Bernoulli **8**(6), 2002, 745–765

The first exit time of Brownian motion from a parabolic domain

MIKHAIL LIFSHITS[1] and ZHAN SHI[2]

[1]*Faculty of Mathematics and Mechanics, St Petersburg State University, 198504 Stary Peterhof, Bibliotechnaya pl. 2, Russia. E-mail: lifts@mail.rcom.ru*
[2]*Laboratoire de Probabilités UMR 7599, Université Paris VI, 4 Place Jussieu, F-75252 Paris Cedex 05, France. E-mail: zhan@proba.jussieu.fr*

Consider a planar Brownian motion starting at an interior point of the parabolic domain $D = \{(x, y) : y > x^2\}$, and let τ_D denote the first time the Brownian motion exits from D. The tail behaviour (or equivalently, the integrability property) of τ_D is somewhat exotic since it arises from an interference of large-deviation and small-deviation events. Our main result implies that the limit of $T^{-1/3} \log \mathbb{P}\{\tau_D > T\}$, $T \to \infty$, exists and equals $-3\pi^2/8$, thus improving previous estimates by Bañuelos *et al*. and Li. The existence of the limit is proved by applying the classical Schilder large-deviation theorem. The identification of the limit leads to a variational problem, which is solved by exploiting a theorem of Biane and Yor relating different additive functionals of Bessel processes. Our result actually applies to more general parabolic domains in any (finite) dimension.

Keywords: Brownian motion; Bessel process; exit time

1. Introduction

Let $(\mathbf{B}(t),\ t \geqslant 0)$ be a Brownian motion taking values in $\mathbb{R}^{d+1}$, and let D be an unbounded Borel subset of $\mathbb{R}^{d+1}$. We assume that $\mathbf{B}$ starts at a point in the interior of D, and we are interested in

$$\tau_D := \inf\{t \geqslant 0 : \mathbf{B}(t) \notin D\},$$

the first exit time of the Brownian motion from D.

Of course, the distribution of τ_D strongly depends on the form of D. Apart from trivial situations, the example which has attracted the most research attention is when D is a (possibly generalized) cone. In this case, the exact distribution of τ_D is known, from which it can be deduced that

$$\mathbb{P}\{\tau_D > T\} \sim cT^{-\kappa}, \qquad T \to \infty,$$

where $c = c(D) > 0$ and $\kappa = \kappa(D) > 0$ are constants whose values can be explicitly formulated in terms of the eigenvalues and eigenfunctions of the Laplacian in D; see Bañuelos and Smits (1997) for a detailed account of the problem. Throughout the paper, we adopt the usual notation $a(T) \sim b(T)$, $T \to T_0$, to denote $\lim_{T \to T_0} a(T)/b(T) = 1$.

The cone in dimension 2 can be thought of as the domain above the graph of a function of the form $y = a|x|$. As pointed out in Bañuelos *et al*. (2001), it is a highly non-trivial matter to find other unbounded domains above graphs of functions for which one can say something deep about the (tail) distribution of τ_D. They studied the natural example of a parabola in dimension 2:

$$D := \{(x, y) \in \mathbb{R}^2 : y > x^2\}. \tag{1.1}$$

Their main result says that τ_D has a subexponential tail. More precisely, they proved the following theorem:

Theorem A. *Let $d = 1$ and let D be as in* (1.1). *There are two constants $A_1 > 0$ and $A_2 > 0$ such that*

$$-A_1 \leqslant \liminf_{T\to\infty} T^{-1/3} \log \mathbb{P}\{\tau_D > T\} \leqslant \limsup_{T\to\infty} T^{-1/3} \log \mathbb{P}\{\tau_D > T\} \leqslant -A_2. \tag{1.2}$$

Therefore, the tail behaviour of τ_D in the case of a two-dimensional parabola differs very much from that in the case of a cone.

Recently, Li (2001) has refined the result of Bañuelos *et al*. (2001) by showing that (1.2) holds with

$$A_1 = (2^{-7/3}3^{4/3})\pi^{4/3}, \qquad A_2 = (2^{-7/3}3)\pi^{4/3}. \tag{1.3}$$

It is our aim to prove that $T^{-1/3} \log \mathbb{P}\{\tau_D > t\}$ has a limit and to determine its value. Our Theorem 1.1 below will imply that

$$\lim_{T\to\infty} T^{-1/3} \log \mathbb{P}\{\tau_D > T\} = -\frac{3\pi^2}{8}. \tag{1.4}$$

(It is easily verified that $(2^{-7/3}3)\pi^{4/3} < 3\pi^2/8 < (2^{-7/3}3^{4/3})\pi^{4/3}$. Thus (1.4) is in agreement with (1.3). These inequalities also indicate that neither of numerical estimates in Li (2001) is optimal.)

Actually, one can consider generalized parabolic domains of arbitrary dimension. It was Li (2001) who studied the generalized parabolic shape in $\mathbb{R}^{d+1}$:

$$D = D_{d,p,a} := \{(\mathbf{x}, y) := (x_1, \ldots, x_d, y) \in \mathbb{R}^{d+1} : y > a\|\mathbf{x}\|^p\}, \tag{1.5}$$

where $p > 1$, and $\|\mathbf{x}\| := [\sum_{i=1}^d x_i^2]^{1/2}$ is the Euclidean norm of $\mathbf{x} := (x_1, \ldots, x_d) \in \mathbb{R}^d$. Of course, if $a = d = 1$ and $p = 2$, then we have the case in (1.1). We mention that the presence of a is superfluous; it can easily be removed with appropriate changes by means of the scaling property of Brownian motion. Li's result is as follows:

Theorem B. *For $d \geqslant 1$, $a > 0$, $p > 1$ and $D = D_{d,p,a}$ as in* (1.5),

$$\begin{aligned} -A_3 &\leqslant \liminf_{T\to\infty} T^{-(p-1)/(p+1)} \log \mathbb{P}\{\tau_D > T\} \\ &\leqslant \limsup_{T\to\infty} T^{-(p-1)/(p+1)} \log \mathbb{P}\{\tau_D > T\} \leqslant -A_4, \end{aligned}$$

where

$$A_3 := \frac{\left(a^2(p+1)^{2p} p^{2-p} j_{(d-2)/2}^{2p}\right)^{1/(p+1)}}{2(p-1)},$$

$$A_4 := \frac{p+1}{2}\left(\frac{a^2 j_{(d-2)/2}^{2p}}{(p-1)^p}\right)^{1/(p+1)}$$

and $j_{(d-2)/2}$ *is the smallest positive zero of the Bessel function* $J_{(d-2)/2}(\cdot)$.

Here is our main result:

Theorem 1.1. *Let* $d \geqslant 1$, $a > 0$ *and* $p > 1$. *Let* $D = D_{d,p,a}$ *be as in* (1.5). *We have*

$$\lim_{T\to\infty} T^{-(p-1)/(p+1)} \log \mathbb{P}\{\tau_D > T\} = -(p+1)\left(\frac{\pi j_{(d-2)/2}^{2p} a^2}{2^{p+3}(p-1)^{p-1}} \frac{\Gamma^2((p-1)/2)}{\Gamma^2(p/2)}\right)^{1/(p+1)}, \tag{1.6}$$

where $j_{(d-2)/2}$ *is, as before, the smallest positive zero of the Bessel function* $J_{(d-2)/2}(\cdot)$, *and* $\Gamma(\cdot)$ *denotes the usual gamma function*.

Since $j_{-1/2} = \pi/2$, Theorem 1.1 immediately yields (1.4) by taking $a = d = 1$ and $p = 2$.

The rest of the paper is as follows. In Section 2, we give (an outline of) the proof of Theorem 1.1. Further technical details are given in Section 3. In Section 4 we give an extension of Theorem 1.1 by studying exit times from non-polynomial shapes.

Throughout the paper, the letter c with subscripts denotes constants which are finite and positive.

