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MARK A. KON
BOSTON UNIVERSITY

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The homology of Hopf spaces, by R. M. Kane, North-Holland, Amsterdam, New York, Oxford and Tokyo, 1988, xv + 479 pp., $105.25. ISBN 0-444-70464-7

In 1988 Richard Kane published his book, *The homology of Hopf spaces*. I remember at the 1986 Arcata Topology Conference (before the International Congress of Mathematicians) when Alex Zabrodsky, John Harper, Clarence Wilkerson and I received mimeographed preprints of Richard's book, we were all pleasantly surprised that someone had taken the time to amass many of the details of this growing field into a coherent book. More recently, Frank Williams commented that Kane's book will probably be the only book on finite H-spaces published in the 1980s. For myself and others who have Ph.D. students working in the area, Kane's book is an excellent first reference for many of the ideas currently used by the experts.

An H-space (or Hopf space) is simply a pointed space X, $*$ together with a binary pairing $X \times X \to X$ such that the two inclusions

$$X \times * \to X \times X \to X$$
$$* \times X \to X \times X \to X$$

are homotopic to the identity.

Mathematicians are interested in these spaces because all topological groups are H-spaces, and further, if one takes an arbitrary topological space, its loop space is an H-space. The interplay between space and loop space has been an important area of study. For example, the homotopy and homology of space and loop space are intimately related by suspension maps. Further, all topological groups have the homotopy type of loop spaces, and those spaces that are infinite loop spaces can be used to define cohomology theories. It would take several volumes (of which there are several references) to expound on these topics; however, it is not the main focus of Kane's book.

Kane's book focuses primarily on the study of *finite* H-spaces. A *finite* H-space is an H-space that has the homotopy type of a finite complex. Recall a theorem of Malcev-Iwasawa that says a Lie group splits into a compact part producted with $\mathbb{R}^n$ so it has the homotopy type of a finite complex. Kane states in his Introduction:

> Finite H-space theory is an outgrowth of the homotopy theory of Lie groups. With the advent of algebraic topology in the 1930s and 1940s, mathematicians quite naturally began to study the homology and cohomology of Lie groups. It became apparent that some of the results obtained did not really depend on the entire Lie group structure but rather only on the much more limited structure captured in the finite H-space concept.
>
> The Hopf structure theorem for the rational cohomology of Lie groups was the first example of such a fact. Given a Lie group G, the H-space structure on G induces, what is now called, a Hopf algebra structure on $H^*(X;\mathbb{Q})$. The fact that $H^*(X;\mathbb{Q})$ is a *finite* Hopf algebra then forces $H^*(X,\mathbb{Q})$ to be an exterior algebra $E(x_1,\dots,x_r)$ where $|x_1| = 2n_i - 1$.
>
> The explicit concept of an H-space is due to Serre. Because of his interest in the path space fibration $\Omega X \to PX \to X$, he was led to consider loop spaces. The usual multiplication of loops is not associative. So he was led to introduce the idea of an H-space to describe the multiplication on ΩX. As we will see, finite loop spaces have been an object of intensive study. They are closely related to compact Lie groups.
>
> Throughout the 1950s and early 1960s mathematicians continued to analyse the homology of the Lie groups. The homology was calculated for the semi-simple compact Lie groups. In addition, a number of interesting general properties were obtained. Notably, we have Borel's mod p version of the Hopf decomposition as well as his study of the cohomology of classifying spaces. Bott's proof that $H_*(\Omega G)$ is torsion free in the 1-connected case and Hodgkin's proof that $K^*(G)$ is torsion free in the same case. We also cannot fail to mention Schereer's theorem that 1-connected compact Lie groups all have distinct homotopy types as well as the fact that, throughout this period, the only known examples of connected finite H-spaces were

> products of Lie groups, S^7 (=the units in the Cayley numbers) and $\mathbb{R}P^7$.
>
> After this period the focus in homotopy theory shifted away from Lie groups in themselves to the more general category of finite H-spaces. So it is at this point that our book begins its tale. We should note, however, the Lie groups have continued to provide a major stimulant to the development of finite H-space theory. As we will see, it has been mainly concerned with understanding the above Lie properties in the more general context of finite H-space theory. So Lie groups might be described as the experimental data of H-space theory.

