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Distributions and analytic functions, by Richard D. Carmichael and Dragiša Mitrović. Pitman Research Notes in Mathematics Series, vol. 206, Longman Scientific & Technical, Harlow, England. ISBN 0-582-01856-0; John Wiley & Sons, New York, (USA only), 1989, xii+347 pp., $42.00. ISBN 0-470-21398-5

The ties between holomorphic functions and the distributions of Laurent Schwartz originated long before distributions were even discovered. According to Daniele Struppa [15] it was Francesco Severi who in 1924 suggested to Luigi Fantappiè to study the functional which associates with a function its derivative at some point, i.e., what we call now the distributional derivative of the Dirac measure.

Inspired by this suggestion, Fantappiè created the theory of analytic functionals [3]. He considers holomorphic functions f, each having as its domain of definition M a region of the Riemann sphere $\mathbf{P}_1(\mathbf{C})$. It is assumed that $M \neq \mathbf{P}_1(\mathbf{C})$, and if the point at infinity ω belongs to M, then $f(\omega) = 0$. An analytic line is a function $y(t, z)$ of two variables, holomorphic in each variable. An analytic functional is a map F which associates with each f a scalar $F[f]$ such that if F acts on the analytic line $y(t, z)$ considered as a function of t, the resulting function $F[y(\cdot, z)]$ shall be holomorphic in z. A particular analytic line is given by $\frac{1}{2\pi i}\frac{1}{z-t}$, and $F_t[\frac{1}{2\pi i}\frac{1}{z-t}]$ is called the Fantappiè indicatrix of F.

The Portuguese mathematician José Sebastião e Silva, who studied in Rome during several years, made the first attempt to apply the methods of modern linear functional analysis to Fantappiè's theory [10]. He considers a fixed closed subset C of $\mathbf{P}_1(\mathbf{C})$ and the holomorphic functions f defined in some neighborhood U of C. In the set of all pairs (f, U) he introduces the following equivalence relation; $(f_1, U_1) \sim (f_2, U_2)$ if there exists an open neighborhood U of C contained in $U_1 \cap U_2$ such that f_1 and f_2 coincide in U. He calls an equivalence class "an analytic function attached to the set C" [12, §16]; it is now called a germ of an analytic function on C. In the vector space $H(C)$ of these germs he defines the convergence of a sequence (f_n) to f as follows: there exists an open neighborhood U of C on which all the functions f_n and f are defined and bounded, and f_n tends to f uniformly on U [12, §17]. An analytic functional can then be defined as a continuous linear form on $H(C)$.

In the meanwhile distributions were discovered, and related to them the theory of locally convex spaces, in particular of $(\mathscr{LF})$-spaces, was developed. G. Köthe [8], A. Grothendieck [4], and the Brazilian mathematician C. L. da Silva Dias [13] simultaneously saw the possibility of applying the new concepts to the ideas of Sebastião e Silva. Let C be, as above, a closed subset of $\mathbf{P}_1(\mathbf{C})$, and A its complement in $\mathbf{P}_1(\mathbf{C})$. Denote by $H(A)$ the vector space of all holomorphic functions in A. Consider a sequence (K_n) of compact subsets of A such that each K_n is contained in the interior of K_{n+1}, and $\bigcup K_n = A$. The sequence of norms $\|f\|_n = \sup_{z \in K_n} |f(z)|$ defines a metrizable, complete, locally convex topology on $H(A)$, i.e., $H(A)$ is a Fréchet space. It is even a Montel space (i.e. every bounded subset is relatively compact), and therefore reflexive. On $H(C)$ one defines a topology as follows: Let (U_n) be a decreasing sequence of open neighborhoods of C such that $\bigcap U_n = C$. Denote by $H(\overline{U}_n)$ the vector space of all functions holomorphic in U_n and continuous on its closure $\overline{U}_n$. Equipped with the max-norm, $H(\overline{U}_n)$ is a Banach space. There is a natural map $j_n \colon H(\overline{U}_n) \to H(C)$ which associates with each element of $H(\overline{U}_n)$ the germ on C it defines, and one puts on $H(C)$ the finest locally convex topology for which the j_n are continuous. The convergence on $H(C)$ introduced by Sebastião e Silva is the convergence in the sense of this topology.