2. Proof of Theorem 1.1

2.1. Preliminaries

It is easily seen that the asymptotic behaviour of $\mathbb{P}\{\tau_D > T\}$ (for $T \to \infty$) does not depend on the starting point of the Brownian motion, as long as it is in the interior of D. Without loss of generality, we assume that $a = 1$ and that our Brownian motion starts at $(0, \ldots, 0, 1) \in \mathbb{R}^{d+1}$. By definition, for any $T > 0$,

$$\{\tau_D > T\} = \{\mathbf{B}(t) \in D, \forall t \in [0, T]\}.$$

Therefore, if we write

$$\gamma_{d,p}(T) := \mathbb{P}\{\tau_D > T\},$$

then

$$\gamma_{d,p}(T) = \mathbb{P}\{\|\mathbf{W}(t)\|^p < \tilde{W}(t) + 1, \forall t \in [0, T]\}, \tag{2.1}$$

where $\mathbf{W} := (W_1, \ldots, W_d)$ is a d-dimensional Brownian motion starting at $\mathbf{0} \in \mathbb{R}^d$, and $\tilde{W}$ is a one-dimensional Brownian motion starting at 0, such that $\mathbf{W}$ and $\tilde{W}$ are independent.

A few simple observations about the asymptotic order of $-\log \gamma_{d,p}(T)$ for large T will be useful here – these were known to Li (2001), who rigorously proved the correct rate $T^{(p-1)/(p+1)}$. The event on the right-hand side of (2.1) is of very small probability: it is hard for the independent Brownian motions $\mathbf{W}$ and $\tilde{W}$ to satisfy the condition $\|\mathbf{W}(t)\|^p < \tilde{W}(t) + 1$ for all $t \in [0, T]$. A sufficiently economical way to meet such a condition is that both $\|\mathbf{W}(t)\|^p$ and $\tilde{W}(t)$ should behave like $T^\alpha f(t/T)$ for some $\alpha > 0$ and $f : [0, 1] \to \mathbb{R}_+$. An easy optimization of polynomial degree yields $\alpha = p/(p+1)$, while the right choice of the profile function f boils down to a functional optimization problem. Summarizing the argument, one would expect that

$$\begin{aligned}
\gamma_{d,p}(T) &\approx \mathbb{P}\left\{ \|\mathbf{W}(t)\| \leqslant T^{1/(p+1)} f^{1/p}\left(\frac{t}{T}\right), 1 \leqslant t \leqslant T \right\} \\
&\quad \times \mathbb{P}\left\{ \tilde{W}(t) \geqslant T^{p/(p+1)} f\left(\frac{t}{T}\right), 1 \leqslant t \leqslant T \right\} \qquad (2.2)\\
&\approx \exp\left(-c_1 \int_0^1 f^{-2/p}(s) \mathrm{d}s \, T^{(p-1)/(p+1)} \right) \times \exp\left(-\frac{1}{2} \int_0^1 \dot{f}^2(s) \mathrm{d}s \, T^{(p-1)/(p+1)} \right) \qquad (2.3)\\
&= \exp(-c_2(f) T^{(p-1)/(p+1)}).
\end{aligned}$$

The function f providing an optimal constant $c_2(f)$ appears via solution of an extremal problem

$$B_0 := \frac{1}{2} \inf_{f \in \mathbb{A}_0^\uparrow} \int_0^1 \dot{f}^2(t) \mathrm{d}t, \tag{2.4}$$

where $\mathbb{A}_0^\uparrow$ is the set of all non-decreasing absolutely continuous functions $f : [0, 1] \to \mathbb{R}_+$ such that $f(0) = 0$ and $\int_0^1 f^{-2/p}(t) \mathrm{d}t \leqslant 1$. Assuming that the infimum on the right-hand side of (2.4) is attained at some f_*, we minimize $c_2(vf_*) = c_1 v^{-2/p} + B_0 v^2$ by taking $v = (c_1/pB_0)^{p/(p+1)}$. Hence, $f = vf_*$ gives the optimal value $c_2(f) = (p+1)(c_1^p B_0/p^p)^{1/(p+1)}$. Since $p > 1$ by assumption, we have $1/(p+1) < 1/2$ and $p/(p+1) > 1/2$. Therefore, on the right-hand side of (2.2), the first probability expression is a so-called 'small-ball probability' for $\mathbf{W}$ (i.e., probability that the Brownian motion stays in a narrow domain for a long time), whereas the second one is a 'large-deviation probability' for $\tilde{W}$ (i.e., probability that the Brownian motion reaches a high level in a short time). Estimation of $\gamma_{d,p}(T)$ thus requires a mixture of small-ball and large-deviation techniques.

The rest of this section is devoted to a rigorous proof of Theorem 1.1. To clarify the presentation, we assume a few technical results whose proofs are postponed to Section 3.

2.2. Proof of Theorem 1.1: upper bound

To obtain a rigorous upper bound for $\gamma_{d,p}(T)$, let $0 = t_0 < t_1 < t_2 < \ldots < t_N \leqslant T$ and observe that

$$\gamma_{d,p}(T) \leqslant \mathbb{P}\left\{ \sup_{t\in[t_{i-1},t_i]} \|\mathbf{W}(t)\| < \left(\sup_{t\in[0,t_i]} \tilde{W}(t) + 1 \right)^{1/p}, \forall i \leqslant N \right\}. \tag{2.5}$$

Let $0 < a_1 < \ldots < a_N$. By Anderson's inequality (see, for example, Lifshits 1995), we have

$$\begin{aligned}\mathbb{P}&\left\{ \sup_{t\in[t_{i-1},t_i]} \|\mathbf{W}(t)\| < a_i, \forall i \leqslant N \right\} \\ &\leqslant \mathbb{P}\left\{ \sup_{t\in[t_{i-1},t_i]} \|\mathbf{W}(t)\| < a_i, \forall i \leqslant N-1 \right\} \times \mathbb{P}\left\{ \sup_{t\in[0,t_N - t_{N-1}]} \|\mathbf{W}(t)\| < a_N \right\},\end{aligned}$$

and, by induction, this leads to

$$\mathbb{P}\left\{ \sup_{t\in[t_{i-1},t_i]} \|\mathbf{W}(t)\| < a_i, \forall_i \leqslant N \right\} \leqslant \prod_{i=1}^{N} \mathbb{P}\left\{ \sup_{t\in[0,t_i - t_{i-1}]} \|\mathbf{W}(t)\| < a_i \right\}.$$

At this stage, it is convenient to recall from Ciesielski and Taylor (1962) that

$$\mathbb{P}\left\{ \sup_{t\in[0,1]} \|\mathbf{W}(t)\| < x \right\} \sim c_3 \exp\left(-\frac{\kappa}{2x^2} \right), \qquad x \to 0+, \tag{2.6}$$

where $\kappa := j^2_{(d-2)/2}$ ($j_{(d-2)/2}$ denoting, as before, the smallest positive zero of the Bessel function $J_{(d-2)/2}$), and $c_3 = c_3(d)$ is a positive constant depending on d (whose value is explicitly known). By scaling, for any $\varepsilon \in (0, 1)$, there exists $c_4 = c_4(\varepsilon, d)$ such that for all s, $y > 0$,

$$\mathbb{P}\left\{ \sup_{t\in[0,s]} \|\mathbf{W}(t)\| < y \right\} \leqslant c_4 \exp\left\{ -\frac{(1-\varepsilon)\kappa s}{2y^2} \right\}. \tag{2.7}$$

Accordingly,

$$\mathbb{P}\left\{ \sup_{t\in[t_{i-1},t_i]} \|\mathbf{W}(t)\| < a_i, \forall i \leqslant N \right\} \leqslant c_4^N \exp\left(-\frac{(1-\varepsilon)\kappa}{2} \sum_{i=1}^{N} \frac{t_i - t_{i-1}}{a_i^2} \right).$$

Going back to (2.5), and by conditioning upon the linear Brownian motion $\tilde{W}$, we obtain

$$\gamma_{d,p}(T) \leqslant c_4^N \mathbb{E}\left\{ \exp\left(-\frac{(1-\varepsilon)\kappa}{2} \sum_{i=1}^{N} \frac{t_i - t_{i-1}}{[1 + \sup_{t\in[0,t_i]} \tilde{W}(t)]^{2/p}} \right) \right\}.$$