The main questions asked are how do the cohomology and homology of such H spaces differ from that of Lie groups? Present knowledge leads us to the following conjectures.

CONJECTURE 1. *The* mod 2 *cohomology of a simply connected finite H-space is isomorphic as a Hopf algebra over the Steenrod algebra to the* mod 2 *cohomology of a Lie group producted with seven spheres.*

CONJECTURE 2. *The rational cohomology of a simply connected finite loop space is isomorphic as a Hopf algebra to the rational cohomology of a Lie group.*

Kane's book is by no means a comprehensive account of developments in the field of finite H-spaces. Instead he chooses to focus on just a few of the major developments in the field. The main body of his book is divided into three subheadings.

A. The study of finite H-spaces with torsion free homology. The main breakthrough in the last 15 years has been the classification of the mod p cohomology rings of finite loop spaces that have no p torsion in their integral cohomology. This is the main focus of Kane's exposition. Most of this work is due to Adams and Wilkerson, Clark and Ewing and a number of others.

If $X = \Omega B$ is a finite loop space that has no p torsion, work of Borel and Browder shows that the following statements are equivalent:

(a) $H^*(X;\mathbb{Z}_p) = \bigwedge(x_1,\dots,x_r)$

$$\deg x_i \leq \deg x_{i+1}, \quad \deg x_i \text{ odd}$$

and

(b) $H^*(B;\mathbb{Z}_p) = \mathbb{Z}_p[y_1,\dots,y_r]$

$$\deg y_i = \deg x_i + 1.$$

This leads us to the study of

Steenrod's problem. Classify all polynomial algebras $\mathbb{Z}_p[y_1,\dots,y_r]$ that can occur as the mod p cohomology of a space.

Implicit in Steenrod's Problem are two smaller problems

(1) The realization of spaces B with

$$H^*(B;\mathbb{Z}_p) = \mathbb{Z}_p[y_1,\dots,y_r]$$

(2) The necessary conditions for $\mathbb{Z}_p[y_1,\dots,y_r]$ to be compatible with the action of the Steenrod Algebra.

In a nutshell, Clark and Ewing construct spaces whose mod p cohomology is polynomial. Adams and Wilkerson show via (2) that the spaces constructed by Clark and Ewing are all of them if the prime p does not divide the degrees of the y_i. This tour de force is no simple matter and an outline of the proofs is givin in the book.

What should appeal to the topologist who is not an expert in finite H-spaces is that here is a general problem in topology: Given an algebra, when it is realizable as the cohomology of a space? In the case of Steenrod's Problem, it can be rephrased in H-space terms: Given a topological space, when does its loop space have mod p cohomology an exterior algebra? The interplay between spaces and their loop spaces is an ongoing theme in Kane's book. The solution of Steenrod's Problem described here must be considered one of the great breakthroughs of the last twenty years.

It should be noted that this work is closely related to Conjecture 2 on the rational cohomology of a finite loop space. If $X = \Omega B$ is a simply connected finite loop space then if X has no p torsion, we have

$$H^*(B;\mathbb{Z}_p) = \mathbb{Z}_p[y_1,\dots,y_r]$$

and $H^*(B;\mathbb{Q}) = \mathbb{Q}[z_1,\dots,z_r]$ where $\deg y_i = \deg z_i = n_i$, $n_1 \le n_2 \le \cdots \le n_r$.

The sequence of numbers $[n_1,\dots,n_r]$ is the "type" of X and this sequence completely determines the rational cohomology of X. Work of Aguade and Lin [**1**, **6**] shows

THEOREM. *Every simply connected finite loop space X has type*
(1) (*Aguade*) *a union of Lie types and* $[4,24]s$ *and* $[12,16]s$.
(2) (*Lin*) *a union of Lie types and* $[4,16]s$, $[4,24]s$ *and* $[4,48]s$.
In particular if $\pi_3(X) = \mathbb{Z}$ then X has Lie type.

