8. R. Smith, *The Riemann problem in gas dynamics*, Trans. Amer. Math. Soc. **249** (1979), 1–50.

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Modular forms have been studied, *accidently or intentionally*, for about 200 years, beginning seriously with Jacobi and Eisenstein. A key word here is "accidentally": Historically, many peculiar things were discovered and studied in an *ad hoc* fashion; a great number of these are now construed as corollaries of a general phenomenology with the unfortunately unevocative appellations "theory of modular forms" or "theory of automorphic forms." This "underlying phenomenology" is distant from more tangible and elementary issues, and so often seems obscurely technical and tiresomely unmotivated (to the uninitiated, at least).

Because it does provide an underlying pattern, the subject is currently of intense research interest. Either provably or conjecturally, a large fraction of the objects of interest in number theory is intimately related to modular forms. There are also pleasantly surprising connections with many other things: string theory, combinatorics, Kac-Moody algebras, and so on.

To develop a sympathy for the subject, it seems necessary to shift what one believes to be the *primary objects of study*. Because of the efficacy of "the theory of modular forms" as a methodology in number theory, one might study modular forms as fundamental objects, rather than directly consider number fields themselves (for example). To add to the confusion of the novice, there is not a single fixed notion of "modular form": The general idea has many different incarnations, whose common spirit is apparent only after considerable reflection (and proof). Yet, each incarnation of "modular form" has its own utility.

With just a few exceptions, the subject languished for the first half of this century. Its resurrection in the late 1950s and early

1960s was possible not only because of the technical adeptness of Shimura, Piatetski-Shapiro, Langlands, and others, but also because *other parts of mathematics (especially number theory) had by that time developed to a point inviting subtler explanations.* Perhaps more so than with many other parts of mathematics, the general theory of modular/automorphic forms has evolved as an explanation of "secret causes" of more perceptible phenomena.

Even the ambitious and assiduous student, in attempting to begin reading on the subject, may be disheartened by the impression that the prerequisites are esoteric and onerous. A bald statement of definitions is neither suggestive nor illuminating. Indeed, a treatment of the subject can achieve inclusiveness and unity only by invoking very powerful auxiliary machinery. Some of the *ideas* (and primitive motivation) *can* be illustrated in simple examples; however, in this subject it is not always easy to discriminate between parlor tricks and entertaining examples of important general ideas.

Motivated by the resurgence of research interest, many books have appeared in the past twenty years or so, offering more-or-less elementary introductions to parts of the theory of automorphic and modular forms. Of necessity, there is considerable overlap. Miyake's text gives the usual classical foundational material on modular forms, but the text also treats some topics found in few other texts. Further, the discussion throughout is quite detailed, which may be appropriate for students not yet oriented to the "standard methods" in the subject.

A historically important motivation for studying modular forms was the study of *quadratic forms*, especially the problem of counting the number $\nu_k(n)$ of ways an integer n is representable as a sum of k squares of integers (with k fixed). Jacobi first saw the relevance of modular forms to such questions. A generating function for counting representations as a sum of k squares is

$$\varphi_k(q) = \sum_{n\geq 0} \nu_k(n) q^n = \sum_{n_1, n_2, \dots, n_k} q^{n_1^2+\cdots+n^2k}$$
$$= \left(\sum_n q^{n^2}\right)^k = \varphi_1(q)^k.$$

Letting $q = e^{\pi i z}$ with z in the upper half-plane $\mathscr{h}$, put $\theta(z) = \varphi_1(q)$; this θ is the simplest example of a *theta series*. From the representability of functions on $\mathbf{R}/\mathbf{Z}$ by their Fourier series one obtains the *Poisson summation formula* for a nice (e.g., Schwartz) function φ on $\mathbf{R}$:

$$\sum_{n\in\mathbf{Z}} \varphi(n) = \sum_{n\in\mathbf{Z}} \hat{\varphi}(n),$$

where $\hat{\varphi}$ is the Fourier transform. Applying this to $\varphi(n) = e^{-\pi n^2 y}$ with $y > 0$, one obtains $\theta(iy) = \theta(-1/iy)/y^{1/2}$, which by analytic continuation gives $\theta(z) = \theta(-1/z)/(-iz)^{1/2}$. Visibly $\theta(z+2) = \theta(z)$, so $\theta(z)$ satisfies certain transformation rules under the group Γ_θ of linear fractional transformations generated by $z \to z+2$ and $z \to -1/z$. This becomes much simpler if we consider powers of θ divisible by 8: It becomes

$$\theta(\gamma z)^{8n} = (cz+d)^{4n}\theta(z)^{8n}, \quad \left(\text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta\right).$$

