(s)\mathrm{d}s, \tag{1.7}$$

where $(I_-^\kappa f)(s) = (\Gamma(\kappa))^{-1} \int_{\mathbb{R}} f(u)(u-s)_+^{\kappa-1}\,\mathrm{d}u$, $s \in \mathbb{R}$, is a fractional integral on $\mathbb{R}$ and $c_1(\kappa)$ is some constant which depends on κ. As shown in Theorem 3.2 of Pipiras and Taqqu (2000), the inner-product space Λ^κ is not complete. We do not know whether one can use this to conclude immediately that Λ_a^κ is incomplete. We shall provide a proof that relies indirectly on the incompleteness of Λ^κ. Obtaining results concerning integration over $[0, a]$ or $[0, \infty)$ is important in practice (most of the papers quoted above consider integration over this range).

Suppose now that the parameter κ belongs to the range $(-\frac{1}{2}, 0)$. (This range corresponds to $H \in (0, \frac{1}{2})$ or no long-range dependence.) It is natural to ask whether there is a class of integrands for the FBM B^κ with $\kappa \in (-\frac{1}{2}, 0)$ on an interval $[0, a]$ which is a complete inner-product space. It turns out that such a class of integrands exists. It is an inner-product space

$$\Lambda_a^\kappa = \{ f : \exists\, \phi_f \in L^2[0, a] \text{ such that } f(u) = u^{-\kappa}(I_{a-}^{-\kappa} s^\kappa \phi_f(s))(u) \}, \tag{1.8}$$

with the inner product

$$(f, g)_{\Lambda_a^\kappa} = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin\pi\kappa}\int_0^a s^{-2\kappa}(I_{a-}^\kappa u^\kappa f(u))(s)(I_{a-}^\kappa u^\kappa g(u))(s)\mathrm{d}s \tag{1.9}$$

$$= \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin\pi\kappa}\int_0^a \phi_f(s)\phi_g(s)\mathrm{d}s, \tag{1.10}$$

where $\phi_f, \phi_g \in L^2[0, a]$ are associated with the functions f and g, respectively, by definition (1.8), and I_{a-}^κ with $\kappa \in (-\frac{1}{2}, 0)$ are fractional derivatives of order $-\kappa$. (The fractional derivatives I_{a-}^κ with $\kappa \in (-\frac{1}{2}, 0)$ are defined in Section 2 below. They satisfy the property $I_{a-}^\kappa I_{a-}^{-\kappa}\phi = \phi$ for any $\phi \in L^1[0, a]$ and $\kappa \in (-\frac{1}{2}, 0)$, which explains the equality of (1.9) and (1.10).) For the construction of the class of integrands Λ_a^κ with $\kappa \in (-\frac{1}{2}, 0)$ and for the proof of its completeness, see Section 4 below.

This work is organized as follows. In Section 2 we provide a quick review of fractional integrals and derivatives that are used in this work. Then, in Section 3, we represent FBM on an interval in terms of these fractional integrals and derivatives. We use this representation in Section 4 to construct the classes of integrands Λ_a^κ with $\kappa \in (-\frac{1}{2}, \frac{1}{2})$ and $|\Lambda|_a^\kappa$ with $\kappa \in (0, \frac{1}{2})$. We also show, in Section 4, that Λ_a^κ is a complete inner-product space when $\kappa \in (-\frac{1}{2}, 0)$ and, in Section 5, that Λ_a^κ and $|\Lambda|_a^\kappa$ are not complete inner-product spaces when $\kappa \in (0, \frac{1}{2})$. In section 6 we deal with the case $a = \infty$. We consider the prediction problem for FBM in Section 7. Finally, in Section 8 we prove some of the results of Section 4.

2. Fractional integrals and derivatives

An exhaustive source on fractional integrals and derivatives is the book by Samko *et al.* (1993). For the reader's convenience, we provide below definitions of those fractional operators that are used throughout this paper and also list a number of their properties.

Consider the interval $[0, a]$ and let $s \in [0, a]$. An integral over $[0, s]$ is called left-sided and one over $[s, a]$ is called right-sided. The right-sided fractional *integral* of order $\alpha > 0$ on an interval $[0, a]$ of a function $f \in L^1[0, a]$ is defined by

$$(I_{a-}^\alpha f)(s) = \frac{1}{\Gamma(\alpha)}\int_0^a f(u)(u-s)_+^{\alpha-1}\,\mathrm{d}u = \frac{1}{\Gamma(\alpha)}\int_s^a f(u)(u-s)^{\alpha-1}\,\mathrm{d}u, \qquad s \in (0, a) \tag{2.1}$$

(see Samko *et al.* 1993, p. 33). The right-sided fractional *derivative* of order $0 < \alpha < 1$ on an interval $[0, a]$ of a function ϕ is defined by

$$(\mathcal{D}_{a-}^\alpha \phi)(u) = -\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}u}\int_0^a \phi(s)(s-u)_+^{-\alpha}\,\mathrm{d}s, \qquad u \in (0, a) \tag{2.2}$$

(see Samko *et al.* 1993, p. 35). If $0 < \alpha < 1$ and

$$\phi(s) = (I_{a-}^\alpha f)(s), \qquad s \in (0, a), \tag{2.3}$$

then

$$f(u) = (\mathcal{D}_{a-}^\alpha \phi)(u), \qquad u \in (0, a) \tag{2.4}$$

(see Samko *et al.* 1993, pp. 29–31). Hence, $\mathcal{D}_{a-}^\alpha$ can be viewed as an inverse of I_{a-}^α.

(Heuristically, performing the differentiation over u in (2.2) yields (2.1) with $\mathcal{D}^{\alpha}_{a-} = I^{-\alpha}_{a-}$.) For this reason, we will often denote the fractional derivative $\mathcal{D}^{\alpha}_{a-}$ with $\alpha \in (0, 1)$ by $I^{-\alpha}_{a-}$ and also use the notation I^{0}_{a-} for the identity operator, that is, $I^{0}_{a-}f = f$. One can also show that, for any $f \in L^1[0, a]$,

$$(\mathcal{D}^{\alpha}_{a-} I^{\alpha}_{a-} f)(s) = f(s), \qquad s \in (0, a) \tag{2.5}$$

(see Samko *et al*. 1993, p. 24).

3. Representation of fractional Brownian motion on an interval

The following proposition relates FBM and the fractional integral and derivative operators on an interval introduced in Section 2. It will be used in Section 4 to construct classes of integrands for FBM on an interval $[0, a]$.

Proposition 3.1. *Let $a > 0$ and B^{κ} be a standard FBM with parameter $\kappa \in (-\frac{1}{2}, \frac{1}{2})$. Then*

$$\{B^{\kappa}(t)\}_{t\in[0,a]} \stackrel{d}{=} \left\{\sigma_1(\kappa)\int_0^a s^{-\kappa}(I^{\kappa}_{a-}u^{\kappa}1_{[0,t)}(u))(s)\mathrm{d}B^0(s)\right\}_{t\in[0,a]}, \tag{3.1}$$

where

$$\sigma_1(\kappa)^2 = \frac{\Gamma(\kappa)^2\kappa(2\kappa+1)}{\mathrm{B}(\kappa, 1-2\kappa)} = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin\pi\kappa} \tag{3.2}$$

and $\mathrm{B}(p, q) = \int_0^1 (1-v)^{p-1}v^{q-1}\,\mathrm{d}v = \Gamma(p)\Gamma(q)/\Gamma(p+q)$, *$p, q > 0$, is the beta function.*

Proof. Suppose first that $\kappa \in (0, \frac{1}{2})$. It follows from (1.1) that $\mathrm{d}^2\Gamma^{\kappa}(u, v) = \kappa(2\kappa+1)|u-v|^{2\kappa-1}\,\mathrm{d}u\,\mathrm{d}v$ and hence that, for any $t_1, t_2 \in \mathbb{R}$,

$$\Gamma^{\kappa}(t_1, t_2) = \kappa(2\kappa+1)\int_{\mathbb{R}}\int_{\mathbb{R}} 1_{[0,t_1)}(u)1_{[0,t_2)}(u)|u-v|^{2\kappa-1}\,\mathrm{d}u\,\mathrm{d}v. \tag{3.3}$$

By making the change of variables $s = (uv)z/(\min(u, v)z + |u-v|)$ below, one can show that, for $\kappa \in (0, \frac{1}{2})$ and $u, v \in [0, a]$,

$$|u-v|^{2\kappa-1} = \frac{(uv)^{\kappa}}{\mathrm{B}(\kappa, 1-2\kappa)}\int_0^a s^{-2\kappa}(v-s)_+^{\kappa-1}(u-s)_+^{\kappa-1}\,\mathrm{d}s. \tag{3.4}$$

Then, for $t_1, t_2 \in [0, a]$, we obtain from (3.3), (3.4) and definition (2.1) that

$$\Gamma^{\kappa}(t_1, t_2) = \frac{\Gamma(\kappa)^2\kappa(2\kappa+1)}{\mathrm{B}(\kappa, 1-2\kappa)}\int_0^a (s^{-\kappa}(I^{\kappa}_{a-}u^{\kappa}1_{[0,t_1)}(u))(s))(s^{-\kappa}(I^{\kappa}_{a-}u^{\kappa}1_{[0,t_2)}(u))(s))\mathrm{d}s.$$

Hence, the process on the right-hand side of (3.1) has the same covariance structure as a standard FBM. Since it is also Gaussian and has zero mean, it is a standard FBM.

The case $\kappa \in (-\frac{1}{2}, 0)$ is more delicate. Observe first that, for $\kappa \in (-\frac{1}{2}, \frac{1}{2})$,

$$-\Gamma(1+\kappa)s^{-\kappa}(I^{\kappa}_{a-}u^{\kappa}1_{[0,t)}(u))(s) = s^{-\kappa}\frac{\mathrm{d}}{\mathrm{d}s}\int_0^t u^{\kappa}(u-s)^{\kappa}_{+}\,\mathrm{d}u \tag{3.5}$$

$$= \kappa s^{-\kappa}\int_s^t u^{\kappa-1}(u-s)^{\kappa}_{+}\,\mathrm{d}u - \left(\frac{t}{s}\right)^{\kappa}(t-s)^{\kappa}_{+}. \tag{3.6}$$

Indeed, by changing, in the relation below, the order of integration and performing integration by parts, we obtain that, for any $v \in [0, a]$,

$$\kappa\int_v^t\int_s^t u^{\kappa-1}(u-s)^{\kappa}_{+}\,\mathrm{d}u\,\mathrm{d}s - \int_v^t t^{\kappa}(t-s)^{\kappa}_{+}\,\mathrm{d}s = \kappa\int_v^t \mathrm{d}u\, u^{\kappa-1}\int_v^u (u-s)^{\kappa}\,\mathrm{d}s - \frac{t^{\kappa}(t-v)^{\kappa+1}}{\kappa+1}$$

$$= \frac{\kappa}{\kappa+1}\int_v^t u^{\kappa-1}(u-v)^{\kappa+1}\,\mathrm{d}u - \frac{t^{\kappa}(t-v)^{\kappa+1}}{\kappa+1}$$

$$= -\int_v^t u^{\kappa}(u-v)^{\kappa}_{+}\,\mathrm{d}u. \tag{3.7}$$

The equality of (3.5) and (3.6) then follows by taking the derivative of both sides in (3.7).