These results, however, fall short of answering Conjecture 2. Adams and Wilkerson note that the type

$$[4,4,4,8,8,8,12,12,16,16,20,24,24,28]$$

admits the structure of a finite loop space at all primes greater than three.

B. Necessary conditions for a finite H-space to have homology torsion. Here is a reasonable question:

Classify the mod p cohomology rings of all finite H-spaces that have p torsion in their integral homology.

The main thrust of Kane's exposition is to describe progress made on this question.

There is a big difference between finite H-spaces with p torsion and finite H-spaces that are p torsion free. Essentially, p torsion free finite H-spaces have cohomology an exterior algebra on generators of odd degree whereas finite H-spaces with p torsion can have even degree generators or generators truncated at heights greater than two.

When a finite H-space X has homology p torsion the approach described in A doesn't work even if X is a loop space. In fact, $H^*(B;\mathbb{Z}_p)$ becomes very complicated and is not even known for the Lie group E_8 at the prime 2. A different approach is needed. Another fact to notice is that an odd sphere localized at an odd prime is an H-space, so one cannot hope to control the degrees of odd generators in the mod p cohomology of a finite H-space when p is odd. All one can hope for at odd primes is some necessary conditions for the existence of even degree generators in the mod p cohomology of a finite H-space.

So the problem divides into two smaller problems

(1) The necessary conditions for the existence of an even degree cohomology generator in $H^*(X;\mathbb{Z}_p)$ for p an odd prime.

(2) The necessary conditions for the existence of any generator in $H^*(X;\mathbb{Z}_2)$.

In the last twenty years, techniques due to Browder, Hubbuck, Kane, Lin, Thomas, Williams and Zabrodsky and others have yielded surprisingly sharp restrictions on the degrees of generators in (1) and (2). Most of the papers (but not all) use the following key facts:

(a) (Kane) Given $t \in PH_{2l}(X;\mathbb{Z}_p)$ for p odd, $t^p = 0$.

(b) (Browder) Given $s \in PH_l(X;\mathbb{Z}_2)$, $s^2 = 0$.

Properties (a) and (b) can be exploited by the use of secondary cohomology operations to yield the following answers to Problems 1 and 2:

THEOREM 1. (1) *(Lin)* $QH^{\text{even}}(X;\mathbb{Z}_p) = \sum \beta_1 \mathscr{P}^l QH^{2l+1}(X;\mathbb{Z}_p)$ *for p odd.*

(2) *(Kane, Lin)* $QH^{\text{even}}(X;\mathbb{Z}_2) = 0$.

(3) *(Kane, Lin)* $H_*(\Omega X;\mathbb{Z})$ *is torsion free.*

The proof is quite lengthy; Kane patiently outlines most of it. In addition, he introduces the use of Morava k-theory to study finite H-spaces. Part (3) of Theorem 1 is a triumph of homological techniques; the corresponding theorem for Lie groups was originally proved by Bott using Morse theory.

This part of the book must be read in conjunction with the actual research papers. Several readers have remarked that it is difficult to see the motives behind the arguments. On the other hand, I'm impressed that Kane had the patience to actually wade through so much of the actual research.

Since the publication of the book, much has been done to also describe the odd cohomology generators of X *when $H_*(X;\mathbb{Z}_2)$ is an associative ring.* We have the following theorem which generalizes an old theorem due to Thomas.

THEOREM 2 *(Lin)* [7]. (1) $QH^{2^r+2^{r+1}k-1}(X;\mathbb{Z}_2) = Sq^{2^r k} QH^{2^r+2^r k-1}(X;\mathbb{Z}_2)$ *for $k > 0$, $r \geq 0$.*

(2) *The first nonvanishing homotopy group of a finite H-space occurs in degrees* 1, 3, 7 *or* 15.

It should be noted that the study of the cohomology rings of finite H-spaces is by no means a closed subject. There are many recent developments that are both exciting and will certainly yield new ideas. I list a few here:

(1) Hemmi's work [3] on mod 3 homotopy associative three torsion free H-spaces. In this context, we have $H^*(X;\mathbb{Z}_3) = \bigwedge(x_1, x_2, \dots, x_r)$ where $\deg x_i$ odd. The assumption of homotopy associativity under certain circumstances places restrictions on the action of the Steenrod algebra.

