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The uniqueness class of continuous local martingales

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We study some properties of the class $\mathcal{U}$ of laws of a continuous local martingale which is determined by the law of its quadratic variation, and we give a 'simple' characterization of this class.

Keywords: Ocone martingale; uniqueness class

1. Introduction

Generally, the law $\mathcal{L}(\langle M \rangle)$ of the quadratic variation process of a continuous local martingale M does not determine the law $\mathcal{L}(M)$ of M. Indeed, two local martingales with different laws may have the same quadratic variation process.

In this paper we characterize the class $\mathcal{U}$ of laws $\mathcal{L}(M)$ of a continuous local martingale M determined by the law $\mathcal{L}(\langle M \rangle)$ of its quadratic variation process. That is, $\mathcal{L}(M) \in \mathcal{U}$ is equivalent to the statement that if M' is a continuous local martingale defined possibly on another filtered space such that $\mathcal{L}(\langle M \rangle) = \mathcal{L}(\langle M' \rangle)$, then $\mathcal{L}(M) = \mathcal{L}(M')$.

Vostrikova and Yor (2000) remarked that the class $\mathcal{U}$ is included in the class of laws of Ocone's local martingales (see Definition 2.1 below) and they have conjectured that *the class $\mathcal{U}$ is the class* of Gaussian martingales (modulo a weak supplementary condition). A proof of their conjecture is given in this paper.

In the following, we use the notation of Vostrikova and Yor (2000) for a continuous local martingale M, C is the right-continuous inverse of $\langle M \rangle$, $C_t = \inf\{s \geqslant 0, \langle M \rangle_s > t\}$, $\{\mathcal{M}_t\}_{t \geqslant 0}$ is the natural filtration of M, $\{\mathcal{N}_t\}_{t \geqslant 0}$ is the filtration of $\langle M \rangle$ and $(\mathcal{C}_t)$ is the filtration of C (the σ-field $\mathcal{C}_0$ will play a crucial role in most of our arguments). All the filtrations considered are complete and right-continuous. Furthermore, unless otherwise mentioned, B is the Dambis–Dubins–Schwarz (DDS) Brownian motion of M.

2. The Ocone martingales

We begin by defining an Ocone (local) martingale as follows:

Definition 2.1. *A continuous local martingale is called an Ocone (local) martingale if it is null at 0 and if its DDS Brownian motion is independent of $\langle M \rangle$, the bracket of M.*

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It will be convenient to suppress the adjective 'local' and write only Ocone martingale, although we work with local martingales.

This definition is equivalent to the following property: if $\{\varepsilon_t\}_{t\geqslant 0}$ is a predictable process which takes only the values -1 and $+1$, then

$$M^{\varepsilon} \overset{\text{(law)}}{=} M, \quad \text{where} \quad M_t^{\varepsilon} := \int_0^t \varepsilon_s \, \mathrm{d}M_s. \tag{1}$$

In fact, this is the property which Ocone (1993) took initially as a definition, and proved to be equivalent to Definition 2.1. For other properties equivalent to (1), see Dubins *et al*. (1993) and Vostrikova and Yor (2000).

The following simple property will be very useful in the sequel:

Proposition 2.1. *Let M be a continuous local martingale. The following two properties are equivalent:*

(i) $\mathcal{L}(M) \in \mathcal{U}$.

(ii) *Whenever M' is a continuous local martingale defined on a filtered space $(\Omega, \mathcal{A}, \mathbf{P}, \mathcal{F})$ such that $\mathcal{L}(\langle M\rangle') = \mathcal{L}(\langle M\rangle)$, then M' is an Ocone martingale.*

Proof. It suffices to prove that if $\mathcal{L}(M) \in \mathcal{U}$ then M is an Ocone martingale. For every predictable process $(\varepsilon_t)_{t\geqslant 0}$ which takes the values -1 and $+1$, if $M_t^{\varepsilon} = \int_0^t \varepsilon_s \, \mathrm{d}M_s$, then $\langle M^{\varepsilon}\rangle \equiv \langle M\rangle$ and $\mathcal{L}(M) = \mathcal{L}(M^{\varepsilon})$. □

Remarks 2.1. (i) It is natural in our study to introduce the notion of the $\mathcal{F}$-Ocone martingale (analogously to the notion of $\mathcal{F}$-Brownian motion, etc.). An $\mathcal{F}$-Ocone martingale is an $\mathcal{F}$-local martingale which satisfies (1) for every $\mathcal{F}$-predictable process $(\varepsilon_t)_{t\geqslant 0}$. Generally, an Ocone martingale is not an $\mathcal{F}$-Ocone martingale. For example, if $\mathcal{F}$ is the natural filtration of a Brownian motion B, then the only $\mathcal{F}$-Ocone martingales with strictly increasing bracket are the Gaussian martingales with respect to this filtration. Indeed, let M be an $\mathcal{F}$-Ocone martingale $M_t = \int_0^t \mu_s \mathrm{d}B_s$ (for an $\mathcal{F}$-predictable process (μ_t)); then $M'_t = \int_0^t |\mu_s| \mathrm{d}B_s$ is an Ocone martingale. Using Theorem 3 of Vostrikova and Yor (2000), the process $(|\mu_t|)_{t\geqslant 0}$ is independent of B and is also adapted to B, hence $\langle M\rangle$ is deterministic. Vostrikova and Yor (2000) give examples of Ocone martingales which are non-Gaussian in the filtration $\mathcal{F}$ (with $\mu_t > 0 \, \mathrm{d}t\mathrm{d}\mathbf{P}$-a.s.).