Let F be a continuous linear form on $H(C)$. For each $z \in A$ the function

$$t \mapsto \frac{1}{2\pi i} \frac{1}{z - t}$$

defines an element of $H(C)$, and the Fantappiè indicatrix

$$\widetilde{F}(z) = F\left[\frac{1}{2\pi i}\frac{1}{z-t}\right]$$

is an element of $H(A)$. Given any $f \in H(C)$ there corresponds to it a function, also denoted by f, defined in some neighborhood U of C, and

$$F[f] = \int_\Gamma \widetilde{F}(z) f(z)\,dz,$$

where Γ is a finite system of closed curves in $U \cap A$ surrounding C. The map $F \mapsto \widetilde{F}$ is an isomorphism of $H(C)'$ equipped with the strong topology onto $H(A)$.

Let now C be a simple closed analytic curve not going through ω. Then $H(C)$ is a dense subspace of the space $\mathscr{E}(C)$ of infinitely differentiable functions on C, hence the transpose of the injection $H(C) \to \mathscr{E}(C)$ is an injective map from the space $\mathscr{E}'(C)$ of distributions on C into $H(C)' = H(A)$. Köthe points out that the elements of $H(C)'$ are more general than distributions [9, p. 15]; it is therefore surprising to read in [1]: "C'est Sato qui le premier a étudié des fonctions généralisées définies a priori comme "valeurs au bord" de fonctions holomorphes." Köthe calls the elements of $H(C)'$ boundary distributions (Randverteilungen) for the following reason: If $F \in H(C)'$, then $\widetilde{F}$ is a pair of functions $(\widetilde{F}_1, \widetilde{F}_2)$, where $\widetilde{F}_1$ is defined in the domain A_1 inside C, and $\widetilde{F}_2$ in the domain A_2 outside C. Let for a moment C be the unit circle $|z| = 1$, so $A_1 = \{z; |z| < 1\}$ and $A_2 = \{z; |z| > 1\}$. Choosing $0 < r < 1$, the functions $t \mapsto \widetilde{F}_1(rt)$ and $t \mapsto \widetilde{F}_2(t/r)$ are elements of $H(C)$, and can be considered as elements of $H(C)'$ through the map which with $g \in H(C)$ associates the linear form $f \mapsto \int_C f(t)g(t)\,dt$. Then $\widetilde{F}_1(rt) + \widetilde{F}_2(t/r)$ converges in the sense of the topology of $H(C)'$ to F as $r \to 1$. The general case, when C is not the unit circle, is taken care of by a conformal mapping, which takes the circles $|z| = r$ and $|z| = 1/r$ into certain curves Γ_1 and Γ_2, respectively.

Not only the elements of $H(C)$ but any continuous function f on C can be considered as an element of $H(C)'$, and therefore we can consider its Fantappiè indicatrix $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$, where

$$\tilde{f}_1(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt, \qquad z \in A_1,$$
$$\tilde{f}_2(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{z-t}\,dt, \qquad z \in A_2.$$

The Slovenian mathematician J. Plemelj proved in 1908 that if f is Hölder-continuous and z tends to a point $t \in C$, then

$$\lim \tilde{f}_1(z) - \lim \tilde{f}_2(z) = f(t)$$

and

$$\lim \tilde{f}_1(t) + \lim \tilde{f}_2(t) = \frac{1}{\pi i} vp \int_C \frac{f(\tau)}{\tau - t} d\tau,$$

where vp indicates the Cauchy principal value of the integral.

The first half of the book under review is concerned with the analogues of the above considerations when C is replaced by the real line $\mathbf{R}$, and also with the distributional analogue of the Plemelj relations. The pioneering work was done by Köthe's student H. G. Tillmann, and continued by his students, the Mainz school: G. Bengel, P. P. Konder, R. Meise, E. Schmidt, D. Vogt, and W. Wild. An account of their work, with references, can be found in [11]. The book surveys in a fairly self-contained way most of these results, and also the extensive work done by the authors themselves.

