

It may not be amiss here to remark on the price of this outstanding book. At approximately \$80 the price comes to roughly 40 cents per page! Ouch! But, don't leave; there are roughly 30 lines per page compared to the  $\approx 40$  lines per page of Van der Waerden's *Algebra* (Viertel Auflage, Springer-Verlag (1959)), which is of comparable size. Thus, set in the denser Springer-Verlag mode, this book would shrink to  $3/4$ 's the present number of 215 pages, to  $\approx 155$  pages, which comes to  $\approx 55$  cents per page while Springer books average  $\leq 20$  cents per page!

I sympathize with an author's plight: He does not set prices! Regardless, I recommend this excellent text for *those who can afford it*. I found nary a typo, and the treatment of the selected topics is not only lucid but impeccable. It brought the same delight that I experienced reading Kaplansky's non-parallel *Commutative Rings* and Lambek's *Lectures on Rings and Modules*, which is to say that the author's love and command of the subject shines on every page.

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*Harmonic analysis in phase space*, by Gerald B. Folland. Princeton University Press, Princeton, NJ, 1989, \$17.50 (paper), \$55.00 (cloth). ISBN 0-691-08528-5

"The phrase *harmonic analysis in phase space* is a concise if somewhat inadequate name for the area of analysis  $\mathbf{R}^n$  that involves the Heisenberg group, quantization, the Weyl operational calculus, the metaplectic representation, wave packets, and related concepts: It is meant to suggest analysis *on* the configuration space  $\mathbf{R}^n$  done by working *in* the phase space  $\mathbf{R}^n \times \mathbf{R}^n$ . The ideas that fall under this rubric have originated in several fields—Fourier analysis, partial differential equations, mathematical physics, representation theory, and number theory, among others. As a result,

although these ideas are individually well known to workers in such fields, their close kinship and the cross-fertilization they can provide have often been insufficiently appreciated. One of the principal objectives of this monograph is to give a coherent account of this material, comprising not just an efficient tour of the major avenues but also an exploration of some picturesque byways."

The paragraph above is taken from the preface to Folland's splendidly written book and is a very good summary of its contents. To put these contents into perspective let me remind you what the representation theory of the Heisenberg group looks like to group theorists. The salient facts are easy to summarize; they consist of the following four statements:

1. The irreducible infinite-dimensional representations of  $H_n$  are in one-to-one correspondence with the nonzero real numbers. Given any nonzero real number,  $\hbar$ , there exists a unique irreducible representation

$$(*) \quad \rho_{\hbar}: H_n \rightarrow U(V)$$

with the property that the restriction of  $\rho_{\hbar}$  to the center  $\mathbf{R}$  of  $H_n$  is the representation

$$(**) \quad t \rightarrow \exp(2\pi i \hbar t) I_V.$$

This result is known as the Stone-Von Neumann theorem.

2. Let  $A$  be an element of the group  $\mathrm{Sp}(n)$  and let  $\tau_A$  be the automorphism of  $H_n$  associated with  $A$ . By composing  $(*)$  with  $\tau_A$  we get another representation of  $H_n$  with the property  $(**)$ . Therefore, by the Stone-Von Neumann theorem, this representation has to be the Stone-Von Neumann representation in a new guise: i.e., there exists a unitary operator  $T_A: V \rightarrow V$  such that

$$\rho_{\hbar}(\tau_A x) = T_A^{-1} \rho_{\hbar}(x) T_A$$

for all  $x \in H_n$ . Moreover,  $T_A$  is unique up to a constant multiple of modulus one; so the map

$$(***) \quad A \rightarrow T_A$$

is a projective representation of  $\mathrm{Sp}(n)$  on  $V$ .

3. Let  $\mathrm{Mp}(n)$  be the double cover of  $\mathrm{Sp}(n)$ . Then, by adjusting the multipliers in front of the  $T_A$ 's one can make  $(***)$  into an honest unitary representation of  $\mathrm{Mp}(n)$ . This representation is known as the metaplectic (or oscillator, or harmonic, or Segal-Shale-Weil) representation.

