This book is a valuable, and indeed unique, addition to the literature and will be a standard reference. The introductory chapters are very clearly written. The authors derive many previously known results by alternative methods. This gives an independent check of these results. I know of no errors in the trees or tables and have great confidence in the general accuracy of the material presented here.

This book will be an essential part of every mathematical library.

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Elements of differentiable dynamics and bifurcation theory, by David Ruelle. Academic Press, New York, 1989, 187 pp., $27.50. ISBN 0-12-601710-7

About a year and a half ago at the Thom Symposium in Paris many of the talks traced their genesis to René Thom's seminar in the Bois Ste. Marie at the Institut des Hautes Etudes Scientifiques. What a wonderful seminar I thought, and recalled my own stay at the I.H.E.S. in 1969–70 and Thom's seminar that year (which was not one of the ones mentioned). Steve Smale, Charles Pugh, Floris Takens, and I were among the long-term visitors. Thom and David Ruelle were permanent members of the institute. The subject of the seminar was dynamics. Smale was doing mechanics. His 1967 paper, "Differential dynamical systems" [1], "a masterpiece of mathematical literature" as Ruelle writes (p. 48), structured ordinary differential equations and discrete time dynamical systems (iterated diffeomorphisms) with the perspective of one of the leading topologists of the time. Poincaré, Birkoff, Morse, Hopf, and others were of course concerned with the relations of global topology and ordinary differential equations. The Gorki school in the Soviet Union, Andronov, Pontryagin, Chaikin, Witt, et al., were concerned with structural stability, nonlinear oscillations and bifurcations in two dimensions. Lefschetz introduced the concept of structural stability in the West and Peixoto proved his classification theorem for two dimensions. But Smale's work on gradient dynamical systems, the generalized Poincaré conjecture and H-cobordism theorem reunited ordinary differential equations, dynamical systems and topology with a force not seen since Poincaré's time. Then came the horseshoe (infinitely many periodic solutions), the new structurally stable attractors (one derived from Anosov and another derived from expanding in the 1967 paper) and the structure of hyperbolic sets which unified them all. Anosov proved the structural stability of what are now called Anosov diffeomorphisms (the hyperbolic set is the whole manifold) and Smale, the dissipative version, the Ω-stability theorem. In addition Anosov and Sinai had initiated the study of the ergodic theory of these systems.

Pugh and I were working on invariant manifold theory, applications to Anosov flows, and generalizations. Takens was studying normal forms at fixed points. I believe that Thom asked Takens to report on Hopf's paper on bifurcation. With the use of center manifold theory, the Banach space theory reduces to the two-dimensional case, the study of which was initiated by Poincaré. Shortly thereafter the Ruelle-Takens scenario for turbulence appeared.

Ruelle writes: "After the papers of Lorenz [2] and Ruelle and Takens [3], the interest in what is now called chaos developed first slowly then explosively.... It is of course natural—and indeed desirable—that there be some gap between rigorous mathematics and the methods used by students of natural phenomena. But the gap should not be too wide, and there is some danger that a subject that arose from close interplay of mathematics and physics would die from their divorce" (pp. 87–88). So he has written a book with "... more emphasis than usual on infinite dimensional systems, non-invertible maps, attractors and bifurcation theory." to prepare the serious reader "... to enter the treacherous jungle of the literature of chaos" (Preface).