(ii) Observe that if $\mathcal{L}(M) \in \mathcal{U}$ and M is an $\mathcal{F}$-local martingale, then M is an $\mathcal{F}$-Ocone martingale. But the hypothesis that a local martingale M is an $\mathcal{F}$-Ocone martingale is a long way from sufficient to obtain $\mathcal{L}(M) \in \mathcal{U}$, because this hypothesis imposes the equality of the increasing processes of a number of martingales related to M, which is much more restrictive than only the equality of their laws (see Proposition 4.3 below).

Here is an example of Ocone martingale such that $\mathcal{L}(M) \notin \mathcal{U}$. Let (B^1, B^2) be a two-dimensional Brownian motion. From Theorem 3 of Vostrikova and Yor (2000), $M_t = \int_0^t |B_s^1| \mathrm{d}B_s^2$ is an Ocone martingale. But the local martingale $N_t = \int_0^t |B_s^1| \mathrm{d}B_s^1$ which satisfies

$\langle M\rangle \equiv \langle N\rangle$ has a different law than M (because N is extremal, so it is not an Ocone martingale).

3. Some elements of the class $\mathcal{U}$

It is obvious that if $\mathcal{N}_\infty = \mathcal{N}_0$ then $\mathcal{L}(M) \in \mathcal{U}$. The following lemma permits us to obtain other elements of $\mathcal{U}$.

Lemma 3.1. *Let M be a continuous local martingale. If M' is a continuous local martingale such that*

$$\langle M\rangle' \overset{\text{(law)}}{=} \mathbb{1}_{[\varepsilon,+\infty[}\langle M\rangle_{\cdot-\varepsilon}, \tag{2}$$

then $\mathcal{L}(M) \in \mathcal{U}$ implies that $\mathcal{L}(M') \in \mathcal{U}$.

Proof. Let $M'_t = B'_{\langle M'\rangle_t}$ be a continuous local martingale and $(\mathcal{M}'_t)$ its natural filtration such that (2) holds. Let A be the continuous increasing process $A_t = \langle M'\rangle_{t+\varepsilon}$. A is a continuous $\mathcal{M}'_{C'}$-time change and $N_t = B'_{A_t}$ is a continuous local martingale such that $\mathcal{L}(\langle M\rangle) = \mathcal{L}(\langle N\rangle)$. Using Proposition 2.1, N is an Ocone martingale, and so is M', hence $\mathcal{L}(M') \in \mathcal{U}$.□

The following proposition gives a sufficient condition for $\mathcal{L}(M) \in \mathcal{U}$.

Proposition 3.1. *Let M be a continuous local martingale and C be the inverse of $\langle M\rangle$. If $\mathcal{N}_\infty = \mathcal{C}_0$, then $\mathcal{L}(M) \in \mathcal{U}$.*

Remark 3.1. In fact, we prove below that this property characterizes the elements of $\mathcal{U}$. To avoid any confusion, let us emphasize that this property is precisely $\mathcal{C}_0 = \mathcal{N}_\infty$ and not $\mathcal{N}_0 = \mathcal{N}_\infty$ (because generally $\mathcal{C}_0 \neq \mathcal{N}_0$). A simple counter-example is given after the proof.

Proof. If M' is a continuous local martingale such that $\mathcal{L}(\langle M'\rangle) = \mathcal{L}(\langle M\rangle)$, then $\mathcal{N}'_\infty = \mathcal{C}'_0$. Consequently, if B' is the DDS Brownian motion of M' then B' is independent of $\mathcal{C}'_0 \subset \mathcal{M}'_{C_0}$, hence of $\mathcal{N}'_\infty$, which proves that M' is an Ocone martingale. From Proposition 2.1, $\mathcal{L}(M) \in \mathcal{U}$. □

Here is a simple example of a non-Gaussian martingale M with $\mathcal{L}(M) \in \mathcal{U}$. Let a_1 and a_2 be two continuous increasing functions from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $a_1(0) = a_2(0) = 0$, and let η be a Bernoulli random variable with $\mathbf{P}(\eta = 1) = \mathbf{P}(\eta = 0) = \frac{1}{2}$. We define the Ocone martingale M associated with the following increasing process:

$$\langle M\rangle_t = a_1(t)\mathbb{1}_{\{\eta=1\}} + a_2(t)\mathbb{1}_{\{\eta=0\}}.$$

We have $\mathcal{N}_\infty = \sigma(\eta) = \mathcal{C}_0$. Then $\mathcal{L}(M) \in \mathcal{U}$. For $\varepsilon > 0$, if M' is the Ocone martingale associated with the increasing process $\langle M'\rangle$, such that (2) holds, then $\mathcal{L}(M') \in \mathcal{U}$ by Lemma 3.1. Observe also that $C'_0 = \varepsilon$, $\mathcal{C}'_0 = \sigma(\eta)$ and $\mathcal{N}'_0$ is trivial. Then $\mathcal{N}'_0 \neq \mathcal{C}'_0$.