Thus, θ^{8n} is *an automorphic form for* Γ_θ *of weight* $4n$. We will see the significance of this peculiar transformation property for the sum-of-squares problem shortly. *The study of such theta series was a large part of the historical motivation for studying automorphic forms.*

Generally, for a subgroup Γ of $\mathrm{SL}(2, \mathbf{R})$ acting on the complex upper half-plane $\mathscr{h}$ by *linear fractional transformations*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = (az+b)/(cz+d),$$

let $\mu : \Gamma \times \mathscr{h} \to \mathbf{C}^x$ be an "automorphy factor", i.e., having the property that for $\gamma, \delta \in \Gamma$ and $z \in \mathscr{h}$

$$\mu(\gamma\delta, z) = \mu(\gamma, 8z)\mu(8, z).$$

The function

$$\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = (cz+d)^k$$

has this property. Define an *automorphic form* for Γ with automorphy factor μ to be a holomorphic function f satisfying $f(\gamma z) = \mu(\gamma, z)f(z)$ (plus some modest growth constraints which we neglect). If Γ is of finite index in $\mathrm{SL}(2, \mathbf{Z})$, then one says that f is an *elliptic modular form.* In particular, if

$$\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = (cz+d)^k,$$

then f is *of weight* k.

When Γ is a subgroup of $\mathrm{SL}(2, \mathbf{Z})$, we can easily construct modular forms of weight $2k \geq 4$ in another way, the *Eisenstein series*: For example (with $2k \geq 4$ for convergence)

$$E_{2k}(z) = \sum_{c, d \text{ not both } 0} (cz + d)^{-2k}$$

is a modular form of weight $2k$ for $\mathrm{SL}(2, \mathbf{Z})$. The Fourier expansion of an Eisenstein series can be determined by elementary means; for the previous example, it is just

$$E_{2k}(z) = 2\zeta(2k) + 2(-2\pi i)^{2k}\Gamma(2k)^{-1}\sum_{n>0}\sigma_{2k-1}(n)e^{2\pi i n z},$$

where $\sigma_{2k-1}(n)$ is the sum of the $2k-1$ powers of the positive divisors of n. The explicit definition of the Eisenstein series and the rather elementary nature of the Fourier coefficients are two of their important features. *Eisenstein series and theta series together are the explicitly constructible example automorphic forms; everything else is subtler.*

A very important technical virtue of the space of modular forms of a fixed weight for a fixed subgroup Γ is that it is *finite-dimensional*, with a dimension calculable in a variety of ways. In particular, for weight 4, the space of modular forms for the subgroup Γ_θ for θ^8 is so small that *this theta series must be an Eisenstein series*. From this, one finds an explicit formula for the number of ways an integer is expressible as a sum of 8 squares, by equating Fourier coefficients; for n odd, $\nu_8(n) = 16\sigma_3(n)$. Standing alone, this formula is a mere amusement, thought it is suggestive. To this reviewer, the *method of proof* is the most piquant aspect of this example.