Now I don't mean to suggest that the hyperbolic theory was the only successful attack on ordinary differential equations of the period. There was after all the Kolmogorov-Arnold-Moser theory and much other work going on; Levinson's on the forced Vander Pol which stimulated the horseshoe, for example. But this work was more local in nature, more part of analysis than topology or more directed to Hamiltonian systems than to the general theory. Nor do I mean to suggest that Ruelle and Takens invented the relation of the Hopf bifurcation to turbulence. Hopf was interested in infinite-dimensional systems and fluid flow in particular. He gives as an example the bifurcations from steady-state flow around a solid body to periodic vortex shedding with increasing velocity. Landau suggested that turbulence might be explained by quasiperiodic flow on a high-dimensional torus created by a succession of Hopf bifurcations. There was work on the Hopf bifurcation for diffeomorphisms by Naimark in the Soviet Union and Sacker in the United States. Arnold [4] recounted more independent work in the Soviet Union on these questions, for example, Sositaisville on normal forms for the n-dimensional Hopf bifurcation and Arnold's own unsuccessful attempt in 1964 to find numerically a hyperbolic attracting set in a six-dimensional Galerkin approximation to a two-dimensional Navier-Stokes equation [4, p. 278]. There were visits back and forth, Smale to Moscow in 1961 [5], Arnold to I.H.E.S. in 1965 [4, p. 282]. By this time Sinai's work on billiards had also appeared, his Markov partitions for Anosov diffeomorphisms and Bowen's generalization to hyperbolic systems. In fact, there was an explosion of work of hyperbolic systems in the United States and the Soviet Union. But I am straying from my intentions; I mean to point out the main features of this development in chaos theory. In 1966–67 Smale discovered the first structurally stable nonmanifold attractors* which in particular exhibited one of the salient features of hyperbolic sets sensitive

* Personally I have always been rather pleased that one of the first was motivated by my thesis on expanding maps.

dependence on initial conditions. These examples were topologically motivated. There were no explicit equations in mind. In 1969 Ruelle and Takens showed that a sequence of Hopf bifurcations could result in an invariant torus with one of Smale's strange structurally stable attractors on it, and proposed this as a scenario for turbulence. The difference with Landau is the strange attractor which is quite a different kettle of fish than quasiperiodic motions. A few years later, the experiments of Swinney and Gollub gave credence to the notion that the onset of chaotic behavior in fluids is related to nonquasiperiodic dynamics in low dimensions.

Part of the charm of the original Ruelle-Takens model is the low-dimension (albeit not completely specified) dynamics which exhibits chaos, but also this dynamics is part of a theory which can explain it and structure it. It is robust in many ways: structurally stable by work of Smale so that the topological behavior does not vary with perturbations, but also statistically table. Sinai introduced Gibbs states for Anosov systems, and Bowen and Ruelle extended the work to general hyperbolic systems. The Sinai-Bowen-Ruelle measures give robust statistics for hyperbolic attractors.

These days it is easy to produce chaotic dynamics or complex bifurcation pictures with computer graphics. I wasn't around in 1961 when Smale announced that he was giving up topology for dynamical systems because "no problem in topology was as important and exciting as the topological conjugacy problem for diffeomorphisms, already on the 2-sphere" [5, p. 150], but in the early 70s I saw Sullivan begin lectures with a sense of wonder as he would point out that while topology studies maps $f : X \to Y$, if X and Y are the same space $f : X \to X$ then you can iterate f and an infinite process is born. Yesterday Matt Grayson programmed the workstation to iterate a simple dynamical system two million times and to make a motion picture of the dynamics. The enormous value of computers for dynamical systems has been mainly as an experimental tool. The digital computer has the ability to produce reproducible data and pictures, some of which are chaotic, from simple explicit equations. But these pictures are scarcely more useful than the ones I produced with my camera last month of the Iguaçu falls, or what you can observe on your home analog computer by opening the faucet. A complex picture does not necessarily elucidate an already complex phenomenon unless we can organize the information the picture represents. Partly because the machines and the pictures are finite while dynamics is infinite, organizing the information usually at least entails giving a plausible dynamical interpretation of what one is apparently seeing. I know of no useful taxonomy of computer graphical representation of dynamical systems at this time which relies on the graphics alone. Anything which survives is more or less rigorously related to the theory. Some complex phenomena which are discovered stand on the side completely or almost forgotten, waiting for adequate description, as was the case of homoclinic orbits from Poincaré to Smale.

In 1963, Lorenz already found and plotted a chaotic attractor which exhibited sensitive dependence on initial conditions. It arose from a simple looking quadratic equation in three variables which was a truncation of convection flow. The physical origins and simple nature of these differential equations make them especially important for study, but I do not believe Lorenz's attractor was very well received before Guckenheimer and Williams did a geometric analysis of the equations very much in the spirit of hyperbolic chaotic attractors. Lorenz's attractor is not hyperbolic by the way, and provides an example that dynamicists were looking for of persistence of a singular point in the closure of the periodic orbits!