First, we will treat the case where $\mathrm{d}\langle M\rangle_t$ is equivalent to $\mathrm{d}t$. In this case, the corresponding elements of $\mathcal{U}$ are the Gaussian martingales. This follows easily from the next proposition:

Proposition 3.2. *Let M be a continuous local martingale defined on the filtered space* $(\Omega, \mathcal{A}, \mathbf{P}, \mathcal{F})$ such that

$$\langle M\rangle_t = \int_0^t H_s^2 \mathrm{d}s \quad \textit{and} \quad H_s > 0 \,\mathrm{d}s\,\mathrm{d}\mathbf{P}\text{-}a.s.$$

If $\mathcal{L}(M) \in \mathcal{U}$, then $\langle M\rangle$ is independent of every $\mathcal{F}$-Brownian motion B.

Proof. Introducing, if necessary, an independent enlargement (i.e. the embedding of Ω into the product of Ω with the Wiener space), we may assume the existence of a Brownian motion B' independent of $\mathcal{F}_\infty$. Let us define $M'_t := \int_0^t H_s \mathrm{d}B_s$ and $M''_t := \int_0^t H_s \,\mathrm{d}B'_s$. M' and M'' are two $\mathcal{F}'$-martingales where $\mathcal{F}' = \mathcal{F} \vee \mathcal{B}'$ ($\mathcal{B}'$ is the natural filtration of B'). One has $B_t = \int_0^t \mathrm{d}M'_s/H_s$ and $B'_t = \int_0^t \mathrm{d}M''_s/H_s$, so B and B' are adapted to $\{\mathcal{M}'_t\}$ and $\{\mathcal{M}''_t\}$, respectively (using the positivity of H). So $(\langle M\rangle, B) \overset{\text{(law)}}{=} (\langle M\rangle, B')$. Thus, $\langle M\rangle$ is independent of B. □

Theorem 3.1. *Let M be a local martingale such that $\mathcal{L}(M) \in \mathcal{U}$ and*

(i) $\langle M\rangle_t = \int_0^t H_s^2 \,\mathrm{d}s$,
(ii) $H_s > 0 \,\mathrm{d}s\,\mathrm{d}\mathbf{P}$-*a.s.*,
(iii) *there exists a d-dimensional Brownian motion B ($d \in \mathbb{N}^* \cup \{+\infty\}$), such that $\langle M\rangle$ is adapted to the filtration of B.*

Then $\langle M\rangle$ is deterministic.

Proof. Let $n \in \mathbb{N}^*$ and $(f_i(s))_{1 \leqslant i \leqslant n}$ be a vector of bounded deterministic Borel functions such that $\sum_{i=1}^n f_i^2(s) > 0$ for every $s \geqslant 0$. Let γ be the Brownian motion

$$\gamma_t = \int_0^t \sum_{i=1}^n f_i(s) \mathrm{d}B_s^i \Bigg/ \left(\sum_{i=1}^n f_i^2(s)\right)^{1/2}.$$

Using Proposition 3.2, $\langle M\rangle$ is independent of γ and so of $\int_0^t \sum_{i=1}^n f_i(s)\mathrm{d}B_s^i$. Then, for $t \geqslant 0$ and any functional $\Phi \geqslant 0$, one has:

$$\mathbf{E}\left[\Phi(\langle M\rangle_s, s \leqslant t)\exp\left(\int_0^t \sum_{i=1}^n f_i(s)\mathrm{d}B_s^i\right)\right] = \mathbf{E}[\Phi(\langle M\rangle_s, s \leqslant t)]\mathbf{E}\left[\exp\left(\int_0^t \sum_{i=1}^n f_i(s)\mathrm{d}B_s^i\right)\right].$$

Thus, $\langle M\rangle$ is independent of B and adapted to B, hence deterministic. □

The following theorem, which is due to M. Émery, allows hypothesis (iii) of Theorem 3.1 to be suppressed. Define a filtration $\mathcal{F}$ to be weakly included in a filtration $\mathcal{G}$ if $\mathcal{F}_t \subset \mathcal{G}_t$, for all $t \geqslant 0$. Recall also that $\mathcal{F}$ is immersed in $\mathcal{G}$ if it is weakly included in $\mathcal{G}$ and the $\mathcal{F}$-local martingales are $\mathcal{G}$-local martingales.

Theorem 3.2. *A filtration $\mathcal{F}$, such that $\mathcal{F}_\infty$ is essentially separable and $\mathcal{F}_0$ trivial, can be included in a one-dimensional Brownian filtration (that is, the Brownian filtration contains a subfiltration which is isomorphic to $\mathcal{F}$).*

Proof. Let $(t_k)_{k\in\mathbb{Z}}$ be a subdivision of $]0, +\infty[$. By the isomorphism theorem (see Vershik 1968; see also Theorem 3 of Émery and Schachermayer 2001), there exists a subsequence $(t'_k)_{k\in\mathbb{Z}}$ of $(t_k)_{k\in\mathbb{Z}}$ such that $t'_k\downarrow 0$ when $k\downarrow -\infty$ and $t'_k\uparrow +\infty$ when $k\uparrow +\infty$ and $(\mathcal{F}_{t'_k})$ is standard. So there exists a sequence of independent random variables with uniform law on $[0, 1]$ such that $\mathcal{F}_{t'_k} = \sigma(\ldots, U_{k-1}, U_k)$. A Brownian motion B whose natural filtration contains $\mathcal{F}$ is defined by the increments $B_{t_{k-1}} - B_{t_{k-2}} = \varphi_k^{-1}(U_k)$, where φ_k is the distribution function of $\mathcal{N}(0, t_{k-1} - t_{k-2})$. □