The representation problem for a distribution $T \in \mathscr{D}'(\mathbf{R})$ consists in finding a function F "locally" holomorphic in $\mathbf{C}\backslash\mathbf{R}$ such that

$$\langle T, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{-\infty} [F(x + i\varepsilon) - F(x - i\varepsilon)]\varphi(x)\, dx \tag{1}$$

for appropriate test functions φ. The adverb "locally" is meant to indicate that F is in fact a pair (F_1, F_2) of holomorphic functions, one defined in the upper and one in the lower half-plane. The solution is simple if $T \in \mathscr{E}'(\mathbf{R})$, i.e., T has compact support, since then one takes for F the Fantappiè indicatrix $\widetilde{T}$ of T defined by

$$\widetilde{T}(z) = \frac{1}{2\pi i} \left\langle T_t, \frac{1}{t - z} \right\rangle. \tag{2}$$

The authors call $\widetilde{T}$ the Cauchy integral of T; the name Stieltjes transform would also be justified, and the expression

$$\frac{1}{2\pi i} \frac{1}{t - z}$$

is the Cauchy kernel. The linear map which associates with F the distribution T given by (1) is a surjection from the space $H_0(\mathbf{C}\backslash\mathbf{R})$ of locally holomorphic functions satisfying $|F(z)| \leq M|y|^{-\nu}$ $(z = x + iy, y \neq 0, M > 0, \nu > 0)$ onto $\mathscr{E}'(\mathbf{R})$. Its kernel is the subspace $H_0(\mathbf{C})$ of those F for which F_1 and F_2 have the same boundary values on $\mathbf{R}$ and can therefore, by a theorem of Painlevé (a distributional version of which is proved in the book), be extended to an entire function. The resulting isomorphism

$$\mathscr{E}'(\mathbf{R}) \simeq H_0(\mathbf{C}\backslash\mathbf{R})/H_0(\mathbf{C}) \tag{3}$$

has far-reaching consequences.

Tillmann proved the existence of functions F representing tempered distributions, some other classes of distributions, and finally the most general distributions. The authors also consider the representation of distributions belonging to a certain space $\mathscr{O}'_\alpha$ introduced by Bremermann.

A completely different connection between Schwartz distributions and functions of complex variables was discovered by L. Ehrenpreis. The Fourier transformation $\mathscr{F}$ maps the Schwartz space $\mathscr{S}(\mathbf{R}^n)$ of rapidly decreasing infinitely differentiable functions isomorphically onto itself. It maps the subspace $\mathscr{D}(\mathbf{R}^n)$ of functions with compact support onto a subspace $Z(\mathbf{R}^n)$ which by the Paley-Wiener-Schwartz theorem consists of restrictions to $\mathbf{R}^n$ of entire functions of exponential type on $\mathbf{C}^n$ and which therefore satisfies $\mathscr{D}(\mathbf{R}^n) \cap Z(\mathbf{R}^n) = \{0\}$. On $Z(\mathbf{R}^n)$ Ehrenpreis defines a locally convex topology such that $\mathscr{F}$ is a topological isomorphism from $\mathscr{D}(\mathbf{R}^n)$ onto $Z(\mathbf{R}^n)$. This makes it possible to introduce the dual space $Z'(\mathbf{R}^n)$, and by transposition the Fourier transform of any distribution which will be an element of $Z'(\mathbf{R}^n)$. This concept helped Ehrenpreis to prove the existence of a fundamental solution of any linear partial differential operator with constant coefficients [2]. The elements of $Z'(\mathbf{R}^n)$ are called sometimes ultradistributions, sometimes analytic functionals, and they were introduced independently also by Gel′fand and Shilov.

