BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 1, July 1989

CONJECTURE "EPSILON" FOR WEIGHT $k > 2$

BRUCE W. JORDAN AND RON LIVNÉ

Introