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Géométrie algébrique réelle, by J. Bochnak, M. Coste and M.-F. Roy. Ergebnisse der Mathematik und ihrer Grenzgebeite, 3 Folge, Band 12, Springer-Verlag, Berlin, Heidelberg, and New York, 1987, x + 375 pp., $118.30. ISBN 3-540-16951-2

Nash manifolds, by Masahiro Shiota. Lecture Notes in Mathematics, vol. 1269, Springer-Verlag, New York, Berlin, Heidelberg, 1987, 273 pp., $20.00. ISBN 0-387-18102-4

One of the basic problems of mathematics is finding the solutions of a system of equations. The easiest case is when the equations are linear and the general form of the set of solutions is well known to every mathematician. The next case one could take is that of polynomials. Suppose we have a collection of polynomials $p_1, p_2, \ldots, p_m$ in n variables with coefficients in some field k. An algebraic set X is the set of common zeroes in k^n of such a set of polynomials, $X = \{x \in k^n \mid p_i(x) = 0,\ i = 1, \ldots, m\}$. Unless one restricts the problem quite a bit (say by taking the p_i's to be quadratic and $n = 3$ and $k =$ the real numbers $\mathbb{R}$) we are nowhere near to completely understanding algebraic sets. Even restricting the polynomials to be quadratic is no help since by adding new variables which are products of the old variables one can reduce the degrees of the polynomials to the point where they are quadratic. (For example $y^2 = x^3$ can be changed to the quadratic $y^2 = xz$, $z = x^2$.)

The study of algebraic sets spawned the field of algebraic geometry which is very active and attracts some of the best mathematicians. However, the natural development of algebraic geometry led to a shift in point of view from the algebraic sets themselves to the algebraic properties of the polynomials which define them. This shift in point of view was quite successful, modern algebraic geometry is a very beautiful and powerful theory with numerous deep results. Furthermore, if the field k is algebraically closed there is a close connection between an algebraic set and the algebraic properties of the polynomials which define it. In fact much of algebraic geometry restricts itself to algebraically closed fields. Take any algebraic geometry text and near the beginning you are very likely to find a statement such as "From now on we will assume our field k is algebraically closed." But a very important field, the real numbers, is not algebraically closed. As a result, if $k = \mathbb{R}$ there is not such a close connection between the geometry of an algebraic set and the algebra of its equations; or in any case the connection is much more subtle.

If an real algebraic set is studied by modern algebraic geometers, a different space is usually studied, namely the set of complex zeroes of p_1, p_k divided out by the complex conjugation involution with a certain non-Hausdorff topology (that is the set of closed points of the scheme

$$\mathrm{Spec}(\mathbb{R}[x_1, \ldots, x_n]/\mathcal{I})$$

where $\mathcal{I}$ is the ideal of real polynomials vanishing on X). While this space is easier to work with from the algebraic point of view, results about it do not always translate into meaningful results about the underlying algebraic set. There are notable exceptions to this, a good example is Hironaka's resolution of singularities which says that you can smooth out the singular points of schemes over certain fields. Since it is true for schemes over $\mathbb{R}$, it is not hard to show that it is true for real algebraic sets. But often the connection between real algebraic sets and schemes is not close enough. As a result, the successful study of real algebraic sets requires more techniques.

Additional techniques required to study real algebraic sets were developed over the years by various random individuals with random points of view. Results were scattered throughout the literature. *Géométrie algébrique réelle* by J. Bochak, M. Coste and M-F. Roy is an attempt to present the basic techniques of many of these points of view in a coherent form for the first time. It succeeds quite well. I will describe below many of the topics in *Géométrie algébrique réelle.*

From the topological point of view, real algebraic sets are quite different from their complex counterparts. For example, in contrast to the complex case, irreducible real algebraic sets need not be connected, their nonsingular points need not be dense and their image under a proper polynomial map need not be an algebraic set. This last property (illustrated by projecting a circle to the x-axis) gives rise to an offshoot of real algebraic geometry. Although the image of a real algebraic set under a polynomial map may not be a real algebraic set it is what is called a semialgebraic set—a set defined by polynomial equalities and inequalities. We may define a semialgebraic map to be a continuous map whose graph is a semialgebraic set. Then the Tarski-Seidenberg theorem says that the image of a semialgebraic set under a semialgebraic map is semialgebraic.

