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AMENABLE GROUP ACTIONS ON THE INTEGERS; AN INDEPENDENCE RESULT

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Issues involving the uniqueness of Lebesgue measure led to questions as to what extent a group G acting on a set X determines the structure (and number) of G-invariant finitely additive probability measures on X (*G-invariant means*). For example, Banach **[B]** showed that there was more than one rotation invariant finitely additive probability measure on the measurable subsets of S^1. More recently Sullivan**[S]** and independently Margulis **[M1, M2]** (for $n \geq 4$) and Drinfield **[D]** (for $n = 2, 3$) showed that Lebesgue measure is the unique finitely additive rotation invariant measure on the Lebesgue measurable subsets of S^n. An easier example is the following: Let μ be any two-valued finitely additive probability measure defined on all subsets of the natural numbers. Let G be the group of all permutations of $\mathbf{N}$ that are equal to the identity function on a set of μ-measure one. Then μ is the unique G-invariant finitely additive probability measure on $\mathscr{P}(\mathbf{N})$. Rosenblatt, noting that all of the known instances of uniqueness involved nonamenable groups, asked whether an amenable group G acting on a set X could uniquely determine a finitely additive invariant probability measure on X. (For amenable groups invariant measures always exist.) The main result of this note (Corollary to Theorem 3) is that in the concrete case of locally finite (hence amenable) groups acting on the natural numbers, the question of whether there is a G with a unique invariant mean is independent of the standard axioms for mathematics (Zermelo-Fraenkel set theory with the axiom of choice; ZFC).

We begin with some negative results. Let G act on a set X. Rosenblatt and Talagrand **[RT]** showed that if G is nilpotent, then G does not determine a unique invariant mean on X. Krasa **[K]** improved this to solvable groups.

Let $\mathbf{N}!$ be the group of permutations of the natural numbers with the topology of pointwise convergence. Recall that an *analytic* subset A of $\mathbf{N}!$ is a projection of a Borel set $B \subseteq \mathbf{N}! \times \mathbf{N}!$ onto the first coordinate. In particular any Borel set is analytic.

THEOREM 1. *If $G \subseteq \mathbf{N}!$ is an analytic amenable group then G does not determine a unique invariant mean on $\mathscr{P}(\mathbf{N})$.*

COROLLARY. *No countable group determines a unique invariant mean.*

The idea of the proof is to show that if G has a unique invariant mean μ, then this mean is determined by G in a very concrete (positive $\Sigma_1^1(G)$)

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way. If G had the property of Baire, then so would μ, but no nonatomic finitely additive probability measure can have the property of Baire. As a corollary of the proof, one sees that Large Cardinal axioms imply that there is no projective subgroup of $\mathbf{N}!$ with a unique invariant mean (although there is such a P.C.A. group in $\mathbf{L}$).

The relationship between Theorem 1 and the results involving the algebraic structure of groups is not known.

Yang [Y] showed that under the Continuum Hypothesis there is a locally finite group of permutations of $\mathbf{N}$ admitting a unique invariant mean $\mu:\mathscr{P}(\mathbf{N}) \to [0,1]$. Yang's mean is surjective. Yang asked whether the result followed from Martin's Axiom.

THEOREM 2. *Assume Martin's Axiom and let μ be any two-valued, nonatomic finitely additive probability measure defined on $\mathscr{P}(\mathbf{N})$. Then there is a locally finite group G of permutations of $\mathbf{N}$ having μ as its unique invariant mean.*

In the proof we use the following well-known fact.

FACT. Assume *Martin's Axiom.* Let U be a nonprincipal ultrafilter on $\mathbf{N}$ and enumerate $\mathscr{P}(\mathbf{N})\backslash U = \langle Y_\alpha : \alpha \in c\rangle$. Then there is an almost disjoint sequence $\langle A_\alpha : \alpha \in c\rangle \subseteq \sim\ U$ such that

(1) $A_\alpha \cap Y_\beta$ is finite for $\beta < \alpha$,

(2) $A_\alpha \cap Y_\alpha = \varnothing$.