In general, there are many other modular forms than Eisenstein series, so θ^{8k} is not generally an Eisenstein series, and the previous "explicit formula" result for a sum of 8 squares has no simple analogue for 16 squares (nor for 24, 36, etc.). Still, something of significance can be salvaged for this sum-of-squares problem. A vector space complement to the space of Eisenstein series can be aptly described by a rapid decay condition (for example); for $\Gamma = \mathrm{SL}(2, \mathbf{Z})$, the condition can easily be described by a condition on Fourier coefficients: In the Fourier expansion $f(z) = \sum_n c_n e^{2\pi i n z}$ of a modular form f, we say that f is a *cuspform* if $c_n = 0$ for $n \leq 0$. An important feature of cuspforms of weight k is *Hecke's* (*elementary*) *estimate on the Fourier coefficients*: nth Fourier coefficient of cuspform $= O(n^{k/2})$. Then, from the expression of θ^{8k} as a sum of a weight-$4k$ Eisenstein series plus a cuspform, we can obtain, not an exact formula, but a good error term: For n odd,

$$\nu_{8k}(n) = (\pi^{4k}/(1-2^{-4k})\Gamma(4k)\zeta(4k))\sigma_{4k-1}(n) + O(n^{2k}).$$

Note that the sum-of-divisors function still occurs as the main term, with an interesting constant; and the error term is *much* smaller than the main term. The irregularity of σ_{4k-1} matches the irregularity of ν_{8k} inexplicably well.

The precise size of the error term is a very subtle matter. Ramanujan made a conjecture, broadened in scope by Petersson, which asserts that the error term should actually be $O(n^{2k-1/2+\varepsilon})$ for every $\varepsilon > 0$, i.e., that the Fourier coefficients of cuspforms of weight k grow like $n^{k/2-1/2}$, not just $n^{k/2}$. The seemingly slight difference belies the profundity of the distinction: The better estimate was proven only as late as the early 1970s by Deligne, as a corollary of his work on the Weil conjecture.

In a different direction, consider Dirichlet series

$$D(s) = \sum_{n\geq 1} c_n/n^s.$$

The simplest case, Riemann's zeta function

$$\zeta(s) = \sum_{n\geq 1} 1/n^s = \prod_{p \text{ prime}} 1/(1-p^{-s})$$

has well-known applications to prime number theory: From this *Euler factorization* one easily proves the infinitude of primes, and by a subtler analysis of ζ'/ζ one proves (by a variety of means) the *prime number theorem*:

Number of primes less than $x \sim x/\log x$.

For the latter, not only the Euler product factorization is important, but also the fact that

$$\xi(s) = s(1-s)\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right)\zeta(s)$$

has an *analytic continuation* to an entire function, and has a *functional equation* $\xi(s) = \xi(1-s)$. To prove these things, the *functional equation of Jacobi's theta function* $\theta(z)$ enters again (!), and also the *integral representation*

$$\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \int_0^\infty y^{s/2}\frac{1}{2}(\theta(iy)-1)\,dy/y.$$

From the proven utility of this zeta function (and kindred Dirichlet L-functions and zeta functions of larger number fields), one might become interested in the general problem of *determination of Dirichlet series with analytic continuations, functional equations, and Euler product factorizations.*

The integral expression for the ζ-function in terms of the theta function and use of the functional equation of the theta function provide a good clue. Let

$$f(z) = \sum_{n\geq 1} c_n e^{2\pi i n z}$$

be a cuspform for $\mathrm{SL}(2, \mathbf{Z})$, of weight $2k$. Then, forming the same sort of integral as for the zeta function (without having to subtract the constant term as with θ), consider

$$\int_0^\infty y^s f(iy)\, dy/y.$$

Expanding f in its Fourier series and integrating term by term, we see that this integral is

$$(2\pi)^{-s}\Gamma(s) \sum_{n\geq 1} c_n/n^s.$$

Now we use the (Weyl) element $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ of $\mathrm{SL}(2, \mathbf{Z})$ and the relation $f(-1/z) = z^{2k} f(z)$, similar to Jacobi's functional equation for the theta function. We have

$$\int_0^\infty y^s f(iy)\, dy/y = \int_1^\infty y^s f(iy)\, dy/y + \int_0^1 y^s f(iy)\, dy/y.$$

Since f is rapidly decreasing as $y \to +\infty$, the first integral on the right is an entire function of $s \in \mathbf{C}$. In the second integral, we use the Weyl element to obtain

$$\begin{aligned}\int_0^1 y^s f(iy)\, dy/y &= \int_0^1 y^s f(-1/iy)(iy)^{2k}\, dy/y \\ &= (-1)^k \int_1^\infty y^{2k2-s} f(iy)\, dy/y.\end{aligned}$$

This integral is again entire in s. Thus, we find that

$$(2\pi)^{-s}\Gamma(s) \sum_{n\geq 1} c_n/n^s$$

has an *analytic continuation to an entire function* of $s \in \mathbf{C}$, and has a *functional equation*: Under $s \to 2k - s$, it is multiplied by $(-1)^k$.