For the sake of brevity, we write $\beta := 2/p$ and

$$S(t) := \sup_{u\in[0,t]} \tilde{W}(u), \qquad t \geqslant 0.$$

Then we obtain

$$\begin{aligned}\gamma_{d,p}(T) &\leqslant c_4^N \mathbb{E}\left\{\exp\left(-\frac{(1-\varepsilon)\kappa}{2}\sum_{i=1}^N \frac{t_i - t_{i-1}}{[1+S(t_i)]^\beta}\right)\right\}\\ &= c_4^N \mathbb{E}\left\{\exp\left(-\frac{(1-\varepsilon)\kappa}{2}\sum_{i=1}^N \frac{(\tau_i - \tau_{i-1})T^{(p-1)/p}}{[T^{-1/2}+S(\tau_i)]^\beta}\right)\right\},\end{aligned} \tag{2.8}$$

where $\tau_i := t_i/T$. We now set $\tau_i = (1-\varepsilon)^{N-i}$, $1 \leqslant i \leqslant N$. In view of the monotonicity of $t \mapsto S(t)$, we have

$$\begin{aligned}\sum_{i=1}^{N-1} \frac{\tau_i - \tau_{i-1}}{[T^{-1/2}+S(\tau_i)]^\beta} &\geqslant \sum_{i=1}^{N-1} \frac{(1-\varepsilon)(\tau_{i+1}-\tau_i)}{[T^{-1/2}+S(\tau_i)]^\beta}\\ &\geqslant (1-\varepsilon)\sum_{i=1}^N \int_{\tau_i}^{\tau_{i+1}} \frac{\mathrm{d}\tau}{[T^{-1/2}+S(\tau)]^\beta}\\ &= (1-\varepsilon)\int_{\tau_1}^1 \frac{\mathrm{d}\tau}{[T^{-1/2}+S(\tau)]^\beta},\end{aligned}$$

so that

$$\gamma_{d,p}(T) \leqslant C_4^N \mathbb{E}\left\{\exp\left(-\frac{(1-\varepsilon)^2\kappa}{2}\, T^{(p-1)/p}\int_{\tau_1}^1 \frac{\mathrm{d}\tau}{[T^{-1/2}+S(\tau)]^\beta}\right)\right\}. \tag{2.9}$$

Observe that, for any $b > 0$,

$$\begin{aligned}&\mathbb{E}\left\{\exp\left(-b\int_{\tau_1}^1 \frac{\mathrm{d}\tau}{[T^{-1/2}+S(\tau)]^\beta}\right)\right\}\\ &\quad\leqslant \mathbb{E}\left\{\exp\left(-b\int_{2\tau_1}^1 \frac{\mathrm{d}\tau}{[T^{-1/2}+S(\tau)]^\beta}\right)\mathbf{1}_{\{T^{-1/2}\leqslant\varepsilon S(2\tau_1)\}}\right\}\\ &\qquad+ \mathbb{E}\left\{\exp\left(-b\int_{\tau_1}^{2\tau_1} \frac{\mathrm{d}\tau}{[T^{-1/2}+S(\tau)]^\beta}\right)\mathbf{1}_{\{T^{-1/2}>\varepsilon S(2\tau_1)\}}\right\}\\ &\quad\leqslant \mathbb{E}\left\{\exp\left(-\frac{b}{(1+\varepsilon)^\beta}\int_{2\tau_1}^1 S(\tau)^{-\beta}\,\mathrm{d}\tau\right)\right\} + \exp\left(-\frac{b\tau_1 T^{1/p}}{(1+\varepsilon^{-1})^\beta}\right).\end{aligned} \tag{2.10}$$

Taking $b = (1-\varepsilon)^2\kappa T^{(p-1)/p}/2$ and letting $T \to \infty$, we arrive at

$$\limsup_{T\to\infty} T^{-(p-1)/(p+1)} \log \gamma_{d,p}(T)$$

$$\leqslant \inf_{\delta>0,\theta>0} \limsup_{T\to\infty} T^{-(p-1)/(p+1)} \log \mathbb{E}\left\{ \exp\left(-\frac{(1-\delta)\kappa T^{(p-1)/p}}{2} \int_\theta^1 S(\tau)^{-\beta}\,\mathrm{d}\tau \right) \right\}. \qquad (2.11)$$

The question is now how to evaluate

$$\mathbb{E}\left\{ \exp\left(-\lambda \int_\theta^1 S^{-\beta}(t)\mathrm{d}t \right) \right\},$$

when $\lambda \to \infty$ and $\beta \in (0, 2)$ is a fixed constant.

To obtain an upper bound for such an expression, we note that by Schilder's theorem (for a justification, see (3.3) in Section 3.1),

$$\limsup_{x\to 0+} x^{2/\beta} \log \mathbb{P}\left\{ \int_\theta^1 S^{-\beta}(t)\mathrm{d}t \leqslant x \right\} - B_\theta, \qquad (2.12)$$

where

$$B_\theta := \frac{1}{2} \inf_{f\in\mathbb{A}_\theta^\uparrow} \int_0^1 \dot{f}^2(t)\mathrm{d}t.$$

Here and in the following, $\dot{f}$ denotes the Radon–Nikodym derivative of f, and $\mathbb{A}_\theta^\uparrow$ is the set of all non-decreasing functions in the set $\mathbb{A}_\theta$ defined by

$$\mathbb{A}_\theta := \left\{ f : [0, 1] \to \mathbb{R}_+,\ f(0) = 0,\ f \text{ absolutely continuous},\ \int_\theta^1 f^{-\beta}(t)\mathrm{d}t \leqslant 1 \right\}.$$

Clearly, $\inf_{f\in\mathbb{A}_\theta} \int_\theta^1 \dot{f}^2(t)\mathrm{d}t = \inf_{f\in\mathbb{A}_\theta^\uparrow} \int_\theta^1 \dot{f}^2(t)\mathrm{d}t$. (Indeed, for any $f \in \mathbb{A}_\theta$, we can take $g(t) := \sup_{s\in[0,t]} f(s)$, $t \in [0, 1]$, to see that $g \in \mathbb{A}_\theta^\uparrow$ and $\int_\theta^1 \dot{g}^2(t)\mathrm{d}t \leqslant \int_\theta^1 \dot{f}^2(t)\mathrm{d}t$.)

We now need the following elementary result.

Lemma 2.1. *Let $X \geqslant 0$ be a random variable. Let $\alpha > 0$ and $B > 0$. If*

$$\limsup_{x\to 0+} x^\alpha \log \mathbb{P}\{X \leqslant x\} \leqslant -B, \qquad (2.13)$$

then

$$\limsup_{\lambda\to\infty} \lambda^{-\alpha/(\alpha+1)} \log \mathbb{E}[\mathrm{e}^{-\lambda X}] \leqslant -\frac{(\alpha+1)B^{1/(\alpha+1)}}{\alpha^{\alpha/(\alpha+1)}}. \qquad (2.14)$$

The proof of Lemma 2.1 is fairly standard: it suffices to write $\mathbb{E}[\mathrm{e}^{-\lambda X}] = \int_0^1 \mathbb{P}\{\mathrm{e}^{-\lambda X} > x\}\mathrm{d}x$, and estimate the integral using Laplace's method. We omit the details.

By combining (2.12) and (2.14), with $\alpha = 2/\beta$, we obtain

$$\limsup_{\lambda\to\infty} \lambda^{-2/(2+\beta)} \log \mathbb{E}\left\{\exp\left(-\lambda \int_\theta^1 S^{-\beta}(t)\mathrm{d}t\right)\right\} \leqslant -\frac{2+\beta}{\beta} \frac{B_\theta^{\beta/(2+\beta)}}{(2/\beta)^{2/(2+\beta)}}. \tag{2.15}$$

Next, plugging (2.15) into (2.11) with $\beta = 2/p$, and letting $\theta \to 0$, we arrive at

$$\limsup_{T\to\infty} T^{-(p-1)/(p+1)} \log \gamma_{d,p}(T) \leqslant -(p+1)\left(\frac{\kappa}{2p}\right)^{p/(p+1)} B_0^{1/(p+1)}, \tag{2.16}$$

where

$$B_0 := \frac{1}{2} \inf_{f\in\mathbb{A}_0} \int_0^1 \dot{f}^2(t)\mathrm{d}t = \lim_{\theta\to 0} B_\theta. \tag{2.17}$$