The other well-known graphics which came much later, the Julia sets, the Mandelbrot set and the period doubling cascade, are well related to the theory of one-dimensional complex and real dynamics. The scaling in the period doubling cascade which was observed numerically by Feigenbaum and Coullet-Tresser might have survived as a sort of physical law on its own because of its universality and ultimate experimental observation by Libchaber. But here the mathematical physicists led the way. Feigenbaum-Cvitanovic and Coullet-Tresser proposed explanations via renormalization which had been very successful in physics. Lanford and then other mathematical physicists have given formal proofs. Finally mathematicians, Sullivan in particular, studied this important dynamical phenomenon.

It is interesting to note that at the 1986 International Congress of Mathematicians, Eckman, Lanford, and Sullivan spoke about the proofs of the renormalization program. Feigenbaum, Cvitanovic, Coullet, and Tresser were not asked to speak. Are mathematicians too parochial? Ralph Abraham [6, p. 3] wrote "... Some of the mathematicians who were the most innovative and radical in the 1960's have become the most conservative today. Perhaps it is time for the mathematical community to take a soul searching look at itself, in connection with two small minded knee jerk reflexes: one against applications of mathematics, the other against publicity. Both of these unconscious reflexes kicked viciously (and fatally) at catastrophe theory in the 1970's. Both are unleashed today against chaos theory and fractal geometry. Will the New Math of space-time patterns fall victim to this unconscious hostility? Or is it the Old Math which will lose its place in the history of consciousness?"

Ralph Abraham would surely include me among the conservatives. A great many physical systems have been shown to contain homoclinic points in the equations for their dynamics and are called chaotic. Others similarly exhibit period doubling according to the scaling law. But frequently catastrophe theory or chaos theory posits an unspecified dynamical system whose properties are analogous to an observed phenomenon. The experiments of Gollub-Swinney and Libchaber can be seen in this context, but we accept them. Others, such as the claim that sensitive dependence on initial conditions for a nonlinear dynamical system might explain the stock market crash of October 1987, are so far out as to be ludicrous. In the Preface, Ruelle laments the "low quality of much of the recent physics literature on 'chaos'." But what makes some of the literature "low" quality and some high? What is one to make of the phenomenological observations of René Thom and their models in catastrophe theory, or Mandelbrot and fractional dimension, or the woman I met late one night in the subway reading Gleick's book *Chaos* [7] who told me that "It is just like real life." We could use serious discussions of these matters! Gleick's book is an excellent starting point because it puts mathematics in a broader scientific context which we sometimes forget. On the negative side I found it overdramatic and insufficiently appreciative of nonlinear dynamics as a rigorous mathematical discipline which structures and gives legitimacy to his subject. Also I thought he did not sort out the good from the bad.

One can read some discussion of these issues in The Mathematical Intelligencer [8] and hopefully we will read more, even in the *Bulletin*. But I will not play Zuckerman in *The Facts* [9] to David Ruelle's Philip Roth. First of all I am not a fictional character, secondly I wasn't asked to. Ruelle does not distinguish the high-from the low-quality chaos literature, which was for me the only (slightly) disappointing aspect of this enjoyable book. Ruelle has written an excellent introduction to dynamical systems and bifurcation theory. Proofs are sketched where possible. The facts are clearly and well presented. Ruelle comes through very clearly in his selection of material, presentation, and comments. Elementary bifurcation theory and the hyperbolic theory are stressed, but the theory is extended to semiflows and endomorphisms. What are the main features of the theory?

The hyperbolic (or Axiom A) theory deals with dynamics which are products of expansions and contractions. Famous examples are: Smale's horseshoe (Figure 1). The rectangle is contracted horizontally, stretched vertically, and laid back down in the plane so that the quadratic or parabolic lies outside the rectangle. The set of points which are always in the rectangle for positive and negative iterates form a Cantor set, which is in one-to-one correspondence with infinite sequence of 0's and 1's, the k th entry is determined by the subrectangle the point is in for the k th iterate.