Conversely, Hecke showed that any Dirichlet series $D(s)$ so that $(2\pi)^{-s}\Gamma(s)D(s)$ has these properties (and is of reasonable vertical growth) must come from a modular form of weight $2k$ for $\mathrm{SL}(2, \mathbf{Z})$. The functional equations satisfied by the Dirichlet series attached to modular forms for proper subgroups of $\mathrm{SL}(2, \mathbf{Z})$ are themselves more subtle, and a more general form of the *converse theorem* (for $\mathrm{GL}(2)$) was not known until Weil proved the following: If a Dirichlet series $\sum_n c_n/n^s$ *and (sufficiently many of) its "twists"* $\sum_n c_n\chi(n)/n^s$ by *Dirichlet characters* χ have suitable analytic continuations and functional equations (with a mild vertical growth condition), then $f(z) = \sum_n c_n e^{2\pi i n z}$ *is a modular form.* This result concludes an early chapter in the classification of Dirichlet series with analytic continuations and functional equations.

The proof of Weil's converse theorem makes very serious use of an idea which is also essential in addressing the question of *Euler factorization* of Dirichlet series attached to modular forms: the so-called *Hecke operators.* In the context of $\mathrm{SL}(2, \mathbf{Z})$ itself, these are easy to understand, though perhaps obscure in motivation; for proper subgroups of $\mathrm{SL}(2, \mathbf{Z})$ some nasty technicalities arise, which cannot be transparently overcome in this classical setting. For $\mathrm{SL}(2, \mathbf{Z})$ itself, the consideration of these operators shows the following: There is a basis $\{f\}$ for the space of cuspforms of weight $2k$ for $\mathrm{SL}(2, \mathbf{Z})$ so that the Dirichlet series $\sum_n c_n/n^s$ attached to such a basis element has a Euler product factorization

$$\prod_{p \text{ prime}} 1/(1 - c_p p^{-s} + p^{2k-1} p^{-2s}).$$

Implicit in this is the multiplicative property $c_{mn} = c_m c_n$ for m and n relatively prime. Further, the coefficients c_p for p prime evidently determine all the rest. A happy coincidence is that *the eigenvalue of the* pth *Hecke operator is essentially the* pth *Fourier coefficient*, so that relations among the Hecke operators imply relations among Fourier coefficients. The analogous Euler product attached to Eisenstein series of weight $2k$ (without constant term, and renormalized) is merely

$$\sum_n \sigma_{2k-1}(n)/n^s = \zeta(s)\zeta(s-2k+1) = \prod_{p \text{ prime}} 1/(1-p^{-s})(1-p^{2k-1-s}).$$

In general the numbers c_p do not have such a ready elementary interpretation, though in a slightly different case we do obtain some

L-functions already known, as follows. Let

$$f(z) = 4^{-1} \sum_{m,n \text{ not both } 0} (im+n)^{4k} e^{2\pi i(m^2+n^2)z}.$$

This *pluriharmonic theta series* is a mild variant of the theta series θ^2 considered earlier, and is a modular form of weight $1+4k$; again, this is proven via the *Poisson summation formula.* The associated Dirichlet series is

$$\begin{aligned} 4^{-1}\sum_{\alpha} \alpha^{4k}|\alpha|^{-2s} &= 4^{-1}\sum_{\alpha}(\alpha/|\alpha|)^{4k}|\alpha|^{-(2s+4k)} \\ &= L_{\mathbf{Q}(i)}(\omega, 2s+4k), \end{aligned}$$

which is a *grossencharacter L-function* over $\mathbf{Q}(i)$ with the grossencharacter

$$\alpha \to (\alpha/|\alpha|)^{4k}.$$

Though the analytical aspects of this type of less elementary L-function had already been treated by Hecke (and in Tate's thesis), the fact that it arises as well from modular forms is essential for more delicate *arithmetic* investigations, as in Shimura's work on special values of such L-functions (extending that of Damerell).