The idea now is to view (3.5) or (3.6) as the definition of the left-hand side of (3.5) for κ complex, $|\kappa| < \frac{1}{2}$. Let $t_1, t_2 \in [0, a]$ be fixed and $\kappa \in \mathbb{C}$ be such that $|\kappa| < \frac{1}{2}$. Then one can verify (see Appendix) that the function

$$f(\kappa) = \sigma_1(\kappa)^2\int_0^a s^{-2\kappa}(I^{\kappa}_{a-}u^{\kappa}1_{[0,t_1)}(u))(s)(I^{\kappa}_{a-}u^{\kappa}1_{[0,t_2)}(u))(s)\mathrm{d}s$$

is analytic on $\{\kappa : |\kappa| < \frac{1}{2}\}$. On the other hand, the function

$$g(\kappa) = \Gamma^{\kappa}(t_1, t_2) = \tfrac{1}{2}\{|t_1|^{2\kappa+1} + |t_2|^{2\kappa+1} - |t_2 - t_1|^{2\kappa+1}\}$$

is also analytic on $\{\kappa : |\kappa| < \frac{1}{2}\}$. Since the analytic functions f and g coincide for real $\kappa \in (0, \frac{1}{2})$, standard results of complex analysis (see, for example, Conway 1995, p. 79) imply that they have to coincide on $\{\kappa : |\kappa| < \frac{1}{2}\}$. In particular, $f(\kappa) = g(\kappa)$ for real $\kappa \in (-\frac{1}{2}, 0)$. But this means that the Gaussian zero-mean process on the right-hand side of (3.1) has the same covariance structure as a standard FBM for $\kappa \in (-\frac{1}{2}, 0)$ as well. Hence, it is a standard FBM. □

Remark 3.1. The representation of FBM with the kernel function on the right-hand side of (3.5) is also given in Kleptsyna *et al.* (1999b). The representation with the kernel function in (3.6) can be found in Norros *et al.* (1999). Using again an analyticity argument, one can also show that, for $\kappa \in (-\frac{1}{2}, \frac{1}{2})$ and $s, t \in [0, a]$,

$$\Gamma(\kappa+1)s^{-\kappa}(I^{\kappa}_{a-}u^{\kappa}1_{[0,t)}(u))(s) = (t-s)^{\kappa}_{+}\,{}_2F_1\left(-\kappa, \kappa, \kappa+1, 1-\frac{t}{s}\right),$$

where ${}_2F_1$ is the so-called Gauss hypergeometric function. For more details on ${}_2F_1$, see Decreusefond and Üstünel (1999).

Remark 3.2. Heuristically, the equality of (3.5) and (3.6) can be obtained as follows. If $\delta_{\{t-s\}}$ denotes the delta function at a point $t-s$, we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\int_0^t u^\kappa(u-s)_+^\kappa\,\mathrm{d}u &= \frac{\mathrm{d}}{\mathrm{d}s}\int_0^a (v+s)^\kappa v^\kappa 1_{\{0<v<t-s\}}(v)\mathrm{d}v \\ &= \kappa\int_0^a (v+s)^{\kappa-1}v^\kappa 1_{\{0<v<t-s\}}(v)\mathrm{d}v - \int_0^a (v+s)^\kappa v^\kappa \delta_{\{t-s\}}(v)\mathrm{d}v \\ &= \kappa\int_s^t u^{\kappa-1}(u-s)_+^\kappa\,\mathrm{d}u - t^\kappa(t-s)_+^\kappa.\end{aligned}$$

4. Integrands for fractional Brownian motion on an interval

Let $\{B^\kappa(u)\}_{u\in[0,a]}$ be a standard FBM with parameter $\kappa\in(-\frac{1}{2},\frac{1}{2})$. Let $\mathcal{E}_a$ denote the set of all elementary (step) functions on an interval $[0, a]$, that is, functions of the following type:

$$f(u)=\sum_{k=1}^n f_k 1_{[u_k,u_{k+1})}(u), \qquad f_k\in\mathbb{R},\ u_k\in[0,a]. \tag{4.1}$$

For an elementary function $f\in\mathcal{E}_a$ in (4.1), define the integral with respect to the FBM B^κ in a natural way by

$$\mathcal{I}_a^\kappa(f)=\int_0^a f(u)\mathrm{d}B^\kappa(u)=\sum_{k=1}^n f_k(B^\kappa(u_{k+1})-B^\kappa(u_k)). \tag{4.2}$$

It follows by Proposition 3.1 that, for $f\in\mathcal{E}_a$,

$$\mathcal{I}_a^\kappa(f)\overset{d}{=}\sigma_1(\kappa)\int_0^a s^{-\kappa}(I_{a-}^\kappa u^\kappa f(u))(s)\mathrm{d}B^0(s) \tag{4.3}$$

and hence that, for all $f, g\in\mathcal{E}_a$,

$$\mathrm{E}(\mathcal{I}_a^\kappa(f)\mathcal{I}_a^\kappa(g))=\sigma_1(\kappa)^2\int_0^a s^{-2\kappa}(I_{a-}^\kappa u^\kappa f(u))(s)(I_{a-}^\kappa u^\kappa g(u))(s)\mathrm{d}s. \tag{4.4}$$

The following theorem will be used to extend the map $\mathcal{I}_a^\kappa$ on $\mathcal{E}_a$ to functions from Λ_a^κ when $\kappa\in(0,\frac{1}{2})$.

Theorem 4.1. *For* $\kappa\in(0,\frac{1}{2})$*, the class of functions* Λ_a^κ*, defined by* (1.2)*, is a linear space with the inner product* (1.4)*. Moreover, the set of elementary functions* $\mathcal{E}_a$ *is dense in the space* Λ_a^κ*.*

We prove Theorem 4.1 in Section 8. Now let $\kappa\in(0,\frac{1}{2})$. By Theorem 4.1, since the set of elementary functions $\mathcal{E}_a$ is dense in the space Λ_a^κ, one can extend the map $\mathcal{I}_a^\kappa : \mathcal{E}_a\mapsto\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ to the map $\mathcal{I}_a^\kappa:\Lambda_a^\kappa\mapsto\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ in a classical way so that relation (4.4) continues to hold for functions $f, g\in\Lambda_a^\kappa$. Since the extended map $\mathcal{I}_a^\kappa$ is linear and preserves inner products, one can say that Λ_a^κ is isometric to a linear subspace of $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$. The question is whether this linear subspace of $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ is the space $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ itself. It is clear that Λ_a^κ is isometric to $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ itself if and only if Λ_a^κ is a complete inner-product space. We will show in Section 5 that Λ_a^κ is not a complete inner-product space. Hence, it is not isometric to $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ itself. It is isometric to a proper linear subspace of $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ only.

Let us now show that $|\Lambda|_a^\kappa$, given by (1.3), is a linear subspace of the class of integrands Λ_a^κ. Suppose that the function f is such that $|f| \in \Lambda_a^\kappa$ so that $f \in \Lambda_a^\kappa$ as well. Then, as in the proof of Proposition 3.1, by the Fubini's theorem we obtain

$$\int_0^a s^{-2\kappa}[(I_{a-}^\kappa u^\kappa f(u))(s)]^2 \,\mathrm{d}s = \frac{\mathrm{B}(\kappa, 1-2\kappa)}{\Gamma(\kappa)^2}\int_0^a\int_0^a f(u)f(v)|u-v|^{2\kappa-1}\,\mathrm{d}u\,\mathrm{d}v, \tag{4.5}$$

which shows that $|\Lambda|_a^\kappa \subset \Lambda_a^\kappa$. A linear space $|\Lambda|_a^\kappa$ becomes an inner-product space if the inner product on $|\Lambda|_a^\kappa$ is defined by $(f, g)_{|\Lambda|_a^\kappa} = (f, g)_{\Lambda_a^\kappa}$ for $f, g \in |\Lambda|_a^\kappa$. It follows by (1.4) and (4.5) that

$$(f, g)_{|\Lambda|_a^\kappa} = \frac{\sigma_1(\kappa)^2\mathrm{B}(\kappa, 1-2\kappa)}{\Gamma(\kappa)^2}\int_0^a\int_0^a f(u)f(v)|u-v|^{2\kappa-1}\,\mathrm{d}u\,\mathrm{d}v. \tag{4.6}$$

Observe, however, that the constant in (4.6) equals $\kappa(2\kappa+1)$ because $(1_{[0,t_1)}, 1_{[0,t_2)})_{|\Lambda|_a^\kappa} = \Gamma^\kappa(t_1, t_2) = \kappa(2\kappa+1)\int_0^a\int_0^a 1_{[0,t_1)}(u)1_{[0,t_2)}(v)|u-v|^{2\kappa-1}\,\mathrm{d}u\,\mathrm{d}v$ for $t_1, t_2 \in [0, a]$ by relation (3.3). In other words, the inner product on $|\Lambda|_a^\kappa$ equals (1.5) as well.

Remark 4.1. An $\mathbb{R}$-analogue of the space of integrands $|\Lambda|_a^\kappa$ is the class of functions

$$|\Lambda|^\kappa = \left\{f : \mathbb{R} \mapsto \mathbb{R} \text{ such that } \int_{\mathbb{R}}\int_{\mathbb{R}} |f(u)||\,f(v)\,||u-v|^{2\kappa-1}\,\mathrm{d}u\,\mathrm{d}v < \infty\right\}, \tag{4.7}$$

for $\kappa \in (0, \frac{1}{2})$ (see Pipiras and Taqqu 2000). It is also an inner-product space with the inner product

$$(f, g)_{|\Lambda|^\kappa} = \kappa(2\kappa+1)\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)|u-v|^{2\kappa-1}\,\mathrm{d}u\,\mathrm{d}v. \tag{4.8}$$

Moreover, as shown in Pipiras and Taqqu (2000), $|\Lambda|^\kappa$ is a strict subspace of the class of integrands Λ^κ, given by (1.6), and the inner products (1.7) and (4.8) coincide when f, g belong to the smaller class $|\Lambda|^\kappa$.