Concerning the limit relation in (2.17), we first note that, for $\beta \in (0, 2)$, the infimum B_0 is finite since the set $\mathbb{A}_0$ contains appropriate power functions. The family of sets $\mathbb{A}_\theta$ being non-decreasing with respect to the parameter θ, the limit $\lim_{\theta\to 0} B_\theta$ exists and

$$\lim_{\theta\to 0} B_\theta \leqslant B_0 < \infty.$$

To see why the second identity in (2.17) holds, we take $\theta_n = 1/n$, and, for arbitrary $\varepsilon > 0$, we take a sequence of functions $f_n \in \mathbb{A}_{\theta_n}$ such that

$$\frac{1}{2}\int_0^1 \dot{f}_n^2(t)\mathrm{d}t \leqslant B_{\theta_n} + \varepsilon \leqslant \lim_{\theta\to 0} B_\theta + \varepsilon.$$

The Strassen ball being compact in the space of continuous functions, (f_n) contains a subsequence uniformly converging to a limit function, say f. By Fatou's lemma, we have, for any m,

$$\int_{\theta_m}^1 f^{-\beta}(t)\mathrm{d}t \leqslant \liminf_{n\to\infty} \int_{\theta_m}^1 f_n^{-\beta}(t)\mathrm{d}t \leqslant 1,$$

from which it follows that $f \in \mathbb{A}_0$. Moreover,

$$B_0 \leqslant \frac{1}{2}\int_0^1 \dot{f}^2(t)\mathrm{d}t \leqslant \lim_{\theta\to 0} B_\theta + \varepsilon.$$

In the limit as $\varepsilon \to 0$, we obtain $B_0 \leqslant \lim_{\theta\to 0} B_\theta$, and (2.17) is completely justified.

Now we only need to identify B_0. This variational problem leads to an ordinary differential equation. However, we choose a different way, representing B_0 as a solution of another, (well-investigated) variational problem.

Let R denote a two-dimensional Bessel process (i.e., the Euclidean modulus of an $\mathbb{R}^2$-valued Brownian motion) starting at $\mathbf{0}$.

Lemma 2.2. *Let $\beta \in (0, 2)$. Then*

$$\lim_{x\to 0+} x^{2/\beta} \log \mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\mathrm{d}t \leqslant x\right\} = -\frac{2^{2/\beta-3}\pi}{(2-\beta)^{2/\beta-1}\beta} \frac{\Gamma^2((2-\beta)/2\beta)}{\Gamma^2(1/\beta)}. \tag{2.18}$$

On the other hand, we also have

$$\lim_{x\to 0+} x^{2/\beta}\log\mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\mathrm{d}t \leqslant x\right\} = -\frac{1}{2}\inf_{f\in\mathbb{A}_0}\int_0^1 \dot{f}^2(t)\mathrm{d}t = -B_0, \tag{2.19}$$

where B_0 is defined in (2.17).

By assuming Lemma 2.2 for the moment (see Section 3.3 for its proof), we are ready to complete the proof of the upper bound in Theorem 1.1. Taking $\beta := 2/p$ in Lemma 2.2, we obtain

$$B_0 = \frac{\pi p^p\Gamma^2((p-1)/2)}{8(p-1)^{p-1}\Gamma^2(p/2)}.$$

Plugging this into (2.16) yields

$$\limsup_{T\to\infty} T^{-(p-1)/(p+1)}\log\gamma_{d,p}(T) \leqslant -(p+1)\left(\frac{\pi\kappa^p\Gamma^2((p-1)/2)}{2^{p+3}(p-1)^{p-1}\Gamma^2(p/2)}\right)^{1/(p+1)},$$

which is the desired upper bound in Theorem 1.1.

2.3. Proof of Theorem 1.1: lower bound

Take a function $h\in\mathbb{A}_0^{\uparrow}$ solving the variational problem in (2.12). That is, let $h\in\mathbb{A}_0^{\uparrow}$ be such that

$$\frac{1}{2}\int_0^1 \dot{h}^2(t)\mathrm{d}t = B_0 \quad\text{and}\quad \int_0^1 h^{-\beta}(t)\mathrm{d}t = 1.$$

For any $\delta > 0$, consider a piecewise linear approximation h_δ of h such that

$$\frac{1}{2}\int_0^1 \dot{h}_\delta^2(t)\mathrm{d}t \leqslant (1+\delta)B_0 \quad\text{and}\quad \int_0^1 h_\delta^{-\beta}(t)\mathrm{d}t = 1. \tag{2.20}$$

We bound our probability $\gamma_{d,p}$ by employing a trick similar to that suggested in (2.2):

$$\begin{aligned}\gamma_{d,p}(T) &\geqslant \mathbb{P}\{\|\mathbf{W}(t)\|^p \leqslant \varrho T^{p/(p+1)}h_\delta(t/T) \leqslant 1+\tilde{W}(t),\ \forall t\in[0,\,T]\}\\ &= \mathbb{P}\{\|\mathbf{W}(t)\| \leqslant \varrho^{1/p}T^{1/(p+1)}h_\delta^{1/p}(t/T),\ \forall t\in[0,\,T]\}\\ &\quad\times\mathbb{P}\{\varrho T^{p/(p+1)}h_\delta(t/T) \leqslant 1+\tilde{W}(t),\ \forall t\in[0,\,T]\},\end{aligned}$$

where the optimal value of the additional parameter $\varrho > 0$ is yet to be chosen. For the first probability on the right-hand side, the small-ball estimate is well known:

$$\lim_{T\to\infty} T^{-(p-1)/(p+1)}\log\mathbb{P}\{\|\mathbf{W}(t)\| \leqslant \varrho^{1/p}T^{1/(p+1)}h_\delta^{1/p}(t/T),\ \forall t\in[0,\,T]\} = -\frac{\kappa}{2\varrho^\beta}, \tag{2.21}$$

where $\kappa = j^2_{(d-2)/2}$ as in (2.7) – see Li (1999) and related work by Berthet and Shi (2000) and Lifshits and Linde (2002). For the second probability, we have, by scaling,

$$\begin{aligned}&\mathbb{P}\{\varrho T^{p/(p+1)} h_\delta(t/T) \leqslant 1 + \tilde{W}(t), \forall t \in [0, T]\}\\ &\quad = \mathbb{P}\{\varrho T^{(p-1)/2(p+1)} h_\delta(t) \leqslant T^{-1/2} + \tilde{W}(t), \forall t \in [0, 1]\}.\end{aligned}$$

This is essentially a large-deviation probability but, unfortunately, the presence of a non-zero starting point on the right-hand side prohibits a direct application of classical large-deviation results. Instead, we offer the following palliative.

Lemma 2.3. *Let W be a standard Brownian motion and let a, b, $u > 0$. Then for every piecewise linear function $f(\cdot)$ with $f(0) = 0$,*

$$\liminf_{r\to\infty} r^{-2} \log \mathbb{P}\{rf(t) \leqslant ar^{-b} + W(t), \forall t \in [0, u]\} \geqslant -\frac{1}{2}\int_0^u \dot{f}^2(t)\mathrm{d}t.$$

By assuming Lemma 2.3 for the moment (see Section 3.2 for its proof), we are ready to complete the proof of the lower bound in Theorem 1.1. With $f = h_\delta$, $r = \varrho T^{(p-1)/2(p+1)}$ in Lemma 2.3, we obtain

$$\begin{aligned}\liminf_{T\to\infty} T^{-(p-1)/(p+1)} &\log \mathbb{P}\{\varrho T^{(p-1)/2(p+1)} h_\delta(t) \leq T^{-1/2} + \tilde{W}(t), \forall t \in [0, 1]\}\\ &\geqslant -\frac{\varrho^2}{2}\int_0^1 \dot{h}_\delta^2(t)\mathrm{d}t \geqslant -(1+\delta)\varrho^2 B_0. \end{aligned}\tag{2.22}$$

The estimates (2.21) and (2.22) together yield

$$\liminf_{T\to\infty} T^{-(p-1)/(p+1)} \log \gamma_{d,p}(T) \geqslant -\frac{\kappa}{2\varrho^\beta} - (1+\delta)\varrho^2 B_0.$$

By letting $\delta \to 0$ and maximizing the term on the right-hand side via the choice

$$\varrho := \left(\frac{\kappa}{2pB_0}\right)^{p/2(p+1)},$$

we obtain:

$$\liminf_{T-\infty} T^{-(p-1)/(p+1)} \log \gamma_{d,p}(T) \geqslant -(p+1)\left(\frac{\kappa}{2p}\right)^{p/(p+1)} B_0^{1/(p+1)},$$

which yields on the right-hand side the same constant as in the upper bound (2.16). This implies the lower bound in Theorem 1.1.