Though the consideration of the quotients $\Gamma\backslash\mathscr{h}$ as Riemann surfaces allows quite explicit study of the dimensions of spaces of cuspforms and analysis of Hecke operators, there is another very interesting idea which has a bearing on these matters and is applicable in much more general circumstances, that of *trace formulas.* For example, with $\Gamma = \mathrm{SL}(2, \mathbf{Z})$ and $2k \geq 4$, the function

$$K(z, w) = c_{2k}\sum_{\gamma\in\Gamma}(\gamma z-\overline{w})^{-2k}(cz+d)^{-2k} \qquad \left(\text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

with

$$c_{2k} = (2k-1)(2i)^{2k}(4\pi)^{-1}$$

is a *reproducing kernel* for the space of cuspforms of weight $2k$, in that

$$\int_{\Gamma\backslash\mathscr{h}} f(z)\overline{K}(z, w)y^{2k-1}\,dx\,dy/y = f(w).$$

Therefore, the *dimension of the space of cuspforms* of weight $2k$ is

$$\dim\{\text{cuspforms of weight } 2k\} = \int_{\Gamma\backslash\mathscr{h}} K(z, z)y^{2k-2}\,dx\,dy.$$

The right-hand side may be "unwound", in the following sense. Let $q(z, w) = (z - \overline{w})^{-k}$. We have the formal property $q(gz, gz) = q(z, z)$. Therefore, letting $\Gamma(\delta)$ be the centralizer in Γ of an element δ of $\Gamma = \mathrm{SL}(2, \mathbf{Z})$, and letting X be a set of representations for conjugacy class in Γ,

$$\begin{aligned}\sum_{\gamma\in\Gamma} &\mu(\gamma, z)q(\gamma z, z) \\ &= c_{2k}\sum_{\delta\in X}\sum_{\gamma\in\Gamma(\delta)\backslash\Gamma} \mu(\gamma^{-1}\delta\gamma, z)q(\gamma^{-1}\delta\gamma z, z) \\ &= c_{2k}\sum_{\delta\in X}\sum_{\gamma\in\Gamma(\delta)\backslash\Gamma} \mu(\gamma^{-1}\delta\gamma, z)q(\delta\gamma z, \gamma z).\end{aligned}$$

If we fail to worry about convergence, then *by changing variables in the integral*

$$\begin{aligned}\int_{\Gamma\backslash\mathscr{H}} &K(z, z)y^{2k-1}\,dx\,dy/y \\ &= c_{2k}\sum_{\delta\in X}\int_{\Gamma(\delta)\backslash\mathscr{H}} q(\delta z, z)y^{2k-1}\,dx\,dy/y.\end{aligned}$$

Letting $G(\delta)$ be the centralizer of δ in $\mathrm{SL}(2, \mathbf{R})$, this is

$$\begin{aligned}\int_{\Gamma\backslash\mathscr{H}} &K(z, z)y^{2k-1}\,dx\,dy/y \\ &= c_{2k}\sum_{\delta\in X}\mathrm{vol}(\Gamma(\delta)\backslash G(\delta))\int_{G(\delta)\backslash\mathscr{H}} q(\delta z, z)y^{2k},\end{aligned}$$

where we must normalize measures on $\Gamma(\delta)\backslash G(\delta)$ and $G(\delta)\backslash\mathscr{H}$ appropriately. A problem here is that we have ignored problems of convergence; since it turns out that the above expression is *not absolutely convergent/*, we may err. *Indeed, the previous formula is not valid.* The error can be repaired (with considerable effort), but let us consider a situation where the previous manipulation is literally correct: We must consider a discrete subgroup Γ of $\mathrm{SL}(2, \mathbf{R})$ so that $\Gamma\backslash\mathscr{H}$ is *compact.*

From the uniformization theorem, any compact Riemann surface of genus ≥ 2 occurs as a quotient $\Gamma\backslash\mathscr{H}$ for some discrete subgroup Γ of $\mathrm{SL}(2, \mathbf{R})$, but this does not give us much information about the group Γ. Rather, we can construct a group Γ by arithmetic means so that the quotient is compact, as follows. Let D be a division algebra of dimension 4 over $\mathbf{Q}$ so that $D\otimes\mathbf{R}$ is isomorphic to the ring $M_2(\mathbf{R})$ of two-by-two real matrices, and fix such an isomorphism. Let $\mathscr{O}$ be an order in D, i.e., a subring of D which is finitely-generated as a $\mathbf{Z}$-module. Let $\Gamma = \mathscr{O} \cap \mathrm{SL}(2, \mathbf{R})$. Then (much as one gives the adelic proof of the units theorem and finiteness of class numbers) one can prove that $\Gamma \backslash \mathrm{SL}(2, \mathbf{R})$ is compact, from which follows the compactness of $\Gamma\backslash\mathscr{h}$. Groups Γ of this sort are *quaternion unit groups.*

The compactness of $\Gamma\backslash\mathscr{h}$ perfectly legitimizes the previous formalism. Let Γ be a quaternion unit group, as above. Further, for simplicity, we assume that there are no torsion elements in Γ other than 1; this can always be accomplished by taking a subgroup defined by a congruence condition. *Then a direct computation shows that the only conjugacy class whose integral makes a nonzero contribution to the trace formula is that of* 1. Therefore, the trace formula yields

$$\begin{aligned}&\text{dimension of space of weight-}k\text{ automorphic forms}\\&\text{for}\\ \Gamma &= \int_{\Gamma\backslash\mathscr{h}} K(z, z) y^{k-1}\, dx\, dy/y\\ &= (k-1)(4\pi)^{-1} \times \mathrm{vol}(\Gamma\backslash\mathscr{h}).\end{aligned}$$

Thus, we have obtained a form of the Riemann-Roch theorem for the compact connected Riemann surface $\Gamma\backslash\mathscr{h}$.

The potential strength of the previous sort of argument invites a more *general definition of holomorphic automorphic form.* Let X be a complex manifold, let G be a semi-simple real Lie group acting transitively on X by holomorphic maps, and suppose that the isotropy group in G of a point of X is compact. Let Γ be a discrete subgroup of G so that $\Gamma\backslash G$ has finite (invariant) measure. Let $\mu : \Gamma \times X \to \mathbf{C}$ be an *automorphy factor*, i.e., satisfying $\mu(gh, z) = \mu(g, hz)\mu(h, z)$. Then a holomorphic function f on X is an *automorphic form* if $f(\gamma z) = \mu(\gamma, z) f(z)$ for all $\gamma \in \Gamma$ and $z \in X$. (Some growth restrictions may be necessary.) This definition includes the particular situations studied classically, such as Siegel modular forms, Hilbert modular forms, and hermitian modular forms. *In this general situation, the formalism of the* (*holomorphic*) *trace formula above still works* (with substantial technical modifications if $\Gamma\backslash X$ is noncompact).

This more general notion of automorphic form does not have an immediately visible application to Dirichlet series. Indeed, the application is much subtler, but is nevertheless of great importance.

For example, by construction, the quaternion unit groups Γ have no unipotent elements, so modular/automorphic forms with respect to them *do not have Fourier expansions.* On the other hand, a suitably enlightened understanding of the idea of "Hecke operators" allows one to define such on these automorphic forms. Then we may certainly write a *formal Dirichlet series* $\sum_n \lambda_n/n^s$, where λ_n is the eigenvalue of "the nth Hecke operator" on a fixed automorphic form f (assumed to be an eigenfunction for the Hecke operators). Unfortunately, we have lost the connection to analytic properties: *We have no obvious integral representation of this Dirichlet series* whereby to demonstrate its analytic properties; cogent treatment of the latter problem requires a more sophisticated viewpoint.