Corollary 3.1. *Let M be a continuous local martingale such that $\mathcal{C}_0$ is trivial. If $\mathcal{L}(M) \in \mathcal{U}$, then $\langle M\rangle$ is deterministic.*

Proof. (In the case where $\mathrm{d}\langle M\rangle_t$ is equivalent to the Lebesgue measure, the result follows immediately from Theorems 3.1 and 3.2 above.) Insert a Brownian motion B with a natural filtration $\mathcal{B}$ such that $\mathcal{C}_t \subset \mathcal{B}_t$ for all t. $\langle M\rangle$ is a $\mathcal{B}$-time change (by Proposition 1.1 in Chapter V of Revuz and Yor 1999). Considering the local martingale $N_t = B_{\langle M\rangle_t}$, one has $\langle N\rangle \equiv \langle M\rangle$, so N is an Ocone martingale (Proposition 2.1). But $\langle M\rangle$ is $\mathcal{B}_\infty$-measurable, so $\langle M\rangle$ is deterministic. □

4. The characterization theorem of the class $\mathcal{U}$

The following proposition is found in Vostrikova and Yor (2000). Here we propose another proof.

Proposition 4.1. *Let M be an Ocone martingale.*

(i) *Every $\{\mathcal{N}_t\}$-martingale (N_t) is an $\{\mathcal{M}_t\}$-martingale and is orthogonal to M, that is, $\langle N, M\rangle = 0$.*

(ii) *M is extremal if and only if M is Gaussian.*

Proof. (i) Every $\{\mathcal{N}_t\}$-martingale is an $\{\mathcal{M}_t\}$-martingale because $\{\mathcal{N}_t \vee \mathcal{B}_\infty\}_{t\geqslant 0}$ is an independent enlargement of $\{\mathcal{N}_t\}$ (so $\{\mathcal{N}_t\}$ is immersed in $\{\mathcal{N}_t \vee \mathcal{B}_\infty\}$) and, for all $t \geqslant 0$,

$$\mathcal{M}_t \subset \mathcal{N}_t \vee \mathcal{B}_\infty, \quad \text{where} \quad \mathcal{B}_\infty \equiv \sigma\{B_u, u \geqslant 0\}$$

(see Émery and Schachermayer 2001). For the orthogonality, we suppose that $\langle M\rangle$ is strictly increasing (for the general case, see Vostrikova and Yor 2000). N is a continuous $\{\mathcal{M}_t\}$-martingale, so, for all $t \geqslant 0$,

$$\langle M, N\rangle_t = \langle B_{\langle M\rangle}, N_{C\circ\langle M\rangle}\rangle_t = (\langle B, N_C\rangle \circ \langle M\rangle)_t = 0,$$

because B and N_C are independent.

(ii) Suppose that M is extremal; if N is an $\{\mathcal{N}_t\}$-martingale, then from (i) there exists a predictable process $(f_t)_{t\geqslant 0}$ such that

$$N_t = N_0 + \int_0^t f_s\,\mathrm{d}M_s \quad \text{and} \quad \langle N, M\rangle_t = \int_0^t f_s\,\mathrm{d}\langle M\rangle_s = 0.$$

So

$$\langle N\rangle_t = \int_0^t f_s^2 \mathrm{d}\langle M\rangle_s = 0.$$

Thus $N_t = N_0$ for all $t \geqslant 0$. All the $\{\mathcal{N}_t\}$-martingales are constants, so $\mathcal{N}_\infty$ is trivial and M is Gaussian. □

Remark 4.1. Proposition 4.1 is also true if M is assumed to be a $\mathbf{P}$-local martingale.

The Ocone martingale property is preserved by an equivalent change of probability with Radon–Nikodym density $D_\infty \in \mathrm{L}^1(\mathcal{N}_\infty)$. This is shown in the following proposition.

Proposition 4.2. *Let M be a continuous local martingale defined on the filtered space $(\Omega, \mathcal{M}_\infty, \mathbf{P}, \mathcal{M})$, and $\mathbf{P}'$ be a probability equivalent to $\mathbf{P}$ such that $D_\infty = \mathrm{d}\mathbf{P}'/\mathrm{d}\mathbf{P}$ is $\mathcal{N}_\infty$-measurable. If M is a $\mathbf{P}$-Ocone martingale, then M is a $\mathbf{P}'$-Ocone martingale.*

Proof. By Proposition 4.1(i), if $D_t := \mathbf{E}[D_\infty|\mathcal{M}_t]$ then $\langle D, M\rangle \equiv 0$. In fact, if $D'_t := \mathbf{E}[D_\infty|\mathcal{N}_t]$, then D' is an $\mathcal{M}_t$-local martingale, and so $D \equiv D'$. Thus, M is a $\mathbf{P}'$-local martingale. Let $H_1 \in \mathrm{L}^\infty(\mathcal{B}_\infty)$ and $H_2 \in \mathrm{L}^\infty(\mathcal{N}_\infty)$; then

$$\begin{aligned}
\mathbf{E}_{\mathbf{P}'}[H_1 H_2] &= \mathbf{E}_{\mathbf{P}}[D_\infty H_1 H_2] \\
&= \mathbf{E}_{\mathbf{P}}[D_\infty H_2]\mathbf{E}_{\mathbf{P}}[H_1]\mathbf{E}_{\mathbf{P}}[D_\infty] \\
&= \mathbf{E}_{\mathbf{P}}[D_\infty H_1]\mathbf{E}_{\mathbf{P}}[D_\infty H_2] \\
&= \mathbf{E}_{\mathbf{P}'}[H_1]\mathbf{E}_{\mathbf{P}'}[H_2],
\end{aligned}$$

since $\mathbf{E}_{\mathbf{P}}[D_\infty] = 1$ and B is independent of $\langle M\rangle$. So M is a $\mathbf{P}'$-Ocone martingale. □

Remark 4.2. Proposition 4.2 is also true if $\mathbf{P}'$ is absolutely continuous with respect to $\mathbf{P}$, with $\mathrm{d}\mathbf{P}'/\mathrm{d}\mathbf{P}$ $\mathcal{N}_\infty$-measurable.