3. Technical details

This section contains the technical details which were left incomplete in Section 2 in the proof of Theorem 1.1. Section 3.1 summarizes the large-deviation theory which we need, and provides a justification of (2.12). Sections 3.2 and 3.3 are devoted to the proofs of Lemmas 2.3 and 2.2, respectively.

3.1. Large deviations

For the sake of completeness, we recall here a small part of large-deviation theory which is used in the main proof. For further details, see Dembo and Zeitouni (1998) or Lifshits (1995).

Let P be a centred Gaussian measure in a separable Banach space E. Let $|\cdot|$ denote the reproducing kernel Hilbert norm associated with P. Then for every measureable $A \subset E$, we have

$$-\frac{1}{2}\inf_{h\in A^\circ}|h|^2 \leqslant \liminf_{r\to\infty}\frac{\log P(rA)}{r^2} \leqslant \limsup_{r\to\infty}\frac{\log P(rA)}{r^2} \leqslant -\frac{1}{2}\inf_{h\in\bar{A}}|h|^2. \tag{3.1}$$

The set A is called regular if $\inf_{h\in A^\circ}|h| = \inf_{h\in\bar{A}}|h|$, where A° and $\bar{A}$ denote the interior and the closure of A, respectively. For regular sets, (3.1) yields

$$\lim_{r\to\infty}\frac{\log P(rA)}{r^2} = -\frac{1}{2}\inf_{h\in A}|h|^2. \tag{3.2}$$

Let $\beta > 0$ and $G: E \to \mathbb{R}^1$ be a β-homogeneous functional, i.e.,

$$G(\lambda y) = \lambda^\beta G(y), \qquad \lambda > 0,\ y \in E.$$

Let $A = \{y \in E : G(y) \geqslant 1\}$. If G is upper semicontinuous – or equivalently, if A is closed – we have, from (3.1),

$$\limsup_{r\to\infty}\frac{\log P(y\in E : G(y) \geqslant r)}{r^{2/\beta}} = \limsup_{r\to\infty}\frac{\log P(r^{1/\beta}A)}{r^{2/\beta}} \leqslant -\frac{1}{2}\inf_{h\in A}|h|^2. \tag{3.3}$$

Moreover, if G is continuous, then, as we will see, A is regular, and we obtain from (3.2) that

$$\lim_{r\to\infty}\frac{\log P(y\in E : G(y) \geqslant r)}{r^{2/\beta}} = -\frac{1}{2}\inf_{h\in A}|h|^2. \tag{3.4}$$

It remains for us to verify that A is regular. For any $\delta > 0$ and $h \in A$, we have $G((1+\delta)h) = (1+\delta)^\beta G(h) > 1$. Hence, by continuity of G, we have $(1+\delta)h \in A^\circ$ and

$$(1+\delta)|h| = |(1+\delta)h| \geqslant \inf_{\ell\in A^\circ}|\ell|.$$

Taking the minimum over all $h \in \bar{A}$, and making use of the fact that $A = \bar{A}$, we obtain

$$\inf_{h\in\bar{A}}|h| = \inf_{h\in A}|h| \geqslant \inf_{\ell\in A^\circ}|\ell|.$$

Since the inverse inequality is trivial, A is indeed regular.

We need here only one particular case of all these inequalities — which is often referred to as the Schilder theorem (Schilder 1966) — when P is the Wiener measure on $E = C([0, 1], \mathbb{R}^d)$ and

$$|h|^2 = \begin{cases} \displaystyle\int_0^1 \dot{h}^2(t)\mathrm{d}t, & \text{if } h \text{ is absolutely continuous with } h(0) = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

3.2. Proof of Lemma 2.3

We prove Lemma 2.3 by induction over the number of linear pieces in the boundary function f. Consider first the linear boundary. Let $A > 0$, $H > 1$. Then

$$\begin{aligned} &\mathbb{P}\left\{A + W(t) \geqslant Ht, \forall t \in [0, 1] \text{ and } W(1) \geqslant H + \frac{1}{H}\right\} \\ &\geqslant \int_{H+1/H}^{H+2/H} \frac{\exp(-y^2/2)}{\sqrt{2\pi}} \mathbb{P}\{W(t) \geqslant Ht - A, \forall t \in [0, 1] | W(1) = y\}\mathrm{d}y \\ &\geqslant \frac{\exp(-(H + 2/H)^2/2)}{\sqrt{2\pi}H} \mathbb{P}\{W(t) \geqslant Ht - A, \forall t \in [0, 1] | W(1) = H\} \\ &\geqslant \frac{c_5 \exp(-H^2/2)}{H} \mathbb{P}\left\{\inf_{t\in[0,1]} W^0(t) \geqslant -A\right\} \\ &\geqslant c_5 H^{-1} \exp\left(-\frac{H^2}{2}\right) \min\{A^2; 1\}, \end{aligned}$$

where $W^0(t) := W(t) - tW(1)$, $t \in [0, 1]$, is a standard Brownian bridge. Moreover, for our induction argument, we need a scaled version of the proved inequality. Namely, for every $\Delta > 0$, we have

$$\begin{aligned} &\mathbb{P}\left\{A + W(t) \geqslant Ht, \forall t \in [0, \Delta] \text{ and } W(\Delta) \geqslant \Delta H + \frac{1}{H}\right\} \\ &\qquad \geqslant c_5(H\sqrt{\Delta})^{-1} \exp\left(-\frac{H^2\Delta}{2}\right) \min\left\{\frac{A^2}{\Delta}; 1\right\}, \end{aligned}$$

which is valid under the assumptions $A > 0$ and $H\sqrt{\Delta} \geqslant 1$.

In particular, for $\Delta = u$, $A = ar^{-b}$ and $H = rK$, we immediately obtain

$$\liminf_{r\to\infty} r^{-2} \log \mathbb{P}\{rKt \leqslant ar^{-b} + W(t), \forall t \in [0, u]\} \geqslant -\frac{K^2 u}{2}. \tag{3.5}$$

Lemma 2.3 is thus proved for linear boundaries $f(t) = Kt$.

We now justify the induction. Let

$$f(t) = \begin{cases} Kt, & 0 \leqslant t \leqslant \Delta, \\ K\Delta + g(t - \Delta), & \Delta \leqslant t \leqslant u, \end{cases}$$

where g is a piecewise linear function having one linear piece less than f. Since W has independent increments, we have

$$\mathbb{P}\{A + W(t) \geqslant rf(t), \forall t \in [0, u]\} \geqslant p_1(r) \times p_2(r),$$

where

$$p_1(r) := \mathbb{P}\left\{\frac{a}{r^b} + W(t) \geqslant rf(t), \forall t \in [0, \Delta] \text{ and } W(\Delta) \geqslant rf(\Delta) + \frac{1}{Kr}\right\},$$

$$p_2(r) := \mathbb{P}\left\{\frac{1}{Kr} + W(t) - W(\Delta) \geqslant r(f(t) - f(\Delta)), \forall t \in [\Delta, u]\right\}.$$

Since $f(t) = Kt$ on $[0, \Delta]$, it follows from (3.5) that

$$\liminf_{r\to\infty} r^{-2} \log p_1(r) \geqslant -\frac{K^2\Delta}{2}.$$

On the other hand, $p_2(r) = \{(Kr)^{-1} + W(s) \geqslant rg(s), \forall s \in [0, u - \Delta]\}$, so that by the induction assumption,

$$\liminf_{r\to\infty} r^{-2} \log p_2(r) \geqslant -\frac{1}{2}\int_0^{u-\Delta} \dot{g}^2(s)\mathrm{d}s.$$

Assembling these pieces gives that

$$\begin{aligned}\liminf_{r\to\infty} r^{-2} \log \mathbb{P}\left\{\frac{a}{r^b} + W(t) \geqslant f(t), \forall t \in [0, u]\right\} &\geqslant -\frac{K^2\Delta}{2} - \frac{1}{2}\int_0^{u-\Delta} \dot{g}^2(s)\mathrm{d}s \\ &= -\frac{1}{2}\int_0^u \dot{f}^2(t)\mathrm{d}t,\end{aligned}$$

and we are done.