Remark 4.2. For $\kappa \in (0, \frac{1}{2})$, the following useful inclusions hold:

$$L^2[0, a] \subset L^{2/2\kappa+1}[0, a] \subset |\Lambda|_a^\kappa \subset \Lambda_a^\kappa. \tag{4.9}$$

The first inclusion in (4.9) is obvious since $2/2\kappa+1 < 2$ for $\kappa > 0$. According to Proposition 4.2 in Pipiras and Taqqu (2000), we have $\|f\|_{|\Lambda|^\kappa} \leqslant c_\kappa \|f\|_{L^{2/2\kappa+1}(\mathbb{R})}$ for some $c_\kappa > 0$ and all $f \in L^{2/2\kappa+1}(\mathbb{R})$. This implies the second inclusion in (4.9), since

$$\|f\|_{|\Lambda|_a^\kappa} = \|f1_{[0,a]}\|_{|\Lambda|^\kappa} \leqslant c_\kappa\|f1_{[0,a]}\|_{L^{2/2\kappa+1}(\mathbb{R})} = c_\kappa\|f\|_{L^{2/2\kappa+1}[0,a]}.$$

Let us also observe that the inclusion $L^2[0, a] \subset |\Lambda|_a^\kappa$ is easy to verify directly. Indeed, by using the inequality $2|f(u)|\,|f(v)| \leqslant |f(u)|^2 + |f(v)|^2$ and symmetry below, we obtain

$$\int_0^a\int_0^a |f(u)|\,|f(v)|\,|u-v|^{2\kappa-1}\,\mathrm{d}u\,\mathrm{d}v \leqslant \int_0^a\int_0^a |f(u)|^2|u-v|^{2\kappa-1}\,\mathrm{d}u\,\mathrm{d}v \leqslant \frac{a^{2\kappa}}{\kappa}\int_0^a |f(u)|^2\,\mathrm{d}u. \tag{4.10}$$

Suppose now that $\kappa \in (-\frac{1}{2}, 0)$. Set $f(u) = u^{-\kappa}(I_{a-}^{-\kappa} s^\kappa \phi(s))(u)$ for some function $\phi \in L^2[0, a]$. It is well defined for $\phi \in L^2[0, a]$ because the function $s^\kappa\phi(s)$ is in $L^1[0, a]$ for $\kappa > -\frac{1}{2}$ by the Hölder's inequality. Property (2.5) then implies

$$s^{-\kappa}(I_{a-}^{\kappa} u^\kappa f(u))(s) = s^{-\kappa}(\mathscr{D}_{a-}^{-\kappa} u^\kappa u^{-\kappa}(I_{a-}^{-\kappa} z^\kappa \phi(z))(u))(s) = \phi(s)$$

and hence that

$$\int_0^a s^{-2\kappa}[(I_{a-}^{\kappa} u^\kappa f(u))(s)]^2\,\mathrm{d}s = \int_0^a \phi(s)^2\,\mathrm{d}s < \infty.$$

Based on this observation and relations (4.3) and (4.4), it is natural to introduce the class of functions Λ_a^κ given by (1.8). The following result is analogous to Theorem 4.1 and deals with the case $\kappa \in (-\frac{1}{2}, 0)$.

Theorem 4.2. *For $\kappa \in (-\frac{1}{2}, 0)$, the class of functions Λ_a^κ, defined by* (1.8)*, is a linear space with the inner product* (1.9) *or* (1.10)*. Moreover, the set of elementary functions $\mathcal{E}_a$ is dense in the space Λ_a^κ.*

We prove Theorem 4.2 in Section 8. By Theorem 4.2, since the set of elementary functions $\mathcal{E}_a$ is dense in Λ_a^κ, one can extend the map $\mathcal{I}_a^\kappa : \mathcal{E}_a \mapsto \overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ to the map $\mathcal{I}_a^\kappa : \Lambda_a^\kappa \mapsto \overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ so that it is linear and preserves inner products.

The space Λ_a^κ can be viewed as a class of integrands for FBM on an interval $[0, a]$. It is also isometric to a linear subspace of $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$. Is this linear subspace the space $\overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ itself? Or, equivalently, is the space of functions Λ_a^κ a complete inner-product space? The answer is an obvious yes. Indeed, suppose that $\{f_n\}_{n\geqslant 1}$ is a Cauchy sequence in Λ_a^κ. Then the sequence of functions ϕ_{f_n}, associated with the functions f_n by definition (1.8), is Cauchy in $L^2[0, a]$. Since the space $L^2[0, a]$ is complete, there is a function $\phi \in L^2[0, a]$ such that $\phi_{f_n} \to \phi$ in $L^2[0, a]$. If $f(u) = u^{-\kappa}(I_{a-}^{-\kappa} s^\kappa \phi(s))(u)$, then $f_n \to f$ in Λ_a^κ since $\phi_{f_n} \to \phi$ in $L^2[0, a]$. This shows the completeness of Λ_a^κ.

Remark 4.3. An $\mathbb{R}$-analogue of the space of integrands Λ_a^κ with $\kappa \in (-\frac{1}{2}, 0)$ is the class of functions

$$\Lambda^\kappa = \{f : \exists\, \phi_f \in L^2(\mathbb{R}) \text{ such that } f = I_-^{-\kappa}\phi_f\} \tag{4.11}$$

(see Pipiras and Taqqu 2000). The class Λ^κ is an inner-product space with the inner product

$$(f,\, g)_{\Lambda^\kappa} = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2}\int_{\mathbb{R}} (\mathbf{D}_-^{-\kappa} f)(s)(\mathbf{D}_-^{-\kappa} g)(s)\mathrm{d}s \tag{4.12}$$

$$= \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2}\int_{\mathbb{R}} \phi_f(s)\phi_g(s)\mathrm{d}s, \tag{4.13}$$

where $\mathbf{D}_-^{-\kappa}$, $\kappa \in (-\frac{1}{2}, 0)$, is the so-called Marchaud fractional derivative of order $-\kappa$ which has the property $\mathbf{D}_-^{-\kappa} I_-^{-\kappa}\phi = \phi$ for $\kappa \in (-\frac{1}{2}, 0)$ and $\phi \in L^2(\mathbb{R})$. Observe that, when $\kappa \in (-\frac{1}{2}, 0)$, both the space Λ^κ and the space Λ_a^κ are built up by exploiting the same idea.

Remark 4.4. We emphasize again that the inner-product space Λ_a^κ is *complete* if $\kappa \in (-\frac{1}{2}, 0)$, and *incomplete* if $\kappa \in (0, \frac{1}{2})$. The proof in Section 5 below and the proof of completeness preceding Remark 4.3 show that this difference in completeness is a consequence of the following two facts:

(a) If $\kappa \in (-\frac{1}{2}, 0)$, then the equation

$$s^{-\kappa}(I_{a-}^{\kappa} u^\kappa f(u))(s) = s^{-\kappa}(\mathscr{D}_{a-}^{-\kappa} u^\kappa f(u))(s) = \phi(s) \tag{4.14}$$

has a solution $f(u) = u^{-\kappa}(I_{a-}^{-\kappa} s^\kappa \phi(s))(u)$ for every $\phi \in L^2[0, a]$.

(b) When $\kappa \in (0, \frac{1}{2})$, however, there are functions $\phi \in L^2[0, a]$ for which the equation

$$s^{-\kappa}(I_{a-}^{\kappa} u^\kappa f(u))(s) = \phi(s) \tag{4.15}$$

is not solvable. The idea here is that, since I_{a-}^{κ} is the *integral* operator, the left-hand side of (4.15) must satisfy some *smoothness* conditions (for example, one can take its weighted fractional derivative $(\mathcal{D}_{a-}^{\kappa} s^\kappa)$) whereas such smoothness conditions need not hold for a general $\phi \in L^2[0, a]$.

5. Incompleteness of classes of integrands when $0 < \kappa < \frac{1}{2}$

The goal of this section is to show that neither the space $|\Lambda|_a^\kappa$ nor the space Λ_a^κ is a complete inner-product space when $\kappa \in (0, \frac{1}{2})$. We will give a proof by first providing an equivalent criterion for the completeness and then showing that it does not hold. We begin with a number of lemmas which will be required. The parameter κ is in the range $(0, \frac{1}{2})$ throughout.

Lemma 5.1. *Let* $0 \leqslant c < b \leqslant a$ *and* $\kappa \in (0, \frac{1}{2})$. *Then there is a function* $f_{c,b}$ *such that*

$$s^{-\kappa}(I_{a-}^{\kappa} u^\kappa f_{c,b}(u))(s) = 1_{[c,b)}(s), \qquad \textit{for all } 0 \leqslant s \leqslant a. \tag{5.1}$$

Proof. Heuristically, by solving equation (5.1), we obtain that

$$f_{c,b}(u) = u^{-\kappa}(I_{a-}^{-\kappa} s^\kappa 1_{[c,b)}(s))(u) = u^{-\kappa}(\mathcal{D}_{a-}^{\kappa} s^\kappa 1_{[c,b)}(s))(u). \tag{5.2}$$

To show that this function indeed satisfies (5.1), we may assume by linearity that $c = 0$. We may also show (5.1) for $s < b$ only, since for $s \geqslant b$ it is obvious (both sides are zero). As in (3.5) and (3.6) (see also Remark 3.2 for heuristics), we have that

$$-\Gamma(1-\kappa)(\mathcal{D}^{\kappa}_{a-}s^{\kappa}1_{[c,b)}(s))(u) = \kappa\int_u^b s^{\kappa-1}(s-u)^{-\kappa}\,\mathrm{d}s - b^{\kappa}(b-u)_+^{-\kappa}.$$

Then, for $s < b$,

$$\begin{aligned}
-\Gamma(1-\kappa)\Gamma(\kappa)&s^{-\kappa}(I^{\kappa}_{a-}u^{\kappa}u^{-\kappa}(\mathcal{D}^{\kappa}_{a-}z^{\kappa}1_{[c,b)}(z))(u))(s)\\
&= s^{-\kappa}\int_s^a\left\{\kappa\int_u^b z^{\kappa-1}(z-u)^{-\kappa}\,\mathrm{d}z - b^{\kappa}(b-u)_+^{-\kappa}\right\}(u-s)^{\kappa-1}\,\mathrm{d}u\\
&= s^{-\kappa}\kappa\int_s^b \mathrm{d}z\, z^{\kappa-1}\int_s^z(z-u)^{-\kappa}(u-s)^{\kappa-1}\,\mathrm{d}u - s^{-\kappa}b^{\kappa}\int_s^b(b-u)_+^{-\kappa}(u-s)^{\kappa-1}\,\mathrm{d}u\\
&= \mathrm{B}(1-\kappa,\kappa)\left(\kappa s^{-\kappa}\int_s^b z^{\kappa-1}\,\mathrm{d}z - s^{-\kappa}b^{\kappa}\right) = -\Gamma(1-\kappa)\Gamma(\kappa),
\end{aligned}$$

since $\int_s^z(z-u)^{-\kappa}(u-s)^{\kappa-1}\,\mathrm{d}u = \mathrm{B}(1-\kappa,\kappa) = \Gamma(1-\kappa)\Gamma(\kappa)$. This shows (5.1). □

The proof of the following lemma is similar to that of Lemma 5.6 in Pipiras and Taqqu (2000). We include it for the reader's convenience.

Lemma 5.2. *Let $\kappa \in (0, \frac{1}{2})$. The inner-product space Λ^{κ}_a is complete if and only if, for every $\phi \in L^2[0, a]$, there is a function $f_{\phi} \in \Lambda^{\kappa}_a$ such that*

$$s^{-\kappa}(I^{\kappa}_{a-}u^{\kappa}f_{\phi}(u))(s) = \phi(s) \text{ almost everywhere } \mathrm{d}s. \tag{5.3}$$

Proof. Suppose that the inner-product space Λ^{κ}_a is complete and let $\phi \in L^2[0, a]$. There is a sequence of elementary functions ϕ_n such that $\phi_n \to \phi$ in $L^2[0, a]$. By Lemma 5.1, we can express the elementary functions ϕ_n as $\phi_n = s^{-\kappa}(I^{\kappa}_{a-}u^{\kappa}f_n(u))(s)$, for some $f_n \in \Lambda^{\kappa}_a$. Since the sequence $\{\phi_n\}_{n\geqslant 1}$ is Cauchy in $L^2[0, a]$, the sequence $\{f_n\}_{n\geqslant 1}$ is Cauchy in Λ^{κ}_a. Then the completeness of Λ^{κ}_a implies that there is $f \in \Lambda^{\kappa}_a$ such that $\phi_n = s^{-\kappa}(I^{\kappa}_{a-}u^{\kappa}f_n(u))(s) \to s^{-\kappa}(I^{\kappa}_{a-}u^{\kappa}f(u))(s)$ in $L^2[0, a]$. Since $\phi_n \to \phi$ in $L^2[0, a]$ as well, relation (5.3) holds with $f_{\phi} = f$.