A semialgebraic set (and hence a real algebraic set) has the structure of a stratified set, i.e. a semialgebraic set is a disjoint union of a finite number of smooth manifolds (called strata) which are themselves semialgebraic sets. This stratification can be taken to satisfy certain niceness conditions called the Whitney conditions which regulate how one stratum can be contained in the closure of another. What is more, a compact semialgebraic set is triangulable, in fact semialgebraically isomorphic to a finite polyhedron. Thus semialgebraic sets form a very attractive collection of spaces, they are not very pathological, they include a large number of topological types and are more graceful than, say, polyhedra since they can be curved. In fact, anything you can draw in $\mathbb{R}^n$ can be closely approximated by a semialgebraic set (witness splines). There is even a movement to develop topology entirely in a semialgebraic setting.

On the other hand, the topology of real algebraic sets is more subtle. Sullivan noticed the even local Euler characteristic property which says that for any real algebraic set V and any $x \in V$ the relative Euler characteristic $\chi(V, V - x)$ is even. Equivalently, in any triangulation a simplex is a proper face of an even number of simplices. One consequence is that a compact real algebraic set has a fundamental homology class with $\mathbb{Z}/2\mathbb{Z}$ coefficients—just take any triangulation and let the fundamental class be the sum of the top dimensional simplices. The even local Euler characteristic condition will then guarantee that the boundary of this class is 0 so it represents a homology class. In particular, a compact k-dimensional real algebraic subset of a real algebraic set V gives an element of the homology group $H_k(V; \mathbb{Z}/2\mathbb{Z})$. The homology classes which come from compact real algebraic subsets of a real algebraic set V form a subgroup $H_*^{\mathrm{alg}}(V; \mathbb{Z}/2\mathbb{Z})$ of $H_*(V; \mathbb{Z}/2\mathbb{Z})$. We shall see that this subgroup turns out to be very important to understand.

The conditions above on triangulability and even local Euler characteristic give necessary conditions on the topology of a real algebraic set. One can also ask which topological spaces are homeomorphic to a real algebraic set. The answer is not yet completely known, however *Géométrie algébrique réelle* gives a nice exposition of the results for compact smooth manifolds. Nash, in a paper very much ahead of its time (1952), was able to show that any compact smooth manifold is a union of components of a real algebraic set. Later, Tognoli was able to show that any compact smooth manifold is diffeomorphic to a real algebraic set. Tognoli's method of getting rid of the unwanted components employed a very useful cobordism device which gives rise to the general principal that cobordism implies isomorphism. More particularly, if a topological entity X is cobordant with an algebraic entity then (after a sometimes considerable bit of work) X is isomorphic to an algebraic entity. Thus Tognoli's proof succeeded because Milnor gave explicit real algebraic generators for unoriented bordism of a point. The above principal explains the importance of $H_*^{\mathrm{alg}}(V; \mathbb{Z}/2\mathbb{Z})$ since unoriented bordism of a space is determined by $H_*(V; \mathbb{Z}/2\mathbb{Z})$.

Géométrie algébrique rélle develops all of the above topics. Other topological topics discussed are algebraic vector bundles and representing cohomotopy classes by polynomial or regular maps.

Another point of view of real algebraic sets is the algebraic point of view, to try to discover the relation between the algebra and the geometry of a real algebraic set. For example, a basic tool of algebraic geometry is the nullstellensatz which allows you to find the ideal of all polynomials which vanish on an algebraic set. For an algebraically closed field, it is an old theorem that this is just the radical of the ideal $\mathcal{J}$ generated by the polynomials $p_1, \dots, p_k$. The real nullstellensatz was not discovered until about twenty years ago, first by Dubois and then in a better form by Risler. For the reals, if the sum of squares of a bunch of polynomials is in $\mathcal{J}$, throw them all in—then take the radical.