3.3. Proof of Lemma 2.2

Recall that R denotes a two-dimensional Bessel process starting at 0. Let $\beta \in (0, 2)$ be a fixed constant, and write $\alpha := 2\beta/(2 - \beta)$.

Our starting point is the following theorem of Biane and Yor (1987), also stated as Corollary XI.1.12 in Revuz and Yor (1999).

Fact 3.1. *The random variables*

$$\int_0^1 R^{-\beta}(t)\mathrm{d}t \quad \text{and} \quad (\alpha/\beta)^\beta \left(\int_0^1 R^\alpha(t)\mathrm{d}t\right)^{-\beta/\alpha}$$

have the same distribution.

A major attraction of this identity in our context is that instead of studying the *lower*-tail behaviour of $\int_0^1 R^{-\beta}(t)\mathrm{d}t$, we need only study the *upper*-tail behaviour of $\int_0^1 R^\alpha(t)\mathrm{d}t$. For the latter problem, we again use the Schilder theorem (see (3.4)) which yields

$$\lim_{y\to\infty} y^{-2/\alpha} \log \mathbb{P}\left\{ \int_0^1 R^\alpha(t)\mathrm{d}t > y \right\}$$

$$= -\frac{1}{2}\inf\left\{ \int_0^1 \dot{h}^2(t)\mathrm{d}t : h : [0, 1] \to \mathbb{R}^2,\ h(0) = 0,\ \int_0^1 |h(t)|^\alpha \,\mathrm{d}t \geqslant 1,\ h \text{ absolutely continuous} \right\}.$$

Next, it is easy to observe that the infimum is attained on the set of *real* positive increasing functions and hence equals $M^{-2/\alpha}$, where

$$M := \sup\left\{ \int_0^1 |h(t)|^\alpha \,\mathrm{d}t : \int_0^1 \dot{h}^2(t)\mathrm{d}t \leqslant 1,\ h \in \mathbb{B} \right\},$$

with $\mathbb{B} := \{h : [0, 1] \to \mathbb{R}^1_+ \text{ absolutely continuous, } h(0) = 0\}$. We thus obtain

$$\lim_{y\to\infty} y^{-2/\alpha} \log \mathbb{P}\left\{ \int_0^1 R^\alpha(t)\mathrm{d}t > y \right\} = -\frac{M^{-2/\alpha}}{2}. \tag{3.6}$$

Recall that the value of M is known: Strassen (1964) showed that $M = M_S$, where

$$M_S := \frac{2(2+\alpha)^{\alpha/2-1}}{\alpha^{\alpha/2}\left[\int_0^1 (1-t^\alpha)^{-1/2}\,\mathrm{d}t\right]^\alpha} = \frac{2(2+\alpha)^{\alpha/2-1}\alpha^{\alpha/2}\Gamma^\alpha(1/2+1/\alpha)}{\Gamma^\alpha(1/\alpha)}. \tag{3.7}$$

Strassen proved this result for $\alpha \geqslant 1$ but it is easy to verify that it is also valid for $\alpha \in (0, 1)$. Indeed, in the latter case the functional

$$G_c(h) := \int_0^1 h^\alpha(t)\mathrm{d}t - c\int_0^1 \dot{h}^2(t)\mathrm{d}t \qquad (c > 0)$$

is concave on $\mathbb{B}$. Strassen's calculations show that, for some $h_* \in \mathbb{B}$, one has

$$\int_0^1 h_*^\alpha(t)\mathrm{d}t = M_S, \qquad \int_0^1 \dot{h}^2(t)\mathrm{d}t = 1,$$

and for some $c > 0$ the derivative of G_c vanishes at h_*. Hence, G_c attains its maximum at h_*, or equivalently,

$$M_S = \sup\left\{ \int_0^1 h^\alpha(t)\mathrm{d}t : \int_0^1 \dot{h}^2(t)\mathrm{d}t = 1,\ h \in \mathbb{B} \right\} = M.$$

Thus $M = M_S$ for every positive α.

Now, sequentially using Fact 3.1, (3.6) and (3.7), we obtain

$$\lim_{x\to 0+} x^{2/\beta}\log \mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\mathrm{d}t < x\right\} = \lim_{x\to 0+} x^{2/\beta}\log \mathbb{P}\left\{\int_0^1 R^{\alpha}(t)\mathrm{d}t > (\alpha/\beta)^{\alpha} x^{-\alpha/\beta}\right\}$$

$$= -\frac{M^{-2/\alpha}\alpha^2}{2\beta^2} = -\frac{2}{M^{2/\alpha}(2-\beta)^2}$$

$$= -\frac{2^{2/\beta-3}\pi}{(2-\beta)^{2/\beta-1}\beta}\,\frac{\Gamma^2((2-\beta)/2\beta)}{\Gamma^2(1/\beta)},$$

as claimed in Lemma 2.2.

Now we prove the second part (identity (2.19)) of the Lemma. The upper bound in (2.19) follows from the Schilder large-deviations theorem (see (3.3)), that is,

$$\limsup_{x\to 0+} x^{2/\beta}\log \mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\mathrm{d}t \leqslant x\right\} \leqslant -\frac{1}{2}\inf_{f\in\mathbb{A}_0}\int_0^1 \dot{f}^2(t)\mathrm{d}t = -B_0.$$

The lower bound in (2.19) follows not from the general theory but from Lemma 2.3. Indeed, take small numbers $a>0$, $\delta>0$ and denote by $\tau_a := \inf\{t : R(t)=a\}$ the first hitting time of the Bessel process. Then by the strong Markov property,

$$\mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\mathrm{d}t \leqslant (1+\delta)x\right\} \geqslant \mathbb{P}\left\{\int_0^{\tau_a} R^{-\beta}(t)\mathrm{d}t \leqslant \delta x;\ \int_{\tau_a}^{\tau_a+1} R^{-\beta}(t)\mathrm{d}t \leqslant x\right\}$$

$$= \mathbb{P}\left\{\int_0^{\tau_a} R^{-\beta}(t)\mathrm{d}t \leqslant \delta x\right\}\mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\mathrm{d}t \leqslant x | R(0)=a\right\}$$

$$\geqslant \mathbb{P}\left\{\int_0^{\tau_a} R^{-\beta}(t)\mathrm{d}t \leqslant \delta x\right\}\mathbb{P}\left\{\int_0^1 |a+W(t)|^{-\beta}\,\mathrm{d}t \leqslant x\right\}. \tag{3.8}$$

By the scaling property of R, we have, for $x<\delta^{-1}a^{2-\beta}$,

$$\mathbb{P}\left\{\int_0^{\tau_a} R^{-\beta}(t)\mathrm{d}t \leqslant \delta x\right\} = \mathbb{P}\left\{\int_0^{\tau_1} R^{-\beta}(t)\mathrm{d}t \leqslant a^{\beta-2}\delta x\right\}$$

$$\geqslant \mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\mathrm{d}t \leqslant a^{\beta-2}\delta x,\ \tau_1\leqslant 1\right\}$$

$$= \mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\mathrm{d}t \leqslant a^{\beta-2}\delta x\right\},$$

the last equality following from the fact that $\{\int_0^1 R^{-\beta}(t)\mathrm{d}t \leqslant a^{\beta-2}\delta x\} \subset \{\tau_1 \leqslant 1\}$ for all $x<\delta^{-1}a^{2-\beta}$. Hence, by (2.18),

$$\liminf_{x\to 0+} x^{2/\beta}\log\mathbb{P}\left\{\int_0^{\tau_a} R^{-\beta}(t)\mathrm{d}t \leqslant \delta x\right\} \geqslant -\frac{2^{2/\beta-3}\pi}{(2-\beta)^{2/\beta-1}\beta}\frac{\Gamma^2((2-\beta)/2\beta)}{\Gamma^2(1/\beta)}\delta^{-2/\beta}a^{2(2-\beta)/\beta}. \tag{3.9}$$

We mention that it is possible, by means of stochastic calculus techniques, to show that $\int_0^{\tau_a} R^{-\beta}(t)\mathrm{d}t$ is distributed as $\{2/(2-\beta)\}^2 a^{2-\beta}/\sup_{0\leqslant t\leqslant 1}R^2(t)$, so that the 'lim inf' expression on the left-hand side of (3.9) is a true limit and its value is actually 0.