Conversely, suppose that (5.3) holds and let $\{f_n\}_{n\geqslant 1}$ be a Cauchy sequence in Λ^{κ}_a. Then the sequence $\phi_n(s) = s^{-\kappa}(I^{\kappa}_{a-}u^{\kappa}f_n(u))(s)$ is Cauchy in $L^2[0, a]$. Since $L^2[0, a]$ is complete, there is a $\phi \in L^2[0, a]$ such that $\phi_n \to \phi$ in $L^2[0, a]$. By the assumption, there is an $f_{\phi} \in \Lambda^{\kappa}$a such that (5.3) holds. Since $\phi_n \to \phi$ in L^2[0, a] implies $\mathrm{f_n} \to \mathrm{f}_{\phi}$ in Λ^{κ}_a, the space Λ^{κ}_a is complete. □

Lemma 5.3. *Let $\kappa \in (0, \frac{1}{2})$. There are continuous functions ψ on $[0, a]$ such that the equation*

$$(I^{\kappa}_{a-}g)(s) = \psi(s) \text{ a.e. } \mathrm{d}s \text{ on } [0, a] \tag{5.4}$$

has no solution in $g \in L^1[0, a]$.

Proof. The proof of the lemma is by contradiction. Suppose that (5.4) has a solution $g_{\psi} \in L^1[0, a]$ for any $\psi \in L^2[0, a]$. By (2.4),

$$g_\psi(u) = -\frac{1}{\Gamma(1-\kappa)}\frac{\mathrm{d}}{\mathrm{d}u}\int_u^a \psi(s)(s-u)_+^{-\kappa}\,\mathrm{d}s, \qquad u \in (0, a)$$

(see also Samko *et al.* 1993, Section 2.1). Since $g_\psi(u)$ is expressed as a derivative, the function

$$U_\psi(u) = \int_0^a \psi(s)(s-u)_+^{-\kappa}\,\mathrm{d}s = \int_{\mathbb{R}} \psi(s+u)1_{\{0<s+u<a\}}s_+^{-\kappa}\,\mathrm{d}s, \qquad u \in (0, a),$$

is differentiable almost everywhere on $(0, a)$. However, as shown in Lemma 5.7 of Pipiras and Taqqu (2000), there are functions $\psi \in L^2[0, a]$ such that U_ψ is not differentiable on a set of positive Lebesgue measure. For example, when $a = 1$ (otherwise, use appropriate scaling), for the function ψ we can take the real or imaginary part of

$$\psi^*(u) = c_0 \sum_{n=1}^\infty b^{-pn}\,\mathrm{e}^{\mathrm{i}b^n u}, \tag{5.5}$$

where $b > 1$, $0 < p < \kappa$ and $c_0 = (\Gamma(1-\kappa))^{-1}\,\mathrm{e}^{-\mathrm{i}\pi(1-\kappa)/2}$. Indeed, for $v \in (0, 1)$, we have

$$\begin{aligned} U_{\psi^*}(v) &= c_0\int_0^{1-v} s_+^{-\kappa}\left[\sum_{n=1}^\infty b^{-pn}\,\mathrm{e}^{\mathrm{i}b^n s}\,\mathrm{e}^{\mathrm{i}b^n v}\right]\mathrm{d}s = c_0\sum_{n=1}^\infty b^{-pn}\left[\int_0^{1-v}\mathrm{e}^{\mathrm{i}b^n s}s_+^{-\kappa}\,\mathrm{d}s\right]\mathrm{e}^{\mathrm{i}b^n v} \\ &= c_0\sum_{n=1}^\infty b^{-pn}\left[\int_0^{\infty}\mathrm{e}^{\mathrm{i}b^n s}s_+^{-\kappa}\,\mathrm{d}s\right]\mathrm{e}^{\mathrm{i}b^n v} - c_0\sum_{n=1}^\infty b^{-pn}\left[\int_{1-v}^{\infty}\mathrm{e}^{\mathrm{i}b^n s}s_+^{-\kappa}\,\mathrm{d}s\right]\mathrm{e}^{\mathrm{i}b^n v} \\ &=: y_1(v) + y_2(v). \end{aligned}$$

As indicated in Pipiras and Taqqu (2000), the idea then is to show that the function y_1 is not differentiable on $[0, 1]$ and that the function y_2 is differentiable on $[0, \frac{1}{2}]$. By making a change of variables $b^n s = z$ in y_1, we obtain that

$$y_1(v) = \sum_{n=1}^\infty b^{-(p-\kappa+1)n}\,\mathrm{e}^{\mathrm{i}b^n v}.$$

Since $b^{-(p-\kappa+1)}b = b^{\kappa-p} > 1$, the function y_1 is a particular case of the well-known Weierstrass function whose real and imaginary parts are nowhere differentiable functions. One can show that the function y_2 is differentiable on $[0, \frac{1}{2}]$ by using standard results from real analysis (for details, see Pipiras and Taqqu 2000). Finally, since the function (5.5) is continuous, we obtain the result. □

By combining the previous lemmas, we obtain:

Theorem 5.1. *Let $\kappa \in (0, \frac{1}{2})$. The inner-product spaces Λ_a^κ and $|\Lambda|_a^\kappa$ are not complete.*

Proof. For the inner-product space Λ_a^κ this theorem follows from Lemmas 5.2 and 5.3. Indeed, by Lemma 5.3, there is a continuous function ψ such that equation (5.4) has no solution in g. Then the function $\phi(s) = s^{-\kappa}\psi(s)$ is in $L^2[0, a]$, since $\kappa < \frac{1}{2}$, and equation (5.3) has no solution in f. Thus, by Lemma 5.2, the inner-product space Λ_a^κ is not complete.

We now turn to the inner-product space $|\Lambda|_a^\kappa$. Since Λ_a^κ is not complete and the set of elementary (step) functions $\mathcal{E}^a$ is dense in Λ_a^κ, there is a Cauchy sequence $\{f_n\}_{n\geq 1} \subset \mathcal{E}^a$ which does not converge in Λ_a^κ. Since $\mathcal{E}^a \subset |\Lambda|_a^\kappa$ and $(g, h)_{|\Lambda|_a^\kappa} = (g, h)_{\Lambda_a^\kappa}$, for $g, h \in \mathcal{E}^a$, the sequence $\{f_n\}_{n\geq 1} \subset \mathcal{E}^a$ is also Cauchy in $|\Lambda|_a^\kappa$. If $|\Lambda|_a^\kappa$ is complete, there is an $f \in |\Lambda|_a^\kappa$ such that $f_n \to f$ in $|\Lambda|_a^\kappa$. But then $f_n \to f$ in Λ_a^κ as well because $|\Lambda|_a^\kappa \subset \Lambda_a^\kappa$ and $(g, h)_{|\Lambda|_a^\kappa} = (g, h)_{\Lambda_a^\kappa}$, for $g, h \in |\Lambda|_a^\kappa$. Since this is a contradiction, we obtain that the inner-product space $|\Lambda|_a^\kappa$ is not complete. □

6. The case of the positive half-axis

As mentioned in Section 1, the results on incompleteness of classes of integrands are also valid when $a = \infty$ and $\kappa \in (0, \frac{1}{2})$. When $\kappa \in (0, \frac{1}{2})$, by using the scheme described in Section 4, one can construct the following two classes of integrands on $[0, \infty)$:

$$\Lambda_\infty^\kappa = \left\{ f : [0, \infty) \mapsto \mathbb{R} \text{ such that } \int_0^\infty [s^{-\kappa}(I_-^\kappa u^\kappa f(u))(s)]^2 \, \mathrm{d}s < \infty \right\},$$

where $(I_-^\kappa \phi)(s) = (\Gamma(\kappa))^{-1} \int_{\mathbb{R}} \phi(u)(u-s)_+^{\kappa-1} \, \mathrm{d}u$, $s > 0$, is a fractional integral of order κ; and

$$|\Lambda|_\infty^\kappa = \left\{ f : [0, \infty) \mapsto \mathbb{R} \text{ such that } \int_0^\infty \int_0^\infty |f(u)|\,|f(v)|\,|u-v|^{2\kappa-1} \, \mathrm{d}u \, \mathrm{d}v < \infty \right\}.$$

As in the case $a > 0$, the classes of integrands Λ_∞^κ and $|\Lambda|_\infty^\kappa$ are inner-product spaces with the inner products

$$(f, g)_{\Lambda_\infty^\kappa} = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin \pi\kappa} \int_0^\infty s^{-2\kappa}(I_-^\kappa u^\kappa f(u))(s)(I_-^\kappa u^\kappa g(u))(s)\mathrm{d}s$$

and

$$(f, g)_{|\Lambda|_\infty^\kappa} = \kappa(2\kappa+1)\int_0^\infty \int_0^\infty f(u)g(v)|u-v|^{2\kappa-1} \, \mathrm{d}u \, \mathrm{d}v,$$

respectively. We also have that $|\Lambda|_\infty^\kappa \subset \Lambda_\infty^\kappa$ and that the corresponding inner products are equal for functions f, g belonging to the smaller class $|\Lambda|_\infty^\kappa$.

Corollary 6.1. *Let $\kappa \in (0, \frac{1}{2})$. The inner-product spaces Λ_∞^κ and $|\Lambda|_\infty^\kappa$ are not complete.*

Proof. We will show by contradiction that the inner-product space Λ_∞^κ is not complete. Suppose that Λ_∞^κ is complete. By Theorem 5.1, the inner-product space Λ_a^κ is not complete. Hence, there is a Cauchy sequence g_n of functions in Λ_a^κ such that it does not converge. Since $(I_-^\kappa f 1_{[0,a]})(s) = (I_{a-}^\kappa f)(s) 1_{[0,a]}(s)$, for any $f \in L^1[0, a]$ and $s > 0$, the sequence of functions $g_n 1_{[0,a]}$ is also Cauchy in Λ_∞^κ. The completeness of Λ_∞^κ then implies that there is a function $g \in \Lambda_\infty^\kappa$ such that $g_n 1_{[0,a]} \to g$ in Λ_∞^κ, that is, $s^{-\kappa}(I_-^\kappa u^\kappa g_n(u) 1_{[0,a]}(u))(s)$ converges to $s^{-\kappa}(I_-^\kappa u^\kappa g(u))(s)$ in the $L^2(\mathbb{R})$ sense. Since $(I_-^\kappa u^\kappa g_n(u) 1_{[0,a]}(u))(s) = 0$, for $s > a$, we obtain that $(I_-^\kappa u^\kappa g(u))(s) = 0$, for $s > a$, as well. By Lemma 6.1 below, $g(u) = 0$ a.e. for $u > a$, that is, $g(u) = g(u)1_{[0,a]}(u)$ a.e. for $u > 0$. Since

$$\int_0^a [s^{-\kappa}(I_{a-}^\kappa u^\kappa(g_n(u) - g(u)))(s)]^2\,\mathrm{d}s = \int_0^\infty [s^{-\kappa}(I_-^\kappa u^\kappa(g_n(u)1_{[0,a]}(u) - g(u)1_{[0,a]}))(s)]^2\,\mathrm{d}s$$

$$= \int_0^\infty [s^{-\kappa}(I_-^\kappa u^\kappa(g_n(u)1_{[0,a]}(u) - g(u)))(s)]^2\,\mathrm{d}s \to 0,$$

as $n \to \infty$, $g_n \to g$ in Λ_a^κ, which is a contradiction. Hence, the inner-product space Λ_∞^κ is not complete.