Another development was the definition by Laurent Schwartz of the Laplace transformation in a paper published in 1952 in honor of Marcel Riesz, which he included as Chapter VIII of the 1966 edition of his book on distribution theory. Let us denote by $z = (z_1, \dots, z_n) = x + iy$ a point of $\mathbf{C}^n$ where $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, $y = (y_1, \dots, y_n) \in \mathbf{R}^n$ and $z_j = x_j + iy_j$ $(1 \le j \le n)$. If $t = (t_1, \dots, t_n) \in \mathbf{R}^n$, then $y \cdot t = y_1 t_1 + \cdots + y_n t_n$ and $z \cdot t = z_1 t_1 + \cdots + z_n t_n$. The Laplace transform $\mathscr{L}T$ of $T \in \mathscr{D}'(\mathbf{R}^n)$ is defined for all $z \in \mathbf{C}^n$ such that $x \in \mathbf{R}^n$ and y belongs to the set

$\Gamma \subset \mathbf{R}^n$ for which $e^{-2\pi y\cdot t}T_t \in \mathscr{S}'(\mathbf{R}^n)$ by the formula

$$(\mathscr{L}T)(z) = \int_{\mathbf{R}^n} e^{2\pi i z\cdot t} T(t)\, dt = \mathscr{F}_t(e^{-2\pi y\cdot t}T_t)_x$$

(this is the book's notation, Schwartz has x and y interchanged and omits the factor 2π in the exponent). Thus $\mathscr{L}T$ associates with each $y \in \Gamma$ the distribution $\mathscr{F}_t(e^{-2\pi y\cdot t}T_t) \in \mathscr{S}_x'(\mathbf{R}^n)$ but under some mild condition can be thought of as a holomorphic function of the variable $z \in \mathbf{R}^n + i\Gamma$.

The ultradistributions of Ehrenpreis and the Laplace transformation have brought us into the realm of holomorphic functions of several complex variables to which the second half of the book is devoted. Let $\sigma = (\sigma_1, \dots, \sigma_n)$ be one of the 2^n vectors such that all the coordinates σ_j are equal to $+1$ or to -1. Denote by Q_σ the set of those $y \in \mathbf{R}^n$ for which $\sigma_j y_j > 0$ $(1 \le j \le n)$, the sets Q_j are called quadrants in the book, though a coinage like "2^n-ants" would describe the situation more properly. The subset $(\mathbf{C}\backslash\mathbf{R})^n$ of $\mathbf{C}^n$ has the 2^n components $T_\sigma = \mathbf{R}^n + iQ_\sigma$, and the analytic representation of a distribution T consists in 2^n functions F_σ, each holomorphic in the corresponding domain T_σ, such that

$$\langle T, \varphi\rangle = \lim_{\varepsilon\to 0}\int_{\mathbf{R}^n}\sum_\sigma F_\sigma(x + i\varepsilon\sigma)\varphi(x)\, dx$$

for appropriate test functions φ. The n-dimensional Cauchy kernel is given by

$$\frac{1}{(2\pi i)^n}\prod_{j=1}^{n}\frac{1}{t_j - z_j}.$$

A. Korányi [7] on one hand and E. Stein, G. Weiss, and M. Weiss [14, Chapter III] on the other studied classes of holomorphic functions in tube domains. The authors of the book under review consider convex open cones C in $\mathbf{R}^n$ and the corresponding tube $T^C = \mathbf{R}^n + iC$ in $\mathbf{C}^n$. The Cauchy kernel corresponding to T^C is defined by

$$K(z - t) = \int_{C^*} \exp 2\pi i(z - t)\cdot\eta\, d\eta,$$

where $C^* = \{\eta \in \mathbf{R}^n;\ \eta\cdot y \ge 0 \text{ for all } y \in C\}$ is the cone dual to C, and $K(z - t)$ gives rise to a Poisson kernel associated with T^C. The authors study the representation problem in the more general setting, where the quadrants Q_σ are replaced by cones C_j

$(1 \le j \le r)$ such that $\bigcup_{j=1}^{r} C_j^*$ has a complement of measure zero in $\mathbf{R}^n$ and two different cones C_j^* intersect in a negligible set. The cases of both scalar-valued and vector-valued distributions with compact support, of distributions in Bremermann's space $\mathscr{O}'_\alpha$ and in Schwartz's space $\mathscr{D}'_{L^p}$ are considered.