The doubling map (Figure 2, see p. 206) $X \to 2X (\mathrm{mod}\, 1)$ (or $z \to z^2$ for $|z| = 1, z \in \mathbf{C}$). The map is purely expanding. Lebesgue measure is invariant, ergodic, and Bernouilli.

Hyperbolic pieces are assembled in a filtered or gradient fashion (Figure 3, see p. 206), where points represent hyperbolic sets (perhaps even the whole manifold) and arrows, directions of movement. With more conditions, hyperbolic diffeomorphisms are

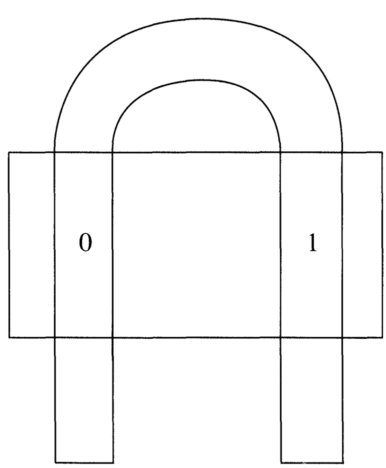


FIGURE 1

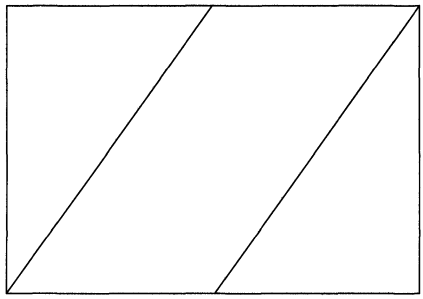


FIGURE 2

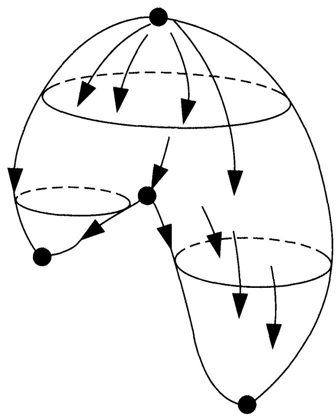


FIGURE 3

stable. That is, perturbations are conjugate, there is a homeomorphism h such that $fh = hg$ for perturbations g of f. If h is defined in the whole manifold, this is structural stability. For h defined only on the nonwandering set, which is a dynamically interesting subset of M, this is Ω-stability. Hyperbolicity conditions guaranteeing structural stability were proved by Peixoto, Anosov, Palis, Smale, and Robbin. Conditions for Ω-stability were proved by Smale. Recently, Mané and Palis proved C^1 converses.

Approximating $z \to z^2$ by an imbedding in R^3 and contracting in the normal direction (Figure 4) maps the solid torus into itself. The intersection of the images, a solenoid, is a stable attractor.

The hyperbolic theory has been extended in several directions. Instead of uniform expansion and contraction, these are allowed to vary measurably. This general situation is analyzed by Pesin and others.

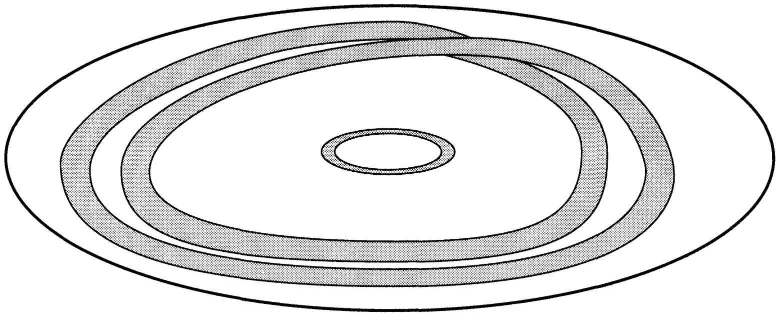
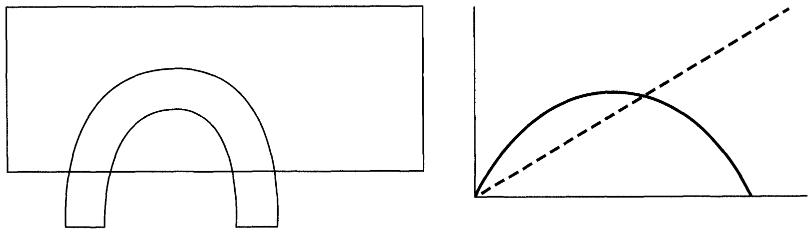