The inner-product space $|\Lambda|_\infty^\kappa$ is not complete either, by arguments similar to those in the proof of Theorem 5.1. □

Lemma 6.1. *Let* $\kappa \in (0, \frac{1}{2})$, $a > 0$ *be fixed and* $g \in \Lambda_\infty^\kappa$. *If, for* $s > a$, $(I_-^\kappa u^\kappa g(u))(s) = 0$ a.e. ds, *then, for* $u > a$, $g(u) = 0$ a.e. du.

Proof. Let $\psi_{c,b}(s) = (\Gamma(1-\kappa))^{-1}((s-c)_+^{-\kappa} - (s-b)_+^{-\kappa})$, $s \in \mathbb{R}$, be a function with $a < c < b$ and observe that it vanishes for $s \leqslant c$. Now define the operator I_+^κ by

$$(I_+^\kappa \psi)(u) = \frac{1}{\Gamma(\kappa)} \int_{\mathbb{R}} \psi(s)(u-s)_+^{\kappa-1}\,\mathrm{d}s, \qquad s \in \mathbb{R}.$$

Heuristically, by changing the order of integration below, we obtain that

$$0 = \int_a^\infty (I_-^\kappa u^\kappa g(u))(s)\psi_{c,b}(s)\mathrm{d}s = \frac{1}{\Gamma(\kappa)} \int_a^\infty \left\{ \int_a^\infty u^\kappa g(u)(u-s)_+^{\kappa-1}\,\mathrm{d}u \right\} \psi_{c,b}(s)\mathrm{d}s$$

$$= \frac{1}{\Gamma(\kappa)} \int_a^\infty u^\kappa g(u) \left\{ \int_a^\infty \psi_{c,b}(s)(s-u)_-^{\kappa-1}\,\mathrm{d}s \right\} \mathrm{d}u = \int_a^\infty u^\kappa g(u)(I_+^\kappa \psi_{c,b})(u)\mathrm{d}u \tag{6.1}$$

and

$$0 = \int_a^\infty u^\kappa g(u) 1_{(c,b]}(u)\mathrm{d}u, \tag{6.2}$$

since, by Lemma 6.2 below, the function $\psi_{c,b}$ has the property $I_+^\kappa \psi_{c,b} = 1_{(c,b]}$. One way to prove relation (6.2) is to justify the change in the order of integration in (6.1) and show that

$$\int_a^\infty \int_a^\infty |u^\kappa g(u)|\,|\psi_{c,b}(s)|(u-s)_+^{\kappa-1}\,\mathrm{d}u\,\mathrm{d}s < \infty.$$

This can be done for almost every b. As in the proof of Lemma 5.4 in Pipiras and Taqqu (2000), by using the relation $|\psi_{c,b}| = \psi_{c,b} - 2\psi_{c,b}1_{(b,\infty)}$, we obtain that

$$
\begin{aligned}
(\Gamma(\kappa))^{-1}\int_a^\infty |\psi_{c,b}(s)|(u-s)_+^{\kappa-1}\,\mathrm{d}s \\
&\leqslant (I_+^\kappa|\psi_{c,b}|)(u) \\
&= (I_+^\kappa\psi_{c,b})(u) - 2(I_+^\kappa\psi_{c,b}1_{(b,\infty)})(u) \\
&= (I_+^\kappa\psi_{c,b})(u) - 2(I_+^\kappa\psi_{c,b}1_{(b,\infty)})(u)1_{(b,\infty)}(u) \\
&= (I_+^\kappa\psi_{c,b})(u) - 2(I_+^\kappa\psi_{c,b})(u)1_{(b,\infty)}(u) + 2(I_+^\kappa\psi_{c,b}1_{(c,b)})(u)1_{(b,\infty)}(u) \\
&= 1_{(c,b]}(u) - 2\,1_{(c,b]}(u)1_{(b,\infty)}(u) \\
&\quad + 2(\Gamma(\kappa)\Gamma(1-\kappa))^{-1}\int_c^b (s-c)_+^{-\kappa}(u-s)_+^{\kappa-1}\,\mathrm{d}s\,1_{(b,\infty)}(u) \\
&\leqslant 1_{(c,b]}(u) + 2(\Gamma(\kappa)\Gamma(1-\kappa))^{-1}\int_c^b (s-c)_+^{-\kappa}\,\mathrm{d}s(u-b)_+^{\kappa-1} \\
&= 1_{(c,b]}(u) + c_\kappa(b-c)^{1-\kappa}(u-b)_+^{\kappa-1}.
\end{aligned}
$$

It is then enough to verify that

$$
\int_a^\infty |u^\kappa g(u)|1_{(c,b]}(u)\mathrm{d}u < \infty \tag{6.3}
$$

and

$$
\int_a^\infty |u^\kappa g(u)|(u-b)_+^{\kappa-1}\,\mathrm{d}u < \infty. \tag{6.4}
$$

Since $g\in\Lambda_\infty^\kappa$ implies that $(I_-^\kappa u^\kappa g(u))(s)$ is well defined a.e. for $s>a$, that is,

$$
\text{for } s>a, \qquad \int_a^\infty |u^\kappa g(u)|(u-s)_+^{\kappa-1}\,\mathrm{d}u < \infty \text{ a.e. } \mathrm{d}s, \tag{6.5}
$$

equality (6.3) follows for any $a<c<b$ (we implicitly assume that integration is in the Lebesgue sense everywhere). If b is such that (6.5) holds, then we immediately obtain (6.4) and hence relation (6.2) is valid. However, if (6.5) does not hold for b, it holds for some b_n such that $b_n\to b$. Then (6.2) with b follows directly by letting $n\to\infty$ in

$$
0 = \int_a^\infty u^\kappa g(u)1_{(c,b_n]}(u)\mathrm{d}u.
$$

Finally, by approximating the set $\{u : a<u<n\}\cap\{u : u^\kappa g(u)>0\}$ by unions of disjoint intervals and letting $n\to\infty$, it follows from (6.2) that $u^\kappa g(u)\leqslant 0$ a.e. $\mathrm{d}u$ for $u>a$ and, by symmetry, that $u^\kappa g(u)\geqslant 0$ a.e. for $u>a$. Hence, $u^\kappa g(u)=0$ a.e. $\mathrm{d}u$ for $u>a$ or $g(u)=0$ a.e. $\mathrm{d}u$ for $u>a$. □

Lemma 6.2. *Let $c<b$ be real numbers and $\kappa\in(0,1)$. Then the function*

$$\psi_{c,b}(s) = (\Gamma(1-\kappa))^{-1}((s-c)_+^{-\kappa} - (s-b)_+^{-\kappa})$$

satisfies the equation

$$(I_+^\kappa \psi_{c,b})(u) = 1_{(c,b]}(u), \qquad u \in \mathbb{R}.$$

Proof. By the definition of I_+^κ (see the proof above), we have to verify that, for $u \in \mathbb{R}$,

$$J_1(u) := \int_{\mathbb{R}} ((s-c)_+^{-\kappa} - (s-b)_+^{-\kappa})(u-s)_+^{\kappa-1}\,\mathrm{d}s$$

$$= \Gamma(\kappa)\Gamma(1-\kappa)1_{[c,b)}(u) =: J_2(u).$$

If $u \leqslant c$, then $J_1(u) = J_2(u) = 0$. If $u > c$, we use the following identity, valid for $t > 0$:

$$\int_0^t v^{-\kappa}(t-v)^{\kappa-1}\,\mathrm{d}v = t^{-\kappa+\kappa-1+1}\int_0^1 s^{-\kappa}(1-s)^{\kappa-1}\,\mathrm{d}s = \mathrm{B}(\kappa, 1-\kappa)$$

$$= \frac{\Gamma(\kappa)\Gamma(1-\kappa)}{\Gamma(\kappa+1-\kappa)} = \Gamma(\kappa)\Gamma(1-\kappa).$$

If $c < u \leqslant b$, then

$$J_1(u) = \int_c^u (s-c)_+^{-\kappa}(u-s)_+^{\kappa-1}\,\mathrm{d}s = \int_0^{u-c} v^{-\kappa}(u-c-v)^{\kappa-1}\,\mathrm{d}v = \Gamma(\kappa)\Gamma(1-\kappa),$$

which is also $J_2(u)$. In the case when $b < u$, we show in a similar way that

$$J_1(u) = \int_c^u (s-c)_+^{-\kappa}(u-s)_+^{\kappa-1}\,\mathrm{d}s - \int_b^u (s-b)_+^{-\kappa}(u-s)_+^{\kappa-1}\,\mathrm{d}s = 0 = J_2(u).$$

□

Remark 6.1. The result of Lemma 6.2 should not be surprising because of the relation $\psi_{c,b} = I_+^{-\kappa}1_{(c,b]}$, where the κ-fractional derivative operator $I_+^{-\kappa}$, defined as

$$(I_+^{-\kappa}f)(s) = \frac{1}{\Gamma(1-\kappa)}\frac{\mathrm{d}}{\mathrm{d}s}\int_{-\infty}^s f(u)(s-u)_+^{-\kappa}\,\mathrm{d}u, \qquad s \in \mathbb{R},$$

is the inverse of I_+^κ (there exist other definitions for an inverse of I_+^κ – see, for example, Section 5 in Samko *et al*. 1993).

7. Prediction of fractional Brownian motion

The prediction problem for FBM is to find an explicit expression for the conditional expectation $X = \mathrm{E}(B^\kappa(t)|B^\kappa(s), s \in [0, a])$ with some fixed $0 < a < t$. As already indicated in the Introduction, we have $X \in \overline{\mathrm{sp}}_{[0,a]}(B^\kappa)$ and therefore we expect $X = \int_0^a f(u)\mathrm{d}B^\kappa(u)$ for some function f. When $\kappa \in (-\frac{1}{2}, 0)$, since the class of integrands Λ_a^κ in (1.8) is complete, we know that the function f exists and belongs to Λ_a^κ. When $\kappa \in (0, \frac{1}{2})$, however, since the class

Λ_a^κ in (1.2) is not complete, we may only hope that there is such a function f which belongs to Λ_a^κ.

In this section, we will show that the above function f exists not only for $\kappa \in (-\frac{1}{2}, 0)$ but also for $\kappa \in (0, \frac{1}{2})$. We will also determine this function explicitely by first providing its heuristic derivation which uses fractional integration ideas and then verifying rigorously that the function obtained is indeed the right one. When $\kappa \in (0, \frac{1}{2})$, the prediction formula was obtained by Gripenberg and Norros (1996). Our approach is different from that of Gripenberg and Norros, where fractional integration is not used, and it also covers the case $\kappa \in (-\frac{1}{2}, 0)$.