Sketch of the proof of Theorem 2. Let μ be a two-valued finitely additive probability measure on $\mathscr{P}(\mathbf{N})$. Then $\{X: \mu(X) = 1\}$ is an ultrafilter. Let $\langle Y_\alpha : \alpha \in c\rangle = \mathscr{P}(\mathbf{N})\backslash U$ and let $\langle A_\alpha : \alpha \in c\rangle$ be as in the fact. (This is the only use of Martin's Axiom.)

Let $\langle A_\alpha^n : n \in \mathbf{N}\rangle$ be a partition of A_α into infinite disjoint sets. Choose bijections $s_\alpha^n : A_\alpha^n \to A_\alpha^{n+1}$ and $t_\alpha : Y_\alpha \to A_\alpha^0$. Our group G will be generated by $\{\sigma_\alpha^n : n \in \mathbf{N}, \alpha < c\} \cup \{\tau_\alpha : \alpha < c\}$ where

$$\sigma_\alpha^n(k) = \begin{cases} k, & k \notin A_\alpha^n \cup A_\alpha^{n+1}, \\ s_\alpha^n(k), & k \in A_\alpha^n, \\ (s_\alpha^n)^{-1}(k), & k \in A_\alpha^{n+1}, \end{cases} \qquad \tau_\alpha(k) = \begin{cases} k, & k \notin Y_\alpha \cup A_\alpha^0, \\ t_\alpha(k), & k \in Y_\alpha, \\ t_\alpha^{-1}(k), & k \in A_\alpha^0. \end{cases}$$

Note that every element of G is equal to the identity on a set of μ-measure one, hence μ is G-invariant. Suppose $\mu(Y) = 0$. Then $Y = Y_\alpha$ some α. But Y_α is G-isomorphic to A_α^n for all n. Hence Y has measure zero for all G-invariant probability measures, so μ is the unique G-invariant probability measure.

It remains to show that for all $\alpha_1, \dots, \alpha_k$ and all N, $\{\sigma_{\alpha_i}^n : i < k, n \leq N\} \cup \{\tau_{\alpha_i} : i < k\}$ generates a finite group H. The proof of this shows by induction on α_k that there is a $B \in \mathbf{N}$ such that for all $m \in \mathbf{N}$, $|Hm| \leq B$. $\square$

Two remarks are appropriate. First, the construction shows that under Martin's Axiom, for any $n \in \mathbf{N}$ there is a locally finite group G such that the collection of G-invariant means has dimension n. Secondly, the construction is only as complicated as a well ordering of the reals. So in $\mathbf{L}$ there is a Σ_2^1 group with a unique invariant mean.

R. McKenzie [M] showed that every locally finite group can be embedded in a locally finite group of the same cardinality with no subgroup of countable index. As a consequence, assuming Martin's Axiom, there are locally finite groups $G < H$ such that there is a G-action on $\mathbf{N}$ admitting a unique invariant mean, but every H-action on $\mathbf{N}$ admits many invariant means. The author conjectures that under Martin's Axiom there is a locally finite group G and two injections $\varphi_1, \varphi_2 : G \to \mathbf{N}!$ such that φ_1"G admits a unique invariant mean but φ_2"G admits many (and both actions are transitive). This would show that "admitting a unique invariant mean" is not an algebraic property.

In the following M will denote a model of the axioms of mathematics and $M[\mathscr{F}]$ will denote another model extending M, constructed by the method of forcing **[Ku]**.

THEOREM 3. *Let $M \models ZFC + C.H.$ Let $\mathscr{F}$ be M-generic for the partial ordering adding $\aleph_2$-Cohen reals. Then $M[\mathscr{F}] \models$ "every locally finite group of permutations of $\mathbf{N}$ has at least two invariant means."*

COROLLARY. *The following statement is independent of ZFC,*

(*) *There is a locally finite group of permutations of $\mathbf{N}$ with a unique invariant finitely additive probability measure.*

PROOF OF COROLLARY. Theorem 2 (or Yang's theorem) shows (*) is consistent with ZFC. Theorem 3 shows that it is consistent with ZFC that (*) is false.