4. The metaplectic representation is not irreducible; however, it splits into two components, an *even* component and an *odd* component; and both of these are irreducible.

The subject of Folland's book is the remarkable world of mathematical reality hiding behind the facade of the simple theory that I've outlined above. The representation  $\rho_{\hbar}$  was, in some sense, "discovered" in the twenties by the physicists when they noticed that the Heisenberg canonical commutation relations could be realized by the scheme:

$$Q_i \rightarrow \text{multiplication by } 2\pi x_i$$

and

$$P_i \rightarrow \frac{1}{\sqrt{-1}} \hbar \frac{\partial}{\partial x_i}.$$

Since then we have become accustomed to thinking of the underlying Hilbert space  $V$  of this representation as being  $L^2(\mathbf{R}^n)$ . However, as Folland points out, this representation occurs in nature in many other guises. For instance, it occurs as holomorphic functions on  $\mathbf{C}^n$  (Fock space), as sections of a certain line bundle over  $T^{2n}$  (the lattice description of  $\rho_{\hbar}$ ), as holomorphic functions on the generalized Siegel upper-half space, as sections of a certain line bundle over the Lagrangian Grassmannian, and (last but not least) as the symplectic analogue of the spin representation of the double cover of  $SO(2n)$ .

In each of these guises the representation theory of the Heisenberg group connects with other areas of mathematics: In its  $L^2(\mathbf{R}^n)$  guise it connects with classical quantum physics and with the theory of pseudodifferential operators. As Fock space it connects with quantum field theory (vacuum states, creation and annihilation operators, etc.) and with complex variables. As a line bundle over  $T^{2n}$  it connects with analytic number theory (in particular, the theory of the theta function) and with phenomena in solid-state physics (such as the de Haas-van Alphen effect). As the Siegel upper half-plane it connects again with analytic number theory (transformation properties of the theta function) and, finally, in its last guise, with such esoteric matters of modern elementary particle physics as BRS cohomology. All these marvelous interconnections are beautifully described in Folland's book. The only place I know of where the picture is laid out as well is in Mackey's book [2]. However, the overlap between these two books is not great. In particular, in Folland's book the focus of interest is on the implications of this whole picture for analysis. One has to go

to Hörmander's beautiful (but formidable) paper [1] to discover the full implications of  $\text{Mp}(n)$  invariance for pseudodifferential operator theory, but one can get a very good idea of what's going on from Folland's Chapters 2, 3, and 4. In these chapters the foundational material on "harmonic analysis on phase space" is presented so clear and accessible a fashion that one could very well use these chapters as a textbook. I have one small quibble with Folland's treatment of pseudodifferential operators: the use of the  $S_{\rho, \delta}^m$  symbol classes. One of Hörmander's major contributions to this subject was to show that, having by  $\text{Mp}(n)$  invariance given the  $p_i$ 's and  $q_i$ 's equal status, one could manufacture a symbol calculus in which this equal status was preserved.  $S_{\rho, \delta}^m$  symbols are, from this point of view, a bit of a cop-out. This small blemish, however, is outweighed by felicities too numerous to cite. (To cite a few: the proof of Calderon–Vaillancourt in §2.5, the discussion of Berezin's ideas in §2.7, the description of the F.B.I. transform in §3.3, and, above all, the wonderful digression on signal transmission in §3.4. This section alone is worth the price of admission.)

A final comment: How nice it would be if other authors of monographs of this length imitated Folland's example and supplied both an index *and* an index of symbols.

#### REFERENCES

1. L. Hörmander, *The Weyl calculus of pseudodifferential operators*, Comm. Pure Appl. Math. **32** (1979), 359–443.
2. G. Mackey, *Induced representations of groups and quantum mechanics*, Benjamin, New York, 1968.

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