The algebraic theory of real algebraic sets really starts with E. Artin's solution of Hilbert's 17th problem showing that a nonnegative polynomial on $\mathbb{R}^n$ is a sum of squares of rational functions. To prove this one needs to consider more general fields than the reals, but these fields share many properties with the reals. Thus for much of *Géométrie algébrique réelle* one considers real closed fields, fields R which are not algebraically closed but for which $R[\sqrt{-1}]$ is algebraically closed. E. Artin and Schreier developed the theory of real closed fields further, showing that being real closed is equivalent to having an order compatible with the field structure. Later, Lang contributed to the theory with the theory of real places. As mentioned above, there is a real nullstellensatz. Other results along this line are the positivstellensatz—characterizing nonnegative or positive polynomials on a closed semialgebraic set. Hilbert's 17th problem leads to a study of quadratic forms and such questions as how many squares are necessary.

The consideration of real closed fields other than $\mathbb{R}$ is not confined to the algebraic parts of *Géométrie algébrique réelle.* A large number of the topological results referred to above are proven for a general real closed field R, although the interpretation would seem strange to a conventional topologist. For example, homology results would refer to simplicial homology defined using simplices in R^n where, for example, R could be the real closed field of Puiseux series $\sum_{i=k}^{\infty} a_i x^{i/q}$, $a_i \in \mathbb{R}$.

One of the more recent developments in real algebraic geometry is the real spectrum. In mainstream algebraic geometry, one studies an algebraic set V by taking the ring A of polynomial functions on V and associating to it a topological space $\operatorname{Spec}(A)$ and a sheaf of functions on $\operatorname{Spec}(A)$. For an algebraically closed field, V and $\operatorname{Spec}(A)$ are quite similar. But for a real closed field, $\operatorname{Spec}(A)$ is very much bigger than V. The real spectrum $\operatorname{Spec}_r(A)$ invented by Coste and Roy seems to be a good replacement in many contexts. One has an injection $V \to \operatorname{Spec}_r(A)$ which actually gives a homeomorphism to its image (with the usual Euclidean topology on V). Moreover, if $S \subset V$ is a semialgebraic set one can associate to it a $\tilde{S} \subset \operatorname{Spec}_r(A)$ and a sheaf of continuous semialgebraic functions on $\tilde{S}$. One could not do this directly to S, since continuous semialgebraic functions might not form a sheaf on S (locally semialgebraic need not imply semialgebraic).

Finally, *Géométrie algébrique réelle* proves various relations between Witt rings and K-theory K_0 for rings of semialgebraic maps and polynomial maps.

Nash manifolds by M. Shiota is a more specialized book. Suppose one wished to develop a theory of algebraic manifolds. Although any compact smooth manifold is diffeomorphic to a nonsingular real algebraic set, one really needs a more convenient algebraic category to work in than that of nonsingular algebraic sets and rational functions. For example, you have no inverse function theorem since a rational function which is a diffeomorphism need not have a rational inverse. A simple example is $V = \operatorname{set}\{(x, y) \in \mathbb{R}^2 | y^3 + y = x\}$ and $\pi: V \to \mathbb{R}$ defined by $\pi(x, y) = x$. Then π is a diffeomorphism, but π^{-1} is not a rational function since the solution to a cubic is not a rational function. So to obtain a more workable theory it is necessary to allow yourself more functions. Thus we may define a C^r Nash map $0 \leq r \leq \omega$ to be a C^r function between Euclidean spaces whose graph is a semialgebraic set. *Géométrie algébrique réelle* develops the basic results of Nash functions for $r = \omega$. Having defined a C^r Nash map we may define a C^r Nash manifold to be a manifold obtained by gluing together a finite number of open semialgebraic sets with C^r Nash diffeomorphisms. A C^r Nash manifold is called affine if it has a C^r Nash imbedding in some $\mathbb{R}^n$. It turns out that any compact C^1 manifold has a unique affine C^ω Nash manifold structure. The nice thing about Nash manifolds is that they are endowed with certain properties not evident when one thinks of them as smooth manifolds. This was exploited by M. Artin and Mazur to obtain an exponential bound on the number of isolated periodic points for a dense set of automorphism of a manifold. Palais studied C^ω Nash manifolds extensively.