Corollary 4.1. *With the same notation and hypotheses as in Proposition* 3.2, we have

$$\mathcal{L}_{\mathbf{P}}(M) \in \mathcal{U} \Leftrightarrow \mathcal{L}_{\mathbf{P}'}(M) \in \mathcal{U},$$

where $\mathcal{L}_{\mathbf{P}}(M)$ is the law of M with respect to $\mathbf{P}$.

Proof. Let M' be a continuous local defined defined on a filtered space $(\Omega, \mathcal{M}'_\infty, \mathbf{Q}', \mathcal{M}')$ such that $\mathcal{L}_{\mathbf{Q}'}(\langle M' \rangle) = \mathcal{L}_{\mathbf{P}'}(\langle M \rangle)$. Write $\mathbf{Q} := D_\infty^{-1} \cdot \mathbf{Q}'$ with $D_\infty^{-1} = D_\infty^{-1}(\langle M' \rangle)$ and $D'_t := \mathbf{E}_{\mathbf{Q}'}[D_\infty^{-1} | \mathcal{M}'_t]$. Then $\mathcal{L}_{\mathbf{Q}}(\langle M' \rangle) = \mathcal{L}_{\mathbf{P}}(\langle M \rangle)$; indeed, for every bounded functional F,

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}}[F(\langle M' \rangle)] &= \mathbf{E}_{\mathbf{Q}'}[D_\infty^{-1} F(\langle M' \rangle)] \\ &= \mathbf{E}_{\mathbf{P}'}[D_\infty^{-1}(\langle M \rangle) F(\langle M \rangle)] \\ &= \mathbf{E}_{\mathbf{P}}[F(\langle M \rangle)]. \end{aligned}$$

We have that $\tilde{M}' := M' - \int \mathrm{d}\langle D', M \rangle / D'$ is a $\mathbf{Q}$-local martingale such that $\langle \tilde{M}' \rangle = \langle M' \rangle$ and $\mathcal{L}_{\mathbf{Q}}(\langle \tilde{M}' \rangle) = \mathcal{L}_{\mathbf{P}}(\langle M \rangle)$. Then $\mathcal{L}_{\mathbf{Q}}(\tilde{M}') = \mathcal{L}_{\mathbf{P}}(M)$ and $\tilde{M}'$ is a $\mathbf{Q}$-Ocone martingale. So $\tilde{M}'$ is a $\mathbf{Q}'$-Ocone martingale (by Proposition 4.2). But $\tilde{M}' = M'$, which completes the proof. □

The following two lemmas will be useful in the proof of the characterization theorem.

Lemma 4.1. *Let* η_0 *be a Bernoulli random variable with* $\mathbf{P}\{\eta_0 = 1\} = \mathbf{P}\{\eta_0 = -1\} = \frac{1}{2}$, $t_0, t_{-1} \in]0, +\infty[$, *where* $t_0 > t_{-1}$ *and* B *is a Brownian motion independent of* η_0. *There exists a Brownian motion* B' *such that* $\mathrm{sgn}(B'_{t_0} - B'_{t_{-1}}) = \eta_0$, $\mathcal{B}'_\infty = \mathcal{B}_\infty \vee \sigma(\eta_0)$ *and* B *is a* $\mathcal{B}'$*-Brownian motion, where* $\mathcal{B}$ *and* $\mathcal{B}'$ *are the natural filtrations of* B *and* B', *respectively.*

Remark 4.3. The Brownian motion B' is a solution of the equation

$$\mathrm{d}B'_s = a(s, B'_s)\mathrm{d}B_s, \quad \text{where} \quad a(t, x) = \mathrm{sgn}(x_{t_k} - x_{t_{k-1}}) \mathbb{1}_{]t_k, t_{k+1}]}(t),$$

with $x \in C_0(\mathbb{R}^+, \mathbb{R})$ and $(t_k)_{k \leqslant 0}$ a sequence of $\mathbb{R}^+$ which decreases strictly to 0. This equation has been studied by Le Gall and Yor (1983); see also Attal *et al*. (1995).