To estimate $\mathbb{P}\{\int_0^1 |a+W(t)|^{-\beta}\,\mathrm{d}t \leqslant x\}$, we take a function h_δ from (2.20) and notice that for $\omega \in \{a+W(t) \geqslant x^{-1/\beta}h_\delta(t), \forall t \in [0, 1]\}$, we have

$$\int_0^1 |a+W(t)|^{-\beta}\mathrm{d}t \leqslant x\int_0^1 h_\delta^{-\beta}(t)\mathrm{d}t = x.$$

Hence, by Lemma 2.3,

$$\begin{aligned}
&\liminf_{x\to 0+} x^{2/\beta}\log\mathbb{P}\left\{\int_0^1 |a+W(t)|^{-\beta}\mathrm{d}t \leqslant x\right\}\\
&\quad\geqslant \liminf_{x\to 0+} x^{2/\beta}\log\mathbb{P}\{a+W(t) \geqslant x^{-1/\beta}h_\delta(t), \forall t\in[0, 1]\}\\
&\quad\geqslant -\frac{1}{2}\int_0^1 \dot h_\delta^2(t)\mathrm{d}t\\
&\quad\geqslant -(1+\delta)B_0.
\end{aligned} \tag{3.10}$$

Combining (3.8)–(3.10) and choosing δ and a sufficiently small, we obtain

$$\liminf_{x\to 0+} x^{2/\beta}\log\mathbb{P}\left\{\int_0^1 R^{-\beta}(t)\mathrm{d}t \leqslant x\right\} \geqslant -B_0.$$

This yields the second part of Lemma 2.2.

4. Slow exit from more general domains

4.1. Domains with regular varying boundary

Let $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing continuous function starting at 0 and p-regularly varying at infinity (with $p > 1$): for any $a > 0$, $\Lambda(ar)/\Lambda(r) \to a^p$, $r\to\infty$. Let $\Lambda^{\leftarrow}$ denote the inverse function of Λ. Then $\Lambda^{\leftarrow}$ is $(1/p)$-regularly varying, so that

$$\nu(r) := r^{-1/p}\Lambda^{\leftarrow}(r)$$

is a slowly varying function. Let

$$D = D_{d,\Lambda} := \{(\mathbf{x}, y) := (x_1, \ldots, x_d, y) \in \mathbb{R}^{d+1} : y > \Lambda(\|\mathbf{x}\|)\}, \tag{4.1}$$

where $\|\mathbf{x}\|$ is the Euclidean norm of $\mathbf{x} := (x_1, \dots, x_d) \in \mathbb{R}^d$. Sharp estimates for exit times from such domains may be found in Li (2001).

We provide the following generalization of Theorem 1.1.

Theorem 4.1. *Let $d \geqslant 1$, Λ and ν be as above. Let $D = D_{d,\Lambda}$ be as in* (4.1). *Assume that*

$$\lim_{T\to\infty} \frac{\nu(\nu^{-p/(p+1)}(T)T)}{\nu(T)} = 1. \tag{4.2}$$

Then

$$\lim_{T\to\infty} \frac{\log \mathbb{P}\{\tau_D > T\}}{f_p(T)} = -(p+1)\left(\frac{\pi j_{(d-2)/2}^{2p}}{2^{p+3}(p-1)^{p-1}} \frac{\Gamma^2((p-1)/2)}{\Gamma^2(p/2)}\right)^{1/(p+1)},$$

where

$$f_p(T) := \frac{T^{(p-1)/(p+1)}}{\nu^{2p/(p+1)}(T^{p/(p+1)})}. \tag{4.3}$$

Remark. The extra assumption (4.2) is verified, for example, by functions $\nu(T) = c(\log T)^\alpha$ (with $\alpha \in \mathbb{R}$) and by functions $\nu(T) = c\exp\{b(\log T)^\alpha\}$ (with $\alpha \in [0, \frac{1}{2}]$).

Proof. We only sketch the proof, starting with that of the lower bound. The splitting argument now reads

$$\begin{aligned}\mathbb{P}\{\tau_D > T\} &\geqslant \mathbb{P}\{\Lambda(\|\mathbf{W}(t)\|) \leqslant \varrho T^{p/(p+1)} h_\delta(t/T) \leqslant 1 + \tilde{W}(t), \forall t \in [0, T]\}\\ &= \mathbb{P}\{\Lambda(\|\mathbf{W}(t)\|) \leqslant \varrho T^{p/(p+1)} h_\delta(t/T), \forall t \in [0, T]\}\\ &\quad \times \mathbb{P}\{\varrho T^{p/(p+1)} h_\delta(t/T) \leqslant 1 + \tilde{W}(t), \forall t \in [0, T]\}\end{aligned}$$

and for the first factor we have

$$\begin{aligned}&\log \mathbb{P}\{\Lambda(\|\mathbf{W}(t)\|) \leqslant \varrho T^{p/(p+1)} h_\delta(t/T), \forall t \in [0, T]\}\\ &\quad = \log \mathbb{P}\{\|\mathbf{W}(t)\| \leqslant \Lambda^{\leftarrow}\Big(\varrho T^{p/(p+1)} h_\delta(t/T)\Big), \forall t \in [0, T]\}\\ &\quad = \log \mathbb{P}\{\|\mathbf{W}(t)\| \leqslant T^{-1/2}\Lambda^{\leftarrow}\Big(\varrho T^{p/(p+1)} h_\delta(t/T)\Big), \forall t \in [0, T]\}\\ &\quad \sim -\frac{\kappa T}{2}\int_0^1 (\Lambda^{\leftarrow})^{-2}\Big(\varrho T^{p/(p+1)} h_\delta(t)\Big)\mathrm{d}t\\ &\quad = -\frac{\kappa}{2\varrho^{2/p}} T^{(p-1)/(p+1)} \int_0^1 h_\delta^{-2/p}(t)\nu^{-2}\Big(\varrho T^{p/(p+1)} h_\delta(t)\Big)\mathrm{d}t\\ &\quad \sim -\frac{\kappa}{2\varrho^{2/p}} T^{(p-1)/(p+1)}\nu^{-2}\Big(\varrho T^{p/(p+1)}\Big).\end{aligned}$$

The rest of the proof strictly follows that of the lower bound in Theorem 1.1 after (2.21) with the replacement of κ by $\kappa\nu^{-2}(T^{p/(p+1)})$. Note that the optimal choice of ϱ is

$$\varrho = \varrho(T) := \left(\frac{\kappa}{2pB_0}\right)^{p/2(p+1)} \nu^{-p/(p+1)}\left(T^{p/(p+1)}\right),$$

and (4.2) provides the equivalence $\nu(\varrho T^{p/(p+1)}) \sim \nu(T^{p/(p+1)})$ which considerably simplifies the calculation.

Now we turn to (a sketch of) the proof of the upper bound. We retain the notation of Section 2. In particular, we use a partition $\{t_0, \dots, t_N\}$, the supremum $S(t)$, and the constant κ. Take a small constant $m > 0$ and a large constant $M > 0$. Introduce three events

$$Q_- = Q_-(T) := \left\{ S(t_1) \leqslant \frac{mT^{p/(p+1)}}{\nu^{p/(p+1)}(T^{p/(p+1)})} \right\},$$

$$Q = Q(T) := \left\{ \frac{mT^{p/(p+1)}}{\nu^{p/(p+1)}(T^{p/(p+1)})} < S(t_1) \leqslant S(T) < \frac{MT^{p/(p+1)}}{\nu^{p/(p+1)}(T^{p/(p+1)})} \right\},$$

$$Q_+ = Q_+(T) := \left\{ S(T) \geqslant \frac{MT^{p/(p+1)}}{\nu^{p/(p+1)}(T^{p/(p+1)})} \right\}.$$

Bounding the parts relating to Q_- and Q_+ is no problem. Indeed, by the classical large-deviation estimate for $S(T)$, we have

$$\limsup_{T\to\infty} \frac{\log \mathbb{P}\{Q_+\}}{f_p(T)} \leqslant -\frac{M^2}{2},$$

and we can choose M as large as possible. On the other hand, taking $t_1 = \tau_1 T$ for any fixed $\tau_1 \in (0, 1)$, we have

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{f_p(T)} \log \mathbb{P}\{\tau_D \geqslant T;\, Q-\} \\
&\leqslant \limsup_{T\to\infty} \frac{1}{f_p(T)} \log \mathbb{P}\left\{ \|\mathbf{W}(t)\| \leqslant \Lambda^{\leftarrow}\left(1 + \frac{mT^{p/(p+1)}}{\nu^{p/(p+1)}(T^{p/(p+1)})}\right),\ \forall t \in [0,\ t_1] \right\} \\
&= -\frac{\kappa\tau_1}{2m^{2/p}},
\end{aligned}$$

the last inequality being a consequence of (2.6) and (4.2). So this part does not present any trouble either, as long as we choose $m > 0$ sufficiently small.