As an introduction to a viewpoint which *does* yield information about analytic properties of more general Dirichlet series, we consider more general *Eisenstein series,* which are of fundamental importance in this and many other regards. We restrict our attention to $\Gamma = \mathrm{SL}(2, \mathbf{Z})$, for simplicity. For $s \in \mathbf{C}$ and $0 < k \in \mathbf{Z}$, define

$$E(z; 2k, s) = \sum_{c, d \text{ not both } 0} (\mathrm{Im}\, z)^s / |cz + d|^{2s} (cz + d)^{2k}.$$

For $2k \geq 4$ and $s = 0$ this is the holomorphic Eisenstein series mentioned earlier; whenever $k + \mathrm{Re}(s) > 1$, this series is nicely convergent. By use of *Poisson summation,* it is not hard to show that $E(z; 2k, s)$ has an *analytic continuation to an entire function* of $s \in \mathbf{C}$ (as $k > 0$), and has a functional equation. For two *cuspforms* f_1, f_2 of weights $2k_1$ and $2k - 2k_1$ (respectively), we can form the integral

$$\mathrm{I}(f_1, f_2, s) = \int_{\Gamma \backslash \mathscr{H}} f_1(z) E(z; 2k, s) f_2^-(z) y^{2k}\, dx\, dy/y^2,$$

which *must be an entire function of* s. (The integral exists for all $s \in \mathbf{C}$ away from the poles of the Eisenstein series, since one can show that for such s the Eisenstein series is "of moderate growth at infinity," and since cuspforms are "of rapid decay.") Now we come to another "unwinding trick," originally due to Rankin. In the region of convergence of the Eisenstein series, by unwinding the Eisenstein series, we have

$$\mathrm{I}(f_1, f_2, s) = \int_{\Gamma_\infty \backslash \mathscr{H}} f_1(z) f_2^-(z) y^{s+2k}\, dx\, dy/y^2,$$

where Γ_∞ is the group of transformations $z \to z+n$ for $n \in \mathbf{Z}$. Let the nth Fourier coefficient of f_1 (resp. f_2) be b_n (resp. c_n). Then by expanding the f_i in their Fourier series and using the orthogonality of distinct exponential functions, we have

$$\begin{aligned} \mathrm{I}(f_1, f_2, s) &= \int_0^\infty \sum_{n>0} b_n \bar{c}_n e^{-4\pi n y} y^{s+2k} \, dy/y^2 \\ &= (4\pi)^{-(s+2k-1)} \Gamma(s+2k-1) \sum_{n>0} b_n \bar{c}_n / n^{s+2k-1}. \end{aligned}$$

Therefore, we find that the "*Rankin convolution L-function*" $\sum_{n>0} b_n \bar{c}_n / n^{s+2k-1}$ has *an analytic continuation to an entire function of* $s \in \mathbf{C}$, *and has a functional equation.* Further, if f_1 and f_2 are both Hecke eigenfunctions, *the Rankin convolution L-function has a Euler product whose p-factor is of degree* 4. This use of Eisenstein series "with parameter s" is the faintest shadow of recent developments.

In this cursory discussion only *analytic aspects* of *holomorphic* automorphic forms have been mentioned, and mainly for *groups acting on the upper half-plane.* This is the scope of Miyake's book, and is a very reasonable limitation for an introduction. However, as briefly indicated in the "more general definition" above, the notion of automorphic form has a much broader sense. Further, for analytic purposes, the condition of holomorphy can be replaced by other significant eigenfunction conditions. A glaring omission in the above remarks is the "arithmetic theory of holomorphic automorphic forms," with its applications to class field theory, special values of L-functions, Hasse-Weil zeta functions of curves, and representations of Galois groups; these omissions do not reflect current interest, but the orientation of Miyake's book.

The parts of Miyake's book of most interest are his treatment of *Weil's converse theorem*, *holomorphic trace formulas* for quaternion unit groups, and *analytic continuation of Eisenstein series* for GL(2). These are tractable illustrations of very important themes, and Miyake treats these topics very carefully, fully justifying his offering yet another introductory book on this subject. *Nevertheless, the student must realize that these are only illustrations which, while suggesting the quality of general phenomena, cannot reveal the provocative richness of "the big picture."* Miyake's bibliography gives good directions for further reading.

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