Proof. Considering the sequence $(\eta_k)_{k<0}$ defined by

$$\eta_{-1} = \eta_0 \, \mathrm{sgn}(B_{t_0} - B_{t_{-1}}),$$

$$\eta_k = \eta_0 \prod_{n=0}^{k+1} \mathrm{sgn}(B_{t_n} - B_{t_{n-1}}), \qquad k \leqslant -1.$$

For all $k \leqslant 0$, the random variable η_k is independent of B; indeed

$$\mathbf{E}[\eta_k] = \mathbf{E}[\eta_0]\mathbf{E}\left[\prod_{n=0}^{k+1} u_n\right] = 0,$$

and

$$\mathbf{E}[F(B)\eta_k] = \mathbf{E}\left[F(B)\eta_0 \prod_{n=0}^{k+1} u_n\right] = \mathbf{E}[\eta_0]\mathbf{E}\left[F(B) \prod_{n=0}^{k+1} u_n\right] = 0.$$

We define B' by $\mathrm{sgn}(B'_{t_0} - B'_{t_{-1}}) = \eta_0$ and $B' = \int \eta \, \mathrm{d}B$, where

$$\eta_t = \sum_{k<0} \eta_k \mathbb{1}_{]t_k, t_{k+1}]}(t) + \eta_0 \mathbb{1}_{]t_0, +\infty[}(t).$$

We observe that $\eta_k = \mathrm{sgn}(B'_{t_k} - B'_{t_{k-1}})$ and $B'_t - B'_{t_k} = \eta_k (B_t - B_{t_k})$ if $t \in [t_k, t_{k+1}[$. The filtration considered is the smallest completed filtration $\mathcal{F}$ which contains $\mathcal{B}_t \vee \sigma(\eta_k)$, where $t \in [t_k, t_{k+1}[$ ($\mathcal{F}$ is a right-continuous filtration).

We now prove that B' is an $\mathcal{F}$-Brownian motion. Let $t \in [t_k, t_{k+1}[$ and $\epsilon \in \{-1, +1\}$. Then

$$\begin{aligned}\mathbf{E}[(B'_t - B'_{t_k}) \mathbb{1}_{\{\eta_k = \epsilon\}} F(B_u, u \leqslant t_k)] &= \mathbf{E}[\eta_k (B_t - B_{t_k}) \mathbb{1}_{\{\eta_k = \epsilon\}} F(B_u, u \leqslant t_k)] \\ &= \epsilon \mathbf{P}(\eta_k = \epsilon) \mathbf{E}[(B_t - B_{t_k}) F(B_u, u \leqslant t_k)] = 0.\end{aligned}$$

Hence $\mathcal{F} = \mathcal{B}'$ because $\mathcal{B}'_\infty = \mathcal{F}_\infty$. B is a $\mathcal{B}'$-Brownian motion because $B = \int \eta \, \mathrm{d}B'$ and η is $\mathcal{F}$-predictable. □

Lemma 4.2. *Let $\varepsilon > 0$ and M be a continuous such that $\mathcal{L}(M) \in \mathcal{U}$. If M^ε is a continuous local martingale (defined possibly on another filtered space) such that*

$$\langle M^\varepsilon \rangle \overset{(\text{law})}{=} \begin{cases} t & \text{if} \quad t \leqslant \varepsilon, \\ \langle M \rangle_{t-\varepsilon} + \varepsilon & \text{if} \quad t \geqslant \varepsilon, \end{cases} \tag{3}$$

then $\langle M^\varepsilon \rangle$ is independent of $\sigma(\beta^\varepsilon_{t+\epsilon} - \beta^\varepsilon_\varepsilon, t \geqslant 0)$, where β^ε is the DDS Brownian motion of M^ε.

Proof. Let M^ε be a local martingale which satisfies hypothesis (3). Write $N_t := \gamma^\varepsilon_{\langle M^\varepsilon \rangle_{t+\varepsilon} - \varepsilon}$, where $\gamma^\varepsilon_t = \beta^\varepsilon_{t+\varepsilon} - \beta^\varepsilon_\varepsilon$. For $s, t \in \mathbb{R}^+$, we have

$$\{\langle N \rangle_t \leqslant s\} = \{\langle M^\varepsilon \rangle_{t+\varepsilon} \leqslant s + \varepsilon\} \in \mathcal{M}^\varepsilon_{C^\varepsilon_{\varepsilon+\varepsilon}}.$$

Hence, $\langle N \rangle$ is a continuous $(\mathcal{M}^\varepsilon_{C^\varepsilon_{s+\varepsilon}})_{s \geqslant 0}$-time change, so N is a continuous local martingale. Since $\mathcal{L}(\langle N \rangle) = \mathcal{L}(\langle M \rangle)$, N is an Ocone martingale and $\langle M^\varepsilon \rangle$ is independent of γ^ε. □

We can now state our characterization theorem.

Theorem 4.1. *Let M be a continuous local martingale and C be the inverse of the increasing process $\langle M \rangle$. We have $\mathcal{L}(M) \in \mathcal{U}$ if and only if $\mathcal{N}_\infty = \mathcal{C}_0$. In particular, $\mathcal{L}(M) \in \mathcal{U}$ is a Gaussian distribution if and only if $\mathcal{C}_0$ is trivial.*

Proof. It is enough to prove that $\mathcal{L}(M) \in \mathcal{U} \Rightarrow \mathcal{N}_\infty = \mathcal{C}_0$. We will present the proof in five steps. In the first three steps, we treat the particular case where $\mathcal{C}_0$ is trivial.