Finally, the estimate relating to the main domain Q follows the scheme of Section 2. Let

$$\gamma_Q(T) := \mathbb{P}\{\tau_D \geqslant T;\, Q\}.$$

Similarly to (2.5), we obtain

$$\gamma_Q(T) \leqslant \mathbb{P}\left\{ \sup_{t\in[t_{i-1},t_i]} \|\mathbf{W}(t)\| < \Lambda^{\leftarrow}\left(\sup_{t\in[0,t_i} \tilde{W}(t) + 1 \right), \forall i \leqslant N;\ Q \right\}.$$

Proceeding as in Section 2, we obtain the counterpart of (2.8), namely,

$$\gamma_Q(T) \leqslant c_4^N \mathbb{E}\left\{ \mathbf{1}_Q \exp\left(-\frac{(1-\varepsilon)\kappa}{2} \sum_{i=1}^{N} \frac{t_i - t_{i-1}}{[1+S(t_i)]^{2/p}\nu^2(1+S(t_i))} \right) \right\}.$$

Note that uniformly for all $i \leqslant N$, we have, on the event Q,

$$\nu(1+S(t_i)) \sim \nu\left(\frac{T^{p/(p+1)}}{\nu^{p/(p+1)}(T^{p/(p+1)})} \right) \sim \nu(T^{p/(p+1)});$$

indeed, the first equivalence follows from the definition of Q and the Karamata representation for slowly varying functions, whereas the second is a consequence of condition (4.2). Accordingly, for all large T,

$$\gamma_Q(T) \leqslant c_4^N \mathbb{E}\left\{ \exp\left(-\frac{(1-\varepsilon)^2\kappa}{2\nu^2(T^{p/(p+1)})} \sum_{i=1}^{N} \frac{t_i - t_{i-1}}{[1+S(t_i)]^{2/p}} \right) \right\}.$$

Now we simply follow Section 2 but with κ replaced by $\kappa\nu^{-2}(T^{p/(p+1)})$. In place of (2.9), we have

$$\gamma_Q(T) \leqslant c_4^N \mathbb{E}\left\{ \exp\left(-\frac{(1-\varepsilon)^3\kappa T^{(p-1)/p}}{2\nu^2(T^{p/(p+1)})} \int_{\tau_1}^{1} \frac{\mathrm{d}\tau}{[T^{-1/2} + S(\tau)]^{2/p}} \right) \right\},$$

and taking $b = (1-\varepsilon)^3\kappa T^{(p-1)/p}/2\nu^2(T^{p/(p+1)})$ in (2.10), we arrive at the following counterpart of (2.11):

$$\limsup_{T\to\infty} \frac{\log \gamma_Q(T)}{f_p(T)} \leqslant \inf_{\delta>0,\theta>0} \limsup_{T\to\infty} \frac{1}{f_p(T)} \log \mathbb{E}\left\{ \exp\left(-\frac{(1-\delta)\kappa T^{(p-1)/p}}{2\nu^2(T^{p/(p+1)})} \int_{\theta}^{1} S(\tau)^{-2/p}\, \mathrm{d}\tau \right) \right\}.$$

We already know from (2.16), via the key estimate (2.15), that the expression on the right-hand side equals $-(p+1)(\kappa/2p)^{p/(p+1)} B_0^{1/(p+1)}$, which yields the desired upper bound. □

4.2. Non-Euclidean norms

Let K be a non-empty connected open set in $\mathbb{R}^d$ that contains $\mathbf{0}$. According to Theorem 7.2 in Port and Stone (1979),

$$P\{\mathbf{W}(t) \in K, \forall t \in [0, T]\} \sim c\exp(-\lambda T), \qquad T \to \infty,$$

with some positive constant c and λ being the principal eigenvalue of the Laplacian $(-\frac{1}{2}\Delta)$ on K (with zero boundary condition).

We can transform this statement into

$$\mathbb{P}\{\mathbf{W}(s) \in xK, \forall s \in [0, 1]\} \sim c \exp\left(-\frac{\lambda}{x^2}\right), \qquad x \to 0+,$$

by ordinary scaling arguments. Using the latter relation instead of our formula (2.6), one can easily obtain the results of this paper for the exit times from a parabolic shape

$$D_{d,p,H} := \{(\mathbf{x}, y) \in \mathbb{R}^{d+1} : y > H(x)^p\},$$

for any norm $H(\cdot)$ in $\mathbb{R}_d$ equivalent to the Euclidean. Obviously, 22 will everywhere replace $j^2_{(d-2)/2}$.

4.3. An open problem: quasi-conic domains

The natural question is what happens if $p = 1$ and ν is an appropriate power of a logarithmic function (using notation from Section 4.1). The answer should be something intermediate between the subexponential behaviour of the exit probability treated in the present paper and the polynomial behaviour which appears for purely conic domains.

The only known results are those of Li (2001), but his upper and lower estimates are still of different orders of magnitude. It does not seem that our methods are appropriate for a complete analysis of that case.

Acknowledgements

We are grateful to the referees for their useful and constructive comments. Part of this work was done while the first author was visiting the Université Paris VI in April 2001. We wish to thank the Laboratoire de Probabilités et Modèles Aléatoires (UMR 7599) for arranging this visit and for its hospitality. The work of the first author was supported by grants INTAS 99-01317 and RFBR/DFG 99-01-04027.

References

Bañuelos, R. and Smits, R.G. (1997) Brownian motion in cones. *Probab. Theory Related Fields*, **108**, 299–319.

Bañuelos, R., DeBlassie, R.D. and Smits, R.G. (2001) The first exit time of planar Brownian motion from the interior of a parabola. *Ann. Probab.*, **29**, 882–901.

Berthet, P. and Shi, Z. (2000) Small ball estimates for Brownian motion under a weighted sup-norm. *Studia Sci. Math. Hungar.*, **36**, 275–289.

Biane, P. and Yor, M. (1987) Valeurs principales associées aux temps locaux browniens. *Bull. Sci. Math.*, **111**, 23–101.

Ciesielski, Z. and Taylor, S.J. (1962) First passage time and sojourn time for Brownian motion in space and the exact Hausdorff measure of the sample path. *Trans. Amer. Math. Soc.*, **103**, 434–450.

Dembo, A. and Zeitouni, O. (1998) *Large Deviation Techniques and Applications*, 2nd edn. New York: Springer-Verlag.

Li, W.V. (1999) Small deviations for Gaussian Markov processes under the sup-norm. *J. Theoret. Probab.*, **12**, 971–984.

Li, W.V. (2001) the first exit time of Brownian motion from unbounded domain. *Ann Probab.* To appear.

Lifshits, M.A. (1995) *Gaussian Random Functions*. Dordrecht: Kluwer.

Lifshits, M.A. and Linde, W. (2002) Approximation and entropy numbers of Volterra operators with application to Brownian motion. *Mem. Amer. Math. Soc.*, **745**.

Port, S.C. and Stone, C.J. (1979) *Brownian Motion and Classical Potential Theory*. New York: Academic Press.

Revuz, D. and Yor, M. (1999) *Continuous Martingales and Brownian Motion*, 3rd edn. Berlin: Springer-Verlag.

Schilder, M. (1966) Some asymptotic formulas for Wiener integrals. *Trans. Amer. Math. Soc.*, **125**, 63–85.

Strassen, V. (1964) An invariance principle for the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **3**, 211–226.

Received October 2001 and revised July 2002