Shiota's book develops basic differential topology type results on C^r Nash manifolds. For example, if $0 < r < \infty$ then any C^r manifold is affine, but if $r = \omega$ it might not be. Notice that a compact PL manifold is a special case of a C^0 Nash manifold. For a while it was hoped that C^0 Nash manifolds might be more general than PL. But Shiota showed that any compact C^0 Nash manifold has a unique PL structure (although the complete proof does not appear in *Nash manifolds*). The most interesting result in Shiota's book is that any noncompact C^0 (or affine C^ω) Nash manifold is the interior of a unique compact C^0 (or affine C^ω) Nash manifold with boundary. This contrasts with the TOP, PL or smooth case where Siebenmann showed an open manifold need not be the interior of a compact manifold with boundary and if it is, there might be many different ways to compactify it. Thus Nash manifolds seem to offer an approach to differential topology which gives you automatic control at infinity.

In summary, both books do a good job in what they set out to do. Shiota's book is largely a research text developing a special topic. *Géométrie algébrique réelle* develops many of the techniques used in studying real algebraic sets. It is not really a fault of *Géométrie algébrique réelle*, but I should point out one topic not covered. Suppose one has a particular collection of real polynomials and one wants specific information about their solution. This is of obvious practical importance but is not addressed directly by the general theory presented in *Géométrie algébrique réelle*. However, it is clear that the development of all areas of real algebraic geometry will benefit greatly from the existence of *Géométrie algébrique réelle.*

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Abelian l-adic representations and elliptic curves, by Jean-Pierre Serre. Addison-Wesley Publ. Co., Reading, Mass., 1989, 140 pp., ISBN 0-201-09384-7

Addison-Wesley has just reissued Serre's 1968 treatise on l-adic representations in their Advanced Book Classics series. This circumstance presents a welcome excuse for writing about the subject, and for placing Serre's book in a historical perspective.

The theory of l-adic representations is an outgrowth of the study of abelian varieties in positive characteristic, which was initiated by Hasse and Deuring (see, e.g., [3, 1]) and continued in Weil's 1948 treatise [12]. Over the complex field $\mathbf{C}$, an abelian variety A of dimension g may be viewed as an (algebrizable) complex torus W/L, where $L \approx \mathbf{Z}^{2g}$ is a lattice in the $\mathbf{C}$-vector space W of dimension g. The classical study of A relies heavily on the lattice L, which is intrinsically the first homology group $H_1(A, \mathbf{Z})$. However, the quotients L/nL (for $n \geq 1$) have a purely algebraic definition. Indeed, over $\mathbf{C}$ the quotient L/nL is canonically the group

$$A[n] = \{P \in A \mid n \cdot P = 0\}$$

of n-division points on A. Over an arbitrary field K, one defines $A[n]$ as the group of points on A (with values in a separable closure $\overline{K}$ of K) of order dividing n. When n is prime to the characteristic of K, $A[n]$ is a free $\mathbf{Z}/n\mathbf{Z}$-module of rank $2g = 2\dim A$, just as in the classical case. Moreover, the module $A[n]$ carries natural commuting actions of the Galois group $\mathrm{Gal}(\overline{K}/K)$ and the ring $\mathrm{End}_K(A)$ of K-endomorphisms of A. Most information provided by L can be extracted from the collection of groups $A[l^\nu]$ $(\nu \geq 1)$, where l is a fixed prime which is different from the characteristic of K.