Step 1. We shall prove that, for all $t \geqslant 0$, $\mathcal{C}_t$ is trivial. Suppose to the contrary that there exists a $t_0 > 0$ and a set $A \in \mathcal{C}_{t_0}$ with $\mathbf{P}(A) \notin \{0, 1\}$. Let $\mathbf{Q}$ be the probability $\mathbf{Q} := D_\infty \cdot \mathbf{P}$, where

$$D_\infty := \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} = \frac{1}{2\mathbf{P}(A)} \mathbb{1}_A + \frac{1}{2\mathbf{P}(A^c)} \mathbb{1}_{A^c}.$$

Observe that $\mathbf{Q}(A) = \frac{1}{2}$. In what follows, we shall work with respect to the probability $\mathbf{Q}$ (by Corollary 3.1, $\mathcal{L}_{\mathbf{Q}}(\mathrm{M}) \in \mathcal{U}$). Let $0 < t_{-1} < t_0$ and $(T_t)_{t \geqslant 0}$ the increasing process

$$T_t = \begin{cases} t & \text{if} \quad t \leqslant t_{-1}, \\ \langle M \rangle_{t-t_{-1}} + t_{-1} & \text{if} \quad t \geqslant t_{-1}, \end{cases}$$

(observe that $T \overset{\text{(law)}}{=} \langle M^{t_{-1}} \rangle$ from Lemma 4.2). Denote by C' the inverse of T and introduce the Brownian motion B' of Lemma 4.1, with $\eta_0 = \mathbb{1}_A - \mathbb{1}_{A^c}$ and B the DDS Brownian motion of M.

Step 2. We need the following lemma:

Lemma 4.3. *Define $\mathcal{F}'_t := \mathcal{B}'_t \vee \mathcal{C}'_t$. If A is independent of $\mathcal{C}'_{t_{-1}}$, then the Brownian motion B' is an $\mathcal{F}'$-Brownian motion ($\mathcal{C}'_t = \bigcap_{\varepsilon > 0} \sigma(C'_s,\, s \leqslant t + \varepsilon)$).*

Proof. Let $t_{k+1} \leqslant t < t_k$ and let F be a bounded $\mathcal{C}'_{t_k}$-measurable function. For $k \leqslant -1$, define $\eta_k = \eta_0 \upsilon_k$, where $\upsilon_k = \prod_{n=0}^{k+1} \operatorname{sgn}(B_{t_n} - B_{t_{n-1}})$. Then

$$\begin{aligned} I &:= \mathbf{E}[(B'_t - B'_{t_k}) F H(B_u,\, u \leqslant t_k) \mathbb{1}_{\{\eta_k = \varepsilon\}}] \\ &= \sum_{\varepsilon' = \pm 1} \mathbf{E}[\upsilon_k (B_t - B_{t_k}) H \mathbb{1}_{\{\upsilon_k = \varepsilon\varepsilon'\}}] \mathbf{E}[\eta_0 F \mathbb{1}_{\{\eta_0 = \varepsilon'\}}]. \end{aligned}$$

But for $k = 0$,

$$I = \mathbf{E}[(B_t - B_{t_0}) H(B_u,\, u \leqslant t_0)] \mathbf{E}[F \eta_0 \mathbb{1}_{\{\eta_0 = \varepsilon\}}] = 0,$$

and for $k < 0$, η_0 is independent of F, so that

$$\begin{aligned} I &= \sum_{\varepsilon' = \pm 1} \mathbf{E}[\upsilon_k (B_t - B_{t_k}) H(B_u,\, u \leqslant t_k) \mathbb{1}_{\{\upsilon_k = \varepsilon\varepsilon'\}}] \mathbf{E}[\eta_0 \mathbb{1}_{\{\eta_0 = \varepsilon'\}}] \mathbf{E}[F] \\ &= \mathbf{E}[F] \mathbf{E}[\eta_k (B_t - B_{t_k}) H(B_u,\, u \leqslant t_k) \mathbb{1}_{\{\eta_k = \varepsilon\}}] \\ &= \mathbf{E}[F] \mathbf{E}[(B'_t - B'_{t_k})(B_u,\, u \leqslant t_k) \mathbb{1}_{\{\eta_k = \varepsilon\}}] = 0. \end{aligned}$$

Hence, B' is an $\mathcal{F}'$-Brownian motion. □

Step 3. In fact, A is independent of $\mathcal{C}'_{t_{-1}} = \mathcal{C}_0$ (because $C'_t = t$ for $t \leqslant t_{-1}$ and $C'_t = C_{t - t_{-1}} + t_{-1}$ for $t > t_{-1}$), so B' is an $\mathcal{F}'$-Brownian motion. It is easily seen that T is a continuous $\mathcal{F}'$-time change, hence $N_t = B'_{T_t}$ is a continuous. Since $\langle N \rangle \overset{\text{(law)}}{=} \langle M^{t_{-1}} \rangle$, we have that $\sigma(B'_{t + t_{-1}} - B'_{t_{-1}},\, t \geqslant 0)$ and $\langle N \rangle$ are independent (Lemma 4.2). But then $\sigma(B'_{t + t_{-1}} - B'_{t_{-1}},\, t \geqslant 0)$ and $\langle M \rangle$ are independent, which is a contradiction (because $A = \{B'_{t_0} - B'_{t_{-1}} > 0\}$ is measurable with respect to $\mathcal{N}_\infty$). This finishes the proof of this particular case.

Step 4 (the general case). In this step, we shall work on the canonical space (we do not use the notation of Theorem 4.2). Let $\mathbb{M}'$ be the convex set of all the probability measures

$\mathbf{P}$ on $\Omega = C(\mathbb{R}^+, \mathbb{R})$ such that the coordinate process $M_t(\omega) = \omega(t)$ is a $\mathbf{P}$-local martingale. We need the following lemma:

Lemma 4.4. *Let* $\mathbf{P} \in \mathbb{M}'$ *and* $\mathbf{P}'$ *be a probability measure on* $\mathcal{M}_\infty$ *such that* $\mathbf{P}' \ll \mathbf{P}$ *and* $D_\infty = \mathrm{d}\mathbf{P}'/\mathrm{d}\mathbf{P}$ *is* $\mathcal{C}_0$*-measurable. Then*

(i) $\mathbf{P}' \in \mathbb{M}'$.
(ii) *If* $\mathbf{P} \in \mathcal{U}$, *then* $\mathbf{P}' \in \mathcal{U}$.