The prediction formula for FBM can be *heuristically* derived as follows. We shall repeatedly use the (heuristic) fractional integration 'rule' $I^\alpha I^\beta = I^{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{R}$, where I^α is some fractional integral or derivative operator. For $s \in [0, t]$, set

$$B^0(s) = \int_0^t u^{-\kappa}(I_{t-}^{-\kappa} z^\kappa 1_{[0,s)}(z))(u)\mathrm{d}B^\kappa(u). \tag{7.1}$$

Since $v^{-\kappa}(I_{t-}^{\kappa} u^\kappa(u^{-\kappa} I_{t-}^{-\kappa} z^\kappa 1_{[0,s)}(z)))(v) = 1_{[0,s)}(v)$, B^0 is the usual Brownian motion. By using the fractional integration by parts formula $\int_0^t \phi(s)(I_{t-}^{\alpha}\psi)(s)\mathrm{d}s = \int_0^t (I_{0+}^{\alpha}\phi)(s)\psi(s)\mathrm{d}s$ with $(I_{0+}^{\alpha}\phi)(s) = \Gamma(\alpha)^{-1}\int_a^b \phi(u)(s-u)_+^{\alpha-1}\,\mathrm{d}u$, we can write (7.1) as $B^0(s) = \int_0^t 1_{[0,s)}(z) z^\kappa (I_{0+}^{-\kappa} u^{-\kappa} \dot{B}^\kappa(u))(z)\mathrm{d}z$, where $\dot{B}^\kappa$ denotes the 'derivative' of B^κ, or

$$\dot{B}^0(s) = s^\kappa (I_{0+}^{-\kappa} u^{-\kappa} \dot{B}^\kappa(u))(s).$$

By applying the operator $u^\kappa I_{0+}^{\kappa} s^{-\kappa}$ to both sides of the last expression, integrating over $[0, s]$ and applying the fractional integration by parts formula again, we obtain

$$B^\kappa(s) = \int_0^t u^{-\kappa}(I_{t-}^{\kappa} z^\kappa 1_{[0,s)}(z))(u)\mathrm{d}B^0(u). \tag{7.2}$$

Since B^0 and B^κ can be defined through each other as in (7.1) and (7.2), we obtain $\sigma\{B^\kappa(s), s \in [0, a]\} = \sigma\{B^0(s), s \in [0, a]\}$ up to sets of measure zero, where $\sigma\{B^\kappa(s), s \in [0, a]\}$ is the σ-algebra generated by the random variables $B^\kappa(s)$, $s \in [0, a]$. Then, since B^0 is the usual Brownian motion,

$$\begin{aligned} \mathrm{E}(B^\kappa(t)|B^\kappa(s), s \in [0, a]) &= \mathrm{E}\left(\int_0^t u^{-\kappa}(I_{t-}^{\kappa} z^\kappa 1_{[0,t)}(z))(u)\mathrm{d}B^0(u)|B^0(s), s \in [0, a]\right) \\ &= \int_0^a u^{-\kappa}(I_{t-}^{\kappa} z^\kappa 1_{[0,t)}(z))(u)\mathrm{d}B^0(u). \end{aligned}$$

By writing $1_{[0,t)} = 1_{[0,a)} + 1_{[a,t)}$ and then using (7.2), we obtain

$$\mathrm{E}(B^\kappa(t)|B^\kappa(s), s \in [0, a]) = B^\kappa(a) + \int_0^a u^{-\kappa}(I_{t-}^{\kappa} z^\kappa 1_{[a,t)}(z))(u)\mathrm{d}B^0(u).$$

We wish to rewrite this last integral as an integral with respect to FBM. Since, heuristically, $u^{-\kappa}(I_{t-}^{\kappa} z^\kappa 1_{[a,t)}(z))(u) = u^{-\kappa} I_{a-}^{\kappa}(v^\kappa v^{-\kappa} I_{a-}^{-\kappa}(I_{t-}^{\kappa} z^\kappa 1_{[a,t)}(z))(v))(u)$ and since, for an integrable function f, $\int_0^a f(u)\mathrm{d}B^\kappa(u) = \int_0^a u^{-\kappa}(I_{a-}^{\kappa} z^\kappa f(z))(u)\mathrm{d}B^0(u)$, we expect that

$$E(B^\kappa(t)|B^\kappa(s),\, s \in [0,\, a]) = B^\kappa(a) + \int_0^a u^{-\kappa}(I^{-\kappa}_{a-}(I^{\kappa}_{t-}z^\kappa 1_{[a,t)}(z)))(u)\mathrm{d}B^\kappa(u).$$

These computations suggest the following prediction formula for FBM:

Theorem 7.1. *Let* $0 < a < t$, $\kappa \in (-\frac{1}{2}, \frac{1}{2})$ *and* B^κ *be an FBM. Then*

$$E(B^\kappa(t)|B^\kappa(s),\, s \in [0,\, a]) = B^\kappa(a) + \int_0^a \Psi_t(a,\, u)\mathrm{d}B^\kappa(u), \tag{7.3}$$

where, for $u \in (0,\, a)$,

$$\Psi_t(a,\, u) = u^{-\kappa}(I^{-\kappa}_{a-}(I^{\kappa}_{t-}z^\kappa 1_{[a,t)}(z)))(u) \tag{7.4}$$

$$= \frac{\sin \pi\kappa}{\pi} u^{-\kappa}(a-u)^{-\kappa} \int_a^t \frac{z^\kappa (z-a)^\kappa}{z-u}\, \mathrm{d}z. \tag{7.5}$$

Proof. Let us first verify that the right-hand sides of (7.4) and (7.5) are equal when $\kappa \in (0, \frac{1}{2})$. Observe that, by (2.1), for $v \in (0,\, a)$,

$$(I^{\kappa}_{t-}z^\kappa 1_{[a,t)}(z))(v) = \frac{1}{\Gamma(\kappa)} \int_a^t z^\kappa (z-v)^{\kappa-1}\, \mathrm{d}z \tag{7.6}$$

and hence, by (2.2),

$$(I^{-\kappa}_{a-}(I^{\kappa}_{t-}z^\kappa 1_{[a,t)}(z)))(u) = -\frac{1}{\Gamma(1-\kappa)\Gamma(\kappa)} \frac{\mathrm{d}}{\mathrm{d}u} \int_u^a \int_a^t z^\kappa (z-v)^{\kappa-1}\, \mathrm{d}z\, (v-u)^{-\kappa}\, \mathrm{d}v \tag{7.7}$$

$$= -\frac{\sin \pi\kappa}{\pi} \frac{\mathrm{d}}{\mathrm{d}u} \int_a^t \mathrm{d}z\, z^\kappa \int_u^a \mathrm{d}v\, (z-v)^{\kappa-1}(v-u)^{-\kappa},$$

since $\Gamma(1-\kappa)\Gamma(\kappa) = \pi/\sin \pi\kappa$. By making the change of variables $v = u + (z-u)s$ and then taking $\mathrm{d}/\mathrm{d}u$ inside the integral, we obtain

$$(I^{-\kappa}_{a-}(I^{\kappa}_{t-}z^\kappa 1_{[a,t)}(z)))(u) = -\frac{\sin \pi\kappa}{\pi} \frac{\mathrm{d}}{\mathrm{d}u} \int_a^t \mathrm{d}z\, z^\kappa \int_0^{(a-u)/(z-u)} \mathrm{d}s\, (1-s)^{\kappa-1} s^{-\kappa}$$

$$= \frac{\sin \pi\kappa}{\pi} (a-u)^{-\kappa} \int_a^t \frac{z^\kappa (z-a)^\kappa}{z-u}\, \mathrm{d}z. \tag{7.8}$$

When $\kappa \in (-\frac{1}{2}, 0)$, the right-hand sides of (7.4) and (7.5) are still equal. To see this, observe first that (7.6) with $\Gamma(\kappa) = \Gamma(1+\kappa)/\kappa$ still holds for $v \in (0,\, a)$, since $z \neq v$ in the range of integration. Then (7.7) becomes (see (2.1))

$$(I^{-\kappa}_{a-}(I^{\kappa}_{t-}z^\kappa 1_{[a,t)}(z)))(u) = \frac{\kappa}{\Gamma(1+\kappa)\Gamma(-\kappa)} \int_u^a \int_a^t z^\kappa (z-v)^{\kappa-1}\, \mathrm{d}z\, (v-u)^{-\kappa-1}\, \mathrm{d}v$$

$$= -\frac{\kappa \sin \pi\kappa}{\pi} \int_a^t \mathrm{d}z\, z^\kappa \int_u^a \mathrm{d}v\, (z-v)^{\kappa-1}(v-u)^{-\kappa-1}.$$

It is now easy to see that

$$\kappa\int_u^a (z-v)^{\kappa-1}(v-u)^{-\kappa-1}\,dv = \frac{d}{du}\int_u^a (z-v)^{\kappa-1}(v-u)^{-\kappa}\,dv = -\frac{(a-u)^{-\kappa}(z-a)^{\kappa}}{z-u},$$

where the derivative can be computed as in the case $\kappa\in(0,\frac{1}{2})$. This yields relation (7.8) in the case $\kappa\in(-\frac{1}{2},0)$ as well.

To verify that the integral in (7.3) is well defined for $\kappa\in(-\frac{1}{2},0)$, it is enough to check, by (1.8), that the function $s^{-\kappa}(I^{\kappa}_{t-}z^{\kappa}1_{[a,t)}(z))(s)$ belongs to $L^2[0,a]$. The same condition needs to be verified in the case $\kappa\in(0,\frac{1}{2})$ by (1.2) and (2.5). In both cases this can be deduced from (7.6). Finally, to prove (7.3), it is enough to verify that, for all $s\in[0,a]$,

$$\mathrm{E}B^{\kappa}(s)(B^{\kappa}(t)-B^{\kappa}(a)) = \mathrm{E}B^{\kappa}(s)\int_0^a \Psi_t(a,u)\mathrm{d}B^{\kappa}(u).$$

This follows from (1.4) and (1.9) by using expression (7.4) for $\Psi_t(a,u)$. □

8. Proofs of Theorem 4.1 and Theorem 4.2

We prove Theorem 4.1 and Theorem 4.2 together. Let $\kappa\in(-\frac{1}{2},\frac{1}{2})$. To show that the maps (1.4) and (1.9) (or (1.10)) define inner products on linear spaces Λ^{κ}_a, given by (1.2) and (1.8) respectively, we check the least obvious condition. If $(f,f)_{\Lambda^{\kappa}_a}=0$ and $\kappa\in(0,\frac{1}{2})$, then $(I^{\kappa}_{a-}u^{\kappa}f(u))(s)=0$ a.e. $s\in[0,a]$. It follows by Lemma 2.5 in Samko *et al*. (1993, p. 40) that $u^{\kappa}f(u)=0$ a.e. $u\in[0,a]$ and hence that $f(u)=0$ a.e. $u\in[0,a]$. If $(f,f)_{\Lambda^{\kappa}_a}=0$ and $\kappa\in(-\frac{1}{2},0)$, then $\phi_f(s)=0$ a.e. $s\in[0,a]$, where $\phi_f\in L^2[0,a]$ is associated with the function f by definition (1.8). It then follows that $f(u)=0$ a.e. $u\in[0,a]$ as well.