Conversely, given a holomorphic function F in a tube T^C, the functions $x \mapsto F(x+iy)$ can be considered as distributions on $\mathbf{R}^n$ depending on the parameter $y \in C$ and it makes sense to ask whether F has a distribution T as its boundary value as $y \to 0$, $y \in C$. Results in this direction, due in part to V. S. Vladimirov, are given under various growth conditions on F, and also concerning the problem of recovering F as the Cauchy integral, the Poisson integral or the Laplace transform of T. The Cauchy integral of a tempered distribution is defined as a class of holomorphic functions, two functions being equivalent if their difference is a pseudopolynomial $P(z) = \sum_{j=1}^{n} \sum_{s=0}^{m_j} g_{js}(\hat{z}_j) z_j^s$, where g_{js} is a holomorphic function of the variables $z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n$, a concept used already by Tillmann, The growth and the representation of functions in H^p classes defined on tubes is another topic touched upon in the book.

Beyond the scope of the book is an important outgrowth of the representation of distributions as boundary values of holomorphic functions. The representation theorems can be expressed as isomorphisms of the form (3), and this led M. Sato to introduce hyperfunctions. For $n = 1$ they are simple to describe. Let Ω be an open subset of $\mathbf{R}$, and V an open neighborhood in $\mathbf{C}$ of Ω in which Ω is relatively closed. The space of hyperfunctions on Ω is the quotient $H(V \backslash \Omega)/H(V)$, which by the Mittag-Leffler theorem is independent of the choice of V. For higher dimensions $H(V \backslash \Omega)$ is equal to $H(V)$, so this definition is worthless. The way out of this dilemma is the observation that the Mittag-Leffler theorem is equivalent to the vanishing of the first cohomology group $H^1(V, \mathscr{O})$ of V with coefficients in the sheaf $\mathscr{O}$ of holomorphic functions. As D. Struppa and C. Turrini carefully explain in their excellent expository article [16], where detailed references can be found, Sato was led to define the space of hyperfunctions on $\Omega \subset \mathbf{R}^n$ as the nth relative cohomology group $H^n(V, V \backslash \Omega; \mathscr{O})$ with coefficients in $\mathscr{O}$, see also [1]. A. Martineau pointed out in a lecture at the Bourbaki seminar that hyperfunctions can also be defined in a more classical way, paralleling that of distributions, and it is in this form that they are discussed in Chapter IX of Hörmander's book [5]. Finally, let me mention that hyperfunctions have led to microfunctions and to "algebraic analysis" for which I refer to [6] which was reviewed in Bull. Amer. Math. Soc. (N.S.) **18** (1988), 104–108.

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Stochastic integration and differential equations—a new approach, by Philip Protter. Springer-Verlag, Berlin and New York, 1990, 302 pp., $48.00. ISBN-0-387-50996-8

Stochastic integration and stochastic differential equations are important for a wide variety of applications in the physical, biological, and social sciences. In particular, the last decade has seen an explosion in applications to financial economics. The need for a theory of *stochastic* integration is readily seen by considering integrals of the form $\int_{[0,t]} X_s \, dM_s$ and noting that these can be defined path-by-path in a Stieltjes sense for all continuous integrands X only if the paths of M are locally of finite variation. This immediately precludes such important processes as Brownian motion and all continuous martingales as integrators, as well as many discontinuous martingales. Consequently, for a large class of martingales M, one must resort to a truly probabilistic or stochastic definition of such integrals. The origins of the theory of stochastic integration lie in the early work of Wiener and the seminal work of Itô [11], where integrals with respect to Brownian motion were defined. Most importantly for applications, Itô developed a change of variables formula for C^2 functions of Brownian motion. In presenting the results of Itô in his book [7], Doob recognized that the two critical properties of Brownian motion B used in Itô's development of the stochastic integral were that B and $\{B_t^2 - t, t \geq 0\}$ are martingales. Extrapolating from this, Doob proposed a general integral with respect to L^2-martingales, which hinged on an as yet unproved decomposition theorem for the square of an L^2-martingale. This is a special case of a decomposition theorem for submartingales (the Doob-Meyer decomposition theorem), which was subsequently proved by Meyer [17, 18]. Using this decomposition result, Kunita and Watanabe