(2) Jeanneret and Suter's (unpublished) use of K-theory to study the algebra

$$\mathbb{Z}_2\frac{[x_{15}]}{x_{15}^4} \otimes \bigwedge(x_{23}, x_{27}, x_{29})$$

and to determine that such an algebra cannot be the mod 2 cohomology of a space. This work suggests that K-theory can, at times, be used to study finite H-spaces even in the presence of torsion.

(3) The work of Goncalves, Hubbuck, Iwase, Lin and Williams [4, 5, 8] on the products of seven spheres with H-spaces. It is well known that S^7 is not homotopy associative, but it is unknown whether or not products of S^7's with H-spaces can become homotopy associative. For example at the prime 2, Spin(8) has the homotopy type of $G_2 \times S^7 \times S^7$.

(4) Slack's work [9] on higher homotopy commutativity obstructions. Early work of Dyer, Lashof and Kudo and Araki, Sugawara, Browder and Cohen created homology operations on iterated loop spaces. One of the real problems with these operations is that it was difficult to prove they were nonzero. Slack, in his thesis, has developed an obstruction theory which detects the dual of the Dyer Lashof operations as higher order cohomology operations which can then be computed.

C. The construction of finite H-spaces with homology torsion. In this section, Kane describes work of Harper (which was later revised with Zabrodsky) to construct non-Lie finite H-spaces for each prime p that have p torsion in their homology.

In particular we have the following theorem.

THEOREM (HARPER). *For each odd prime p there exists a* mod p *H-space $X(p)$ with*

$$H^*(X(p);\mathbb{Z}_p) = \bigwedge(x_3, x_{2p+1}) \otimes \mathbb{Z}_p\frac{[x_{2p+2}]}{(x_{2p+2})^p}.$$

Harper's theorem is important for several reasons. First, prior to this result all known finite H-spaces with p torsion were Lie groups and they have torsion only at the primes 2, 3 and 5. Thus, here were examples of finite H-spaces with torsion at any prime. Secondly, this result fits with work of Kane and Lin on the loop space conjecture which states that the even degree generators must occur in the form $\beta_1\mathscr{P}^l x_{2l+1}$. Here, $x_{2p+2} = \beta_1\mathscr{P}^1 x_3$. Finally, a subtle result of Kane shows that such a complex cannot be homotopy associative.

Kane uses the proof of this result developed by Harper and Zabrodsky. This allows him to introduce the concept of "power space," which is another technique used by H-space theorists (Harper, McGibbon, Zabrodsky) to restrict liftings.

Many of the techniques described in the book have found their way into the study of nonfinite H-spaces, and would be worthy of recording in another book. For example;

(1) The work of Lemaire and Anick on the irrationality of the Poincaré Series for the loops on a finite complex.

(2) The work of Cohen, Neisendorfer, Moore, Selick and others on the exponents for the homotopy groups of spheres.

(3) The study of H-spaces whose $\bmod p$ cohomology is finitely generated as an algebra versus those H-spaces whose $\bmod p$ cohomology is finite dimensional.

It should be mentioned that throughout the book one senses the overriding influence of a great friend and mathematician, Alex Zabrodsky, who died in a car accident. It's reasonable to say that Alex introduced the concepts of mixing homotopy type, H-deviation in conjunction with secondary operations, power spaces to study liftings, and deviations from homotopy commutativity and homotopy associativity. Several of Alex's unpublished manuscripts still sit on many of our shelves. If only he were here to lend us some more insight! (John Harper has mentioned that Alex is probably smiling at us now from above.)

In summary, in the span of 480 pages, Kane gives us a brief overview of recent developments in the field that are not described in any other book. Students of mine who read the book for the first time say that it is an excellent companion to the research papers and has just enough of a summary to assist in filling the gaps in their understanding. In addition there is a comprehensive reference list and an appendix that gathers most of the known information about the cohomology of Lie groups.

Even though Kane has restricted himself to finite H-spaces, I would imagine that many of the techniques described in his book will be of use to any algebraic topologist who finds himself looping a topological space.

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James P. Lin
University of California, San Diego