FIGURE 4

Parabolic behavior without any hyperbolicity, for example on the way to producing a horseshoe (Figure 5, see p. 208), produces period doubling. The mix of parabolic behavior and hyperbolicity (Figure 6, see p. 208) produces wild hyperbolic sets with infinitely many sinks (Newhouse) absolutely continuous measures in one dimension (Yakobson) and strange attractors in the plane (Benedicks-Carleson). For the quadratic family of maps $Z^2 + C$ in one dimension, Sullivan's distortion lemma treats iterates that involve the singularity essentially.

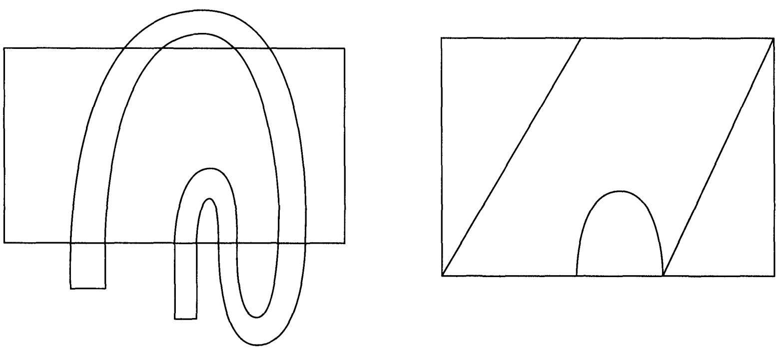
Bifurcation theory studies dynamical systems depending on a parameter. Bifurcation points are points where the behavior is not constant in a neighborhood. The simplest bifurcations hardly involve dynamics at all. They are the bifurcation of the zeros of vector fields (a zero is fixed by the flow). Since a vector field is locally a map $V : R^n \to R^n$, dependence on a parameter P gives a map $V : PxR^n \to R^n$, we see that the bifurcation of the zeros or equilibria is a part of singularity theory, contact equivalence. If $V(p, \_)$ is the gradient of a real valued function $f(p, \_)$, then the local structure of f gives more information and the generic bifurcations for $\dim P \leq 4$ are the elementary catastrophes [10], but even then the bifurcation theory of the functions does not give a complete picture for the dynamics. Among gradient flows the structurally stable* are open and dense, as are the stable one- and two-parameter families; for eight-parameter families this is no longer the case (Smale, Palis, Takens, Dias Carneiro).

* For flows the homeomorphisms are to take unparameterized orbits of one system to those of the other.



In the plane On the interval

FIGURE 5



In the plane On the interval

FIGURE 6

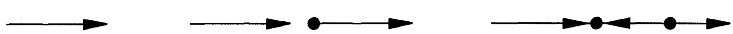


FIGURE 7

FIGURE 8

The local picture for the generic one-parameter families of vector fields at a zero is fairly clear. A pair of equilibria is made (Figure 7).

A single eigenvalue of the linearized equation changes sign (Figure 8) or a complex conjugate pair changes sign (Figure 9), creating a Hopf bifurcation. The center manifold theorem reduces the general situation to one- or two-dimension product with hyperbolic behaviors in the complementary dimensions.