Proof. (i) Let B be the DDS Brownian motion of M, $0 \leqslant s < t$ and $F \in \mathrm{L}^\infty \leqslant (\mathcal{M}_{C_s})$. Then

$$\mathbf{E}_{\mathbf{P}'}[(B_t - B_s)F] = \mathbf{E}_{\mathbf{P}}[(B_t - B_s)FD_\infty] = 0.$$

Hence B is a $(\mathbf{P}', \mathcal{M}_C)$-Brownian motion and M is a $\mathbf{P}'$-local martingale.

(ii) The case where $\mathbf{P}' \sim \mathbf{P}$ has been treated in Corollary 4.1. Suppose that $\mathbf{P}(D_\infty = 0) > 0$ and consider $\mathbf{Q}' \in \mathbb{M}'$ such that $\mathcal{L}_{\mathbf{Q}'}(\langle M \rangle) = \mathcal{L}_{\mathbf{P}'}(\langle M \rangle)$. Define

$$\mathbf{Q} := \frac{\mathbb{1}_{\{D_\infty \neq 0\}}}{D_\infty} \cdot \mathbf{Q}' + \mathbb{1}_{\{D_\infty = 0\}} \cdot \mathbf{P}.$$

Using (i), we have

$$\frac{1}{D_\infty \mathbf{P}(D_\infty > 0)} \cdot \mathbf{Q}' \in \mathbb{M}' \quad \text{and} \quad \frac{\mathbb{1}_{\{D_\infty = 0\}}}{\mathbf{P}(D_\infty = 0)} \cdot \mathbf{P} \in \mathbb{M}'.$$

So, using the convexity of $\mathbb{M}'$, $\mathbf{Q} \in \mathbb{M}'$. Observe that $\mathcal{L}_{\mathbf{Q}}(\langle M \rangle) = \mathcal{L}_{\mathbf{P}}(\langle M \rangle)$, so $\mathbf{P} = \mathbf{Q}$ and $\mathbf{P}' = \mathbf{Q}'$. □

Step 5. We shall show that $\{\mathrm{Mult}(\mathcal{C}_\infty | \mathcal{C}_0) > 1\} = \varnothing$ a.s. Suppose to the contrary that there exists $t_0 > 0$ such that $B := \{\mathrm{Mult}(\mathcal{C}_{t_0} | \mathcal{C}_0) > 1\} \neq \varnothing$ a.s. Using Lemma 4.4,

$$\mathbf{P}' := \frac{\mathbb{1}_B}{\mathbf{P}(B)} \cdot \mathbf{P} \in \mathcal{U}.$$

Arguing as in the particular case (steps 1, 2 and 3) with $\mathbf{P}'$ instead of $\mathbf{P}$ and

$$\mathbf{Q}' := \frac{1}{2} \left(\frac{\mathbb{1}_A}{\mathbf{P}'(A|\mathcal{C}_0)} + \frac{\mathbb{1}_{A^c}}{\mathbf{P}'(A^c|\mathcal{C}_0)} \right) \cdot \mathbf{P}'$$

instead of $\mathbf{Q}$, we obtain the result (observe that A and $\mathcal{C}_0$ are $\mathbf{Q}'$-independent, $\mathbf{Q}'(A) = \frac{1}{2}$, $\mathbf{Q}'(E) = \mathbf{P}'(E)$ and $\mathbf{Q}'(A \cap E) = \frac{1}{2}\mathbf{Q}'(E)$ for every $E \in \mathcal{C}_0$). □

We now present another characterization of the uniqueness class.

Proposition 4.3. $\mathcal{L}(M) \in \mathcal{U}$ *if and only if the DDS Brownian motion* B *of* M *has the predictable representation property in the filtration* $\mathcal{F} = \mathcal{B} \vee \mathcal{C}$, *and* M *is an Ocone martingale*.

Proof. We suppose that B has the predictable representation property in $\mathcal{F}$ and M is an Ocone martingale, and we prove that $\mathcal{C}_0 = \mathcal{C}_\infty = \mathcal{N}_\infty$. The filtrations $\mathcal{C}$ and $\mathcal{B}$ are immersed in $\mathcal{F}$, and if N is a $(\mathcal{C}_t)$-local martingale, then $N_t = N_0 + \int_0^t f_s \,\mathrm{d}B_s$, for an $\mathcal{F}$-predictable process (f_t). But $\langle N, B\rangle \equiv 0$, so $\int_0^t f_s \, ds = 0$ for all $t \geqslant 0$. Hence, for all $t \geqslant 0$, $\int_0^t f_s^2 \, ds = 0$ and $N_t = N_0$. Consequently, $\mathcal{C}_0 = \mathcal{C}_\infty = \mathcal{N}_\infty$ and $\mathcal{L}(M) \in \mathcal{U}$ (by Proposition 3.1).

Conversely, if $\mathcal{L}(M) \in \mathcal{U}$ then $\mathcal{C}_0 = \mathcal{N}_\infty$ (Theorem 4.1) and $\mathcal{F}_t = \mathcal{C}_0 \vee \mathcal{B}_t$. □

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