Let us show that the set of elementary functions $\mathcal{E}_a$ is dense in Λ^{κ}_a when $\kappa\in(-\frac{1}{2},\frac{1}{2})$. Assume without loss of generality that $a>1$. Since any function $\phi\in L^2[0,a]$ can be approximated in $L^2[0,a]$ by functions $s^{-\kappa}\sum_{k=1}^n b_k 1_{[c_k,d_k)}(s) = s^{-\kappa}\sum_{k=1}^n b_k(1_{[0,d_k)}(s)-1_{[0,c_k)}(s))$ with $b_k\in\mathbb{R}$ and $0<c_k<d_k<a$, $k=1,\ldots,n$, it is enough to show, for example, that there is a sequence of elementary functions $f_n\in\mathcal{E}_a\subset\Lambda^{\kappa}_a$ such that

$$h_n(\kappa) := \int_0^a s^{-2\kappa}|1_{[0,1)}(s)-(I^{\kappa}_{a-}u^{\kappa}f_n(u))(s)|^2\,\mathrm{d}s \to 0, \qquad n\to\infty. \tag{8.1}$$

Since $1_{[0,1)}(s)=(I^{\kappa}_{a-}u^{\kappa}f(u))(s)$ with $f(u)=u^{-\kappa}(I^{-\kappa}_{a-}1_{[0,1)})(u)=(\Gamma(1-\kappa))^{-1}u^{-\kappa}(1-u)^{-\kappa}_{+}$, the convergence (8.1) for some elementary functions f_n follows from the following lemma.

Lemma 8.1. *Let $\kappa\in(-\frac{1}{2},\frac{1}{2})$, $\{B^{\kappa}(u)\}_{u\in[0,1]}$ be a standard FBM and*

$$f(u)=u^{-\kappa}(I^{-\kappa}_{1-}1_{[0,1)})(u)=(\Gamma(1-\kappa))^{-1}u^{-\kappa}(1-u)^{-\kappa}_{+} \text{ for } u\in[0,1]$$

Then the integral $\int_0^1 f(u)\mathrm{d}B^{\kappa}(u)$ is the limit in the $L^2(\Omega)$ sense of the integrals

$$\int_0^1 f_n(u)\mathrm{d}B^{\kappa}(u) = \frac{1}{\Gamma(1-\kappa)}\sum_{l=2}^{n-2}\left(\frac{l}{n}\right)^{-\kappa}\left(1-\frac{l}{n}\right)^{-\kappa}\left(B^{\kappa}\left(\frac{l+1}{n}\right)-B^{\kappa}\left(\frac{l}{n}\right)\right),$$

where, for $u \in [0, 1]$,

$$f_n(u) = \frac{1}{\Gamma(1-\kappa)} \sum_{l=2}^{n-2} \left(\frac{l}{n}\right)^{-\kappa} \left(1 - \frac{l}{n}\right)^{-\kappa} 1_{[\frac{l}{n}, \frac{l+1}{n})}(u) = \frac{1}{\Gamma(1-\kappa)} \sum_{l=2}^{n-2} f\left(\frac{l}{n}\right) 1_{[\frac{l}{n}, \frac{l+1}{n})}(u).$$

Remark 8.1. The integral $\int_0^1 f(u) \mathrm{d}B^\kappa(u)$ in Lemma 8.1 is of particular interest because, if $f_t(u) = u^{-\kappa}(I_{t-}^{-\kappa} 1_{[0,t)})(u) = (\Gamma(1-\kappa))^{-1} u^{-\kappa}(t-u)_+^{-\kappa}$, $t, u \in [0, a]$, then by (4.3) (and Theorems 4.1 and 4.2)

$$\int_0^t f_t(u) \mathrm{d}B^\kappa(u) \stackrel{d}{=} \sigma_1(\kappa) \int_0^t s^{-2\kappa} \, \mathrm{d}B^0(s), \qquad t \in [0, a].$$

The process on the left-hand side is known as the fundamental martingale (see Norros *et al.* 1999).

Proof. It is enough to prove the convergence (8.1) with $a = 1$. If $\kappa \in (0, \frac{1}{2})$, by using relation (4.5) and estimate (4.10), we obtain that

$$h_n(\kappa) \leqslant \int_0^1 s^{-2\kappa} [(I_{1-}^\kappa u^\kappa |f(u) - f_n(u)|)(s)]^2 \, \mathrm{d}s \leqslant c(\kappa) \|f - f_n\|_{L^2[0,1]}^2 \to 0,$$

as $n \to \infty$.

The case $\kappa \in (-\frac{1}{2}, 0)$ is more delicate. We will show that $h_n(\kappa) \to 0$ as well, by applying the dominated convergence theorem. Let $l_n = l/n$ for $l, n \in \mathbb{N}$. As in (3.5) and (3.6), we obtain

$$\Gamma(1-\kappa)\Gamma(1+\kappa)(I_{1-}^\kappa u^\kappa f_n(u))(s) = \Gamma(1+\kappa) \sum_{l=2}^{n-2} l_n^{-\kappa} (1 - l_n)^{-\kappa} (I_{1-}^\kappa u^\kappa 1_{[l_n, (l+1)_n)}(u))(s)$$

$$= -\kappa \int_s^1 \sum_{l=2}^{n-2} l_n^{-\kappa} (1 - l_n)^{-\kappa} 1_{[l_n, (l+1)_n)}(u) u^{\kappa - 1} (u - s)_+^\kappa \, \mathrm{d}u$$

$$+ \sum_{l=2}^{n-2} l_n^{-\kappa} (1 - l_n)^{-\kappa} ((l+1)_n^\kappa ((l+1)_n - s)_+^\kappa - l_n^\kappa (l_n - s)_+^\kappa)$$

$$=: g_n^1(s) + g_n(s).$$

The summation by parts formula $\sum_{l=2}^{n-2} a_l (b_{l+1} - b_l) = -\sum_{l=2}^{n-2} b_l (a_l - a_{l-1}) + a_{n-2} b_{n-1} - a_1 b_2$ implies that

$$
\begin{aligned}
g_n(s) = &-\sum_{l=2}^{n-2} l_n^{\kappa}(l_n - s)_+^{\kappa}(l_n^{-\kappa}(1-l_n)^{-\kappa} - (l-1)_n^{-\kappa}(1-(l-1)_n)^{-\kappa}) \\
&+ (n-2)_n^{-\kappa}(1-(n-2)_n)^{-\kappa}(n-1)_n^{\kappa}((n-1)_n - s)_+^{\kappa} - 1_n^{-\kappa}(1-1_n)^{-\kappa}2_n^{\kappa}(2_n - s)_+^{\kappa} \\
= &-\sum_{l=2}^{n-2} l_n^{\kappa}(l_n - s)_+^{\kappa}(l_n^{-\kappa} - (l-1)_n^{-\kappa})(1-l_n)^{-\kappa} - \sum_{l=2}^{n-2} l_n^{\kappa}(l_n - s)_+^{\kappa}(l-1)_n^{-\kappa}((1-l_n)^{-\kappa} \\
&- (1-(l-1)_n)^{-\kappa}) + (n-2)_n^{-\kappa}(1-(n-2)_n)^{-\kappa}(n-1)_n^{\kappa}((n-1)_n - s)_+^{\kappa} \\
&- 1_n^{-\kappa}(1-1_n)^{-\kappa}2_n^{\kappa}(2_n - s)_+^{\kappa} \\
=: &\ g_n^2(s) + g_n^3(s) + g_n^4(s) + g_n^5(s).
\end{aligned}
$$

(Recall that l_n stands for l/n.) In order to establish (8.1), we will show that, for $\kappa \in (-\frac{1}{2}, 0)$,

$$
\int_0^1 s^{-2\kappa}|g_n^5(s)|^2\,\mathrm{d}s \to 0, \qquad n \to \infty, \tag{8.2}
$$

$$
\begin{aligned}
&\Gamma(1-\kappa)\Gamma(1+\kappa)(I_{1-}^{\kappa}u^{\kappa}f_n(u))(s) - g_n^5(s) = g_n^1(s) + g_n^2(s) + g_n^3(s) + g_n^4(s) \\
&\to \Gamma(1-\kappa)\Gamma(1+\kappa)(I_{1-}^{\kappa}u^{\kappa}f(u))(s) = \Gamma(1-\kappa)\Gamma(1+\kappa)1_{[0,1)}(s) \text{ a.e. } s \in [0, 1],
\end{aligned} \tag{8.3}
$$

and

$$
\sup_n |g_n^j(s)| \leqslant g^j(s), \qquad 1 \leqslant j \leqslant 4, \tag{8.4}
$$

where

$$
\int_0^1 s^{-2\kappa}|g^j(s)|^2\,\mathrm{d}s < \infty, \qquad 1 \leqslant j \leqslant 4. \tag{8.5}
$$

Then, by the dominated convergence theorem, we will have the required convergence $h_n(\kappa) \to 0$ for $\kappa \in (-\frac{1}{2}, 0)$ as $n \to \infty$.

The convergence (8.2) follows, since

$$
\int_0^1 s^{-2\kappa}|g_n^5(s)|^2\,\mathrm{d}s \leqslant c\int_0^{2/n} s^{-2\kappa}\left(\frac{2}{n} - s\right)^{2\kappa}\mathrm{d}s = c\frac{2}{n}\int_0^1 z^{-2\kappa}(1-z)^{2\kappa}\,\mathrm{d}z \to 0,
$$

as $n \to \infty$. Let us show (8.4) and (8.5) with $j = 1$. We have

$$
\begin{aligned}
|g_n^1(s)| &\leqslant c_1\int_s^1 \sum_{l=2}^{n-2} l_n^{-\kappa}(1-(l+1)_n)^{-\kappa}1_{[l_n,(l+1)_n)}(u)u^{\kappa-1}(u-s)_+^{\kappa}\,\mathrm{d}u \\
&\leqslant c_1\int_s^1 u^{-\kappa}(1-u)^{-\kappa}u^{\kappa-1}(u-s)_+^{\kappa}\,\mathrm{d}u = c_1\int_s^1 u^{-1}(1-u)^{-\kappa}(u-s)_+^{\kappa}\,\mathrm{d}u =: g^1(s),
\end{aligned}
$$

where the constant c_1 depends on κ only. It is enough to check the convergence of the integral in (8.5) around $s=0$ only. Since $\kappa<0$, observe that, as s is close to 0, the function $g^1(s)$ is bounded (up to a constant) by

$$\int_s^1 u^{-1}(u-s)^{\kappa}\,\mathrm{d}u = s^{\kappa}\int_1^{1/s} v^{-1}(v-1)^{\kappa}\,\mathrm{d}v \leqslant cs^{\kappa}\int_1^{\infty} v^{\kappa-1}\,\mathrm{d}v \leqslant \frac{c}{|\kappa|}s^{\kappa}.$$