For diffeomorphisms the local situation at a fixed point is

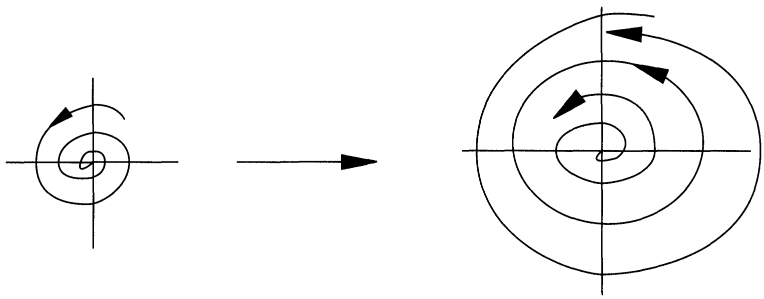


FIGURE 9

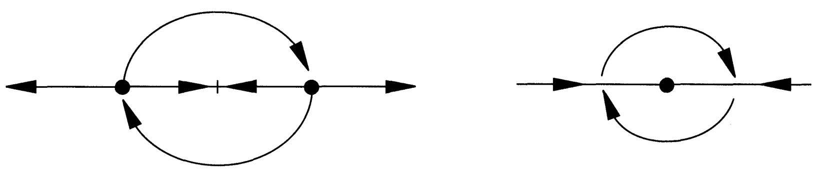


FIGURE 10

much the same, with the addition of a flip with eigenvalue -1 (Figure 10).

More technical conditions are required for the Hopf bifurcation, and now the induced map on the invariant circle is a diffeomorphism so even the local picture involves the bifurcation analysis of circle diffeomorphisms (Herman, Yoccoz).

The general bifurcation picture for surface diffeomorphisms is more complicated and involves the entire collection of phenomena discussed above.

I believe that every picture I've drawn is drawn, discussed, and explained in Ruelle's excellent introduction.

The only proper submanifolds of two manifolds are 0 and one-dimensional. So for two-dimensional diffeomorphisms, the only generic lack of transversality we see is the same as the generic singularity of maps from the line to the line, i.e., the X^2 singularity. This explains the importance of quadratic or parabolic behavior in these dimensions. In higher dimensions we have hardly begun.

Two of my favorite examples are to study diffeomorphism perturbations of the time one map of the geodesic flow on a compact surface of negative curvature, and to study perturbations of the automorphism of the four torus $A : T^4 \to T^4$ where A is given by the 4×4 matrix [11]

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix},$$

even the family

$$X \to AX + \begin{pmatrix} a \sin 2\Pi x_1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

and $a \in \mathbf{R}$. It is possible that for both examples all small perturbations remain transitive, and the ergodic theory should be interesting.

The fine structure of dynamical systems may be too difficult to describe in general. We may have to content ourselves with certain gross features, but we don't have many. One of the best is the topological entropy of Adler, Konheim, and McAndrew. The entropy measures the "chaos" of the system. For the quadratic family $X \to 1 - \lambda X^2$, $0 \leq \lambda \leq 2$, on the interval $[-1, +1]$, it is monotone increasing (Douady, Hubbard, Thurston). In general for C^∞ maps we have the proof of the C^∞ entropy conjecture by Yomdin, that gives a lower bound for the topological entropy, $h(f)$, $f : M \to M$

$$h(f) \geq \ln |\lambda|,$$

where λ is any eigenvalue the induced map on real homology, $f_* : H_*(M, R) \to H_*(M, R)$.

It is a pleasure for me to acknowledge conversations on these matters with Marty Golubitsky, Jacob Palis, and especially Charles Tresser.

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2-Knots and their groups, by Jonathan Hillman. Austral. Math. Soc. Lect. Ser., vol. 5, Cambridge University Press, New York, 1989, 164 pp., $19.95. ISBN 0-521-37812-5

Whereas the algebraic characterization of n-knot groups (fundamental group of the complement $S^{n+2}\backslash K^n$ of an n-knot K^n) is easy for $n \geqq 3$, and apparently hopeless for $n = 1$, the problem of characterizing algebraically the 2-knot groups is very challenging: It is certainly a difficult one, but with some optimism, may perhaps be viewed as not totally hopeless. This book gives, in rather condensed form, an essentially complete survey of the subject and a very good account of the status of the problem.

After an introductory Chapter 1 and a slick exposition of the classic background (i.e., of the results, which are more than ten years old) in Chapter 2, the book starts for good on page 36 with five chapters on the recent rather prolific developments of the algebraic study of 2-knot groups in the last ten years. Many of these results are in fact due to the author.

Thus, the book is intended for the working research topologist who wants to acquire a comprehensive idea of the status of the