SKETCH OF PROOF OF THEOREM 3. Towards a contradiction, let $G \in M[\mathscr{F}]$ be a locally finite group with a unique invariant mean μ, where $\mathscr{F} = \langle C_\alpha : \alpha < \omega_2 \rangle$. We view each Cohen real C_α as a subset of $\mathbf{N}$. We may assume that $\aleph_2$ of the Cohen reals have $\mu(C_\alpha) < 1$, otherwise we replace each real by its complement. Standard amenability considerations show that for each Cohen real C_α with $\mu(C_\alpha) < 1$ there is a finite subgroup $H_\alpha \subseteq G$ such that for all $n \in \mathbf{N}$, $H_\alpha \cdot n \nsubseteq C_\alpha$. Let τ_α be a term for H_α.

Usual arguments show that for a set $S \subseteq \aleph_2$ of size $\aleph_2$,

(1) The supports of $\{\tau_\alpha : \alpha \in S\}$ for a Δ-system with kernel K.

(2) $M[\langle C_\alpha : \alpha \in K \rangle] \models$ C.H. so we may assume $K = \varnothing$ and the supports are disjoint.

(3) Each of the supports, ρ_α, has the same order type and for all $\alpha, \beta \in S$, $\tau_\alpha | \rho_\alpha$ is isomorphic to $\tau_\beta | \rho_\beta$ (in the obvious sense).

By taking a bijection $b : \omega \to \text{o.t.}(\operatorname{supp} \tau_\beta)$ we show:

There is a term τ in $M^{\mathbf{P}}$, where $\mathbf{P}$ is the partial ordering for adding ω Cohen reals $\langle x_n : n \in \omega \rangle$ such that: if $\mathscr{F}_1 \times \mathscr{F}_2 \subseteq \mathbf{P} \times \mathbf{P}$ is M-generic, then

(a) $\tau^{M[\mathscr{F}_i]}$ is a finite group of permutations of $\mathbf{N}$,

(b) for all $m \in \mathbf{N}$, $x_0 \not\supseteq \tau^{M[\mathscr{F}_i]} m$,

(c) $\langle \tau^{M[\mathscr{F}_1]}, \tau^{M[\mathscr{F}_2]} \rangle \subseteq \mathbf{N}!$ is finite.

We claim that this yields a contradiction. Let $(p, q) \in \mathbf{P} \times \mathbf{P}$,

$$(p, q) \models |\langle \tau^{M[\mathscr{F}_1]}, \tau^{M[\mathscr{F}_2]} \rangle| \leq B.$$

Using (b) one proves:

SUBCLAIM. If $p' \leq p$ and $\operatorname{supp} p' \subseteq N \times N$ and if $X \subseteq \mathbf{N}$, $|X| \geq BN + 1$ there is an $m \in X$, such that $\{k: ||k \in \tau^{M[\mathscr{F}_1]}m|| \wedge p' \neq 0\}$ is infinite (and similarly for $q' \leq q$).

Applying the subclaim, one builds a sequence $k_0, k_1, \ldots, k_{B+1} \in \mathbf{N}$ and a descending sequence $\langle (p_i, q_i): i \leq B+1\rangle \subseteq \mathbf{P} \times \mathbf{P}$ with $(p_0, q_0) = (p, q)$ such that $(p_{2i}, q_{2i}) \Vdash k_{2i+1} \in \tau^{M[\mathscr{F}_1]}k_i$ and $(p_{2i+1}, q_{2i+1}) \Vdash k_{2i+2} \in \tau^{M[\mathscr{F}_2]}k_{2i+1}$. This implies that $(p_{B+1}, q_{B+1}) \Vdash$ "the $\langle \tau^{M[\mathscr{F}_1]}, \tau^{N[\mathscr{F}_2]}\rangle$-orbit of k_0 has size $> B$." This contradicts the choice of (p, q).

One can also find other information about groups with invariant means, in particular:

THEOREM 4. *(a) Martin's Axiom implies that every locally finite group of cardinality $< c$ has 2^{2^c} invariant means.*

(b) *In the model $M[\aleph_2$ iterated Sacks reals $]$* **[B-L]** *(where $M \models ZFC +$ C.H.), there is a locally finite group of cardinality $\aleph_1$ with a unique invariant mean. Note that in this model $\aleph_1 < c$.*

So,while it is not possible for a countable amenable group to determine a unique invariant means, it is consistent and independent that there exists an amenable group of cardinality $< c$ which has a unique invariant mean.

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