It then follows that the integral in (8.5) with $j=1$ converges around $s=0$. For functions g_n^2, by using the inequality $|l_n^{-\kappa}-(l-1)_n^{-\kappa}| \leqslant |\kappa|(l-1)_n^{-\kappa-1}/n$, we have

$$\begin{aligned}
|g_n^2(s)| &\leqslant |\kappa|\sum_{l=2}^{n-2} l_n^{\kappa}(l_n-s)_+^{\kappa}(l-1)_n^{-\kappa-1}(1-l_n)^{-\kappa}\frac{1}{n}\\
&\leqslant c_2\sum_{l=2}^{n-2}(l+1)_n^{-1}(l_n-s)_+^{\kappa}(1-(l+1)_n)^{-\kappa}\frac{1}{n}\\
&\leqslant c_2\int_0^1 u^{-1}(u-s)_+^{\kappa}(1-u)^{-\kappa}\,\mathrm{d}u =: g^2(s),
\end{aligned}$$

where the constant c_2 depends on κ only. Relation (8.5) with $j=2$ follows as in the case $j=1$ since the functions g_1 and g_2 are equal up to a constant. As for the functions g_n^3, one can similarly show that

$$|g_n^3(s)| \leqslant c_3\int_0^1 (u-s)_+^{\kappa}(1-u)^{-\kappa-1}\,\mathrm{d}u = c_3\int_0^1 z^{\kappa}(1-z)^{-\kappa-1}\,\mathrm{d}u = c_3\mathrm{B}(1+\kappa,\,-\kappa) =: g^3(s)$$

and

$$\int_0^1 s^{-2\kappa}|g^3(s)|^2\,\mathrm{d}s = c_3^2\mathrm{B}(1+\kappa,\,-\kappa)^2\int_0^1 s^{-2\kappa}\,\mathrm{d}s<\infty.$$

The proof of (8.4) and (8.5) with $j=4$ is obvious. Finally, to show the convergence (8.3), observe that

$$\begin{aligned}
&\lim_{n\to\infty}(\Gamma(1-\kappa)\Gamma(1+\kappa)(I_{1-}^{\kappa}u^{\kappa}f_n(u))(s)-g_n^5(s))\\
&=\lim_{n\to\infty}(g_n^1(s)+g_n^2(s)+g_n^3(s)+g_n^4(s))\\
&=-\kappa\int_0^1 u^{-\kappa}(u-s)_+^{\kappa}u^{\kappa-1}(1-u)^{-\kappa}\,\mathrm{d}u+\kappa\int_0^1 u^{\kappa}(u-s)_+^{\kappa}u^{-\kappa-1}(1-u)^{-\kappa}\,\mathrm{d}u\\
&\quad-\kappa\int_0^1 u^{\kappa}(u-s)_+^{\kappa}u^{-\kappa}(1-u)^{-\kappa-1}\,\mathrm{d}u+0=-\kappa\int_0^1(u-s)_+^{\kappa}(1-u)^{-\kappa-1}\,\mathrm{d}u\\
&=-\kappa\mathrm{B}(\kappa+1,\,-\kappa)1_{[0,1)}(s)=\Gamma(1-\kappa)\Gamma(1+\kappa)1_{[0,1)}(s),\ \text{a.e. } s\in[0,\,1].
\end{aligned}$$

□

Appendix

In the proof of Proposition 3.1, we used the fact that the function

$$f(\kappa)=\sigma_1(\kappa)^2\int_0^a s^{-2\kappa}(I^{\kappa}_{a-}u^{\kappa}1_{[0,t_1)}(u))(s)(I^{\kappa}_{a-}u^{\kappa}1_{[0,t_2 0}(u))(s)\mathrm{d}s,$$

where $t_1, t_2\in[0, a]$, is analytic for complex $|\kappa|<\frac{1}{2}$. The analyticity of f can be verified directly as follows. By (3.6), the function f equals

$$f(\kappa)=(\Gamma(1+\kappa))^{-2}\sigma_1(\kappa)^2\int_0^a s^{-2\kappa}\left\{\kappa\int_s^{t_1}u^{\kappa-1}(u-s)_+^{\kappa}\,\mathrm{d}u-t_1^{\kappa}(t_1-s)_+^{\kappa}\right\}$$

$$\times\left\{\kappa\int_s^{t_2}u^{\kappa-1}(u-s)_+^{\kappa}\,\mathrm{d}u-t_2^{\kappa}(t_2-s)_+^{\kappa}\right\}\mathrm{d}s,$$

where $\sigma_1(\kappa)^2$ is defined by (3.2). Since $(\Gamma(1+\kappa))^{-2}\sigma_1(\kappa)^2$ is analytic on $|\kappa|<\frac{1}{2}$, it is enough to show that the function

$$\begin{aligned}
&\int_0^a s^{-2\kappa}\left\{\kappa\int_s^{t_1}u^{\kappa-1}(u-s)_+^{\kappa}\,\mathrm{d}u-t_1^{\kappa}(t_1-s)_+^{\kappa}\right\}\left\{\kappa\int_s^{t_2}u^{\kappa-1}(u-s)_+^{\kappa}\,\mathrm{d}u-t_2^{\kappa}(t_2-s)_+^{\kappa}\right\}\mathrm{d}s\\
&=\kappa^2\int_0^a s^{-2\kappa}\int_s^{t_1}u^{\kappa-1}(u-s)_+^{\kappa}\,\mathrm{d}u\int_s^{t_2}u^{\kappa-1}(u-s)_+^{\kappa}\,\mathrm{d}u\,\mathrm{d}s-\kappa\int_0^a s^{-2\kappa}\int_s^{t_1}u^{\kappa-1}(u-s)_+^{\kappa}\,\mathrm{d}u\,t_2^{\kappa}(t_2-s)_+^{\kappa}\,\mathrm{d}s\\
&\quad-\kappa\int_0^a s^{-2\kappa}\int_s^{t_2}u^{\kappa-1}(u-s)_+^{\kappa}\,\mathrm{d}u\,t_1^{\kappa}(t_1-s)_+^{\kappa}\,\mathrm{d}s+\kappa\int_0^a s^{-2\kappa}t_1^{\kappa}(t_1-s)_+^{\kappa}t_2^{\kappa}(t_2-s)_+^{\kappa}\,\mathrm{d}s\\
&=:\kappa^2 f_1(\kappa)-\kappa f_2(\kappa)-\kappa f_3(\kappa)+f_4(\kappa)
\end{aligned}$$

is analytic on $|\kappa|<\frac{1}{2}$, or, equivalently, that the functions $f_1(\kappa)$, $f_2(\kappa)$, $f_3(\kappa)$ and $f_4(\kappa)$ are analytic on $|\kappa|<\frac{1}{2}$. Assume without loss of generality that $t_1<t_2$ and consider the function $f_4(\kappa)=\int_0^{t_1}\mathrm{e}^{\kappa\ln s^{-2}t_1(t_1-s)t_2(t_2-s)}\,\mathrm{d}s$ first. For $h\in\mathbb{C}$, $h\neq 0$, we have that

$$\frac{1}{h}(f_4(\kappa+h)-f_4(\kappa))=\int_0^{t_1}f_4(\kappa, h, s)\mathrm{d}s,$$

where

$$f_4(\kappa, h, s)=\mathrm{e}^{\kappa\ln s^{-2}t_1(t_1-s)t_2(t_2-s)}\frac{\mathrm{e}^{h\ln s^{-2}t_1(t_1-s)t_2(t_2-s)}-1}{h}.$$

Observe now that, for $u>0$ and $h=h_1+\mathrm{i}h_2$,

$$\frac{\mathrm{e}^{\mathrm{i}h\ln u}-1}{h}=\frac{\mathrm{e}^{h_1\ln u}\mathrm{e}^{\mathrm{i}h_2\ln u}-1}{h_1+\mathrm{i}h_2}=\mathrm{e}^{\mathrm{i}h_2\ln u}\frac{\mathrm{e}^{h_1\ln u}-1}{h_1+\mathrm{i}h_2}+\frac{\mathrm{e}^{\mathrm{i}h_2\ln u}-1}{h_1+\mathrm{i}h_2}$$

and hence that $|h|^{-1}|\mathrm{e}^{h\ln u}-1|\leqslant 2|\ln u|\mathrm{e}^{|h_1\ln u|}\leqslant 2|\ln u|\mathrm{e}^{\epsilon|\ln u|}$, if $|h|<\epsilon$. This implies that

$$|h|^{-1}|\mathrm{e}^{h\ln s^{-2}t_1(t_1-s)t_2(t_2-s)}-1|\leqslant 2|\ln s^{-2}t_1(t_1-s)t_2(t_2-s)|\,\mathrm{e}^{\epsilon|\ln s^{-2}t_1(t_1-s)t_2(t_2-s)|},$$

for $|h|<\epsilon$. Choose $\epsilon>0$ so that $\int_0^1 s^{-2\kappa}s^{-2\epsilon}\,\mathrm{d}s<\infty$. Since

$$\int_0^{t_1} s^{-2\kappa_1} t_1^{\kappa_1}(t_1-s)_+^{\kappa_1} t_2^{\kappa_1}(t_2-s)_+^{\kappa_1} |\ln s^{-2} t_1(t_1-s)t_2(t_2-s)| \, \mathrm{e}^{\epsilon|\ln s^{-2}t_1(t_1-s)t_2(t_2-s)|} \, \mathrm{d}s < \infty,$$

where κ_1 is the real part of κ and $|\kappa| < \frac{1}{2}$, we obtain from the dominated convergence theorem that

$$\frac{\mathrm{d}}{\mathrm{d}\kappa} f_4(\kappa) = \lim_{h\to 0} \frac{f_4(\kappa+h) - f_4(\kappa)}{h} = \int_0^{t_1} s^{-2\kappa} t_1^{\kappa}(t_1-s)_+^{\kappa} t_2^{\kappa}(t_2-s)_+^{\kappa} \ln s^{-2} t_1(t_1-s)t_2(t_2-s) \mathrm{d}s$$

and hence that the function f_4 is analytic on $|\kappa| < \frac{1}{2}$.

For the function $f_3(\kappa) = \int_0^{t_1} \mathrm{e}^{\kappa \ln s^{-2} t_1(t_1-s)} \int_s^{t_2} u^{-1} \mathrm{e}^{\kappa \ln u(u-s)} \mathrm{d}u \, \mathrm{d}s$, write

$$\frac{f_3(\kappa+h) - f_3(\kappa)}{h} = \int_0^{t_1} \mathrm{e}^{\kappa \ln s^{-2} t_1(t_1-s)} \frac{\mathrm{e}^{h \ln s^{-2} t_1(t_1-s)} - 1}{h} \int_s^{t_2} u^{-1} \mathrm{e}^{(\kappa+h)\ln u(u-s)} \mathrm{d}u \, \mathrm{d}s$$

$$+ \int_0^{t_1} \mathrm{e}^{\kappa \ln s^{-2} t_1(t_1-s)} \int_s^{t_2} u^{-1} \mathrm{e}^{\kappa \ln u(u-s)} \frac{\mathrm{e}^{h \ln u(u-s)} - 1}{h} \mathrm{d}u \, \mathrm{d}s$$

and use the arguments above to conclude that

$$\frac{\mathrm{d}}{\mathrm{d}\kappa} f_3(\kappa) = \int_0^{t_1} s^{-2\kappa} t_1^{\kappa}(t_1-s)^{\kappa} \ln s^{-2} t_1(t_1-s) \int_s^{t_2} u^{\kappa-1}(u-s)^{\kappa} \mathrm{d}u \, \mathrm{d}s$$

$$+ \int_0^{t_1} s^{-2\kappa} t_1^{\kappa}(t_1-s)^{\kappa} \int_s^{t_2} u^{\kappa-1}(u-s)^{\kappa} \ln u(u-s) \mathrm{d}u \, \mathrm{d}s$$

and hence that the function f_3 is analytic on $|\kappa| < \frac{1}{2}$. The proof that the functions f_2 and f_1 are analytic on $|\kappa| < \frac{1}{2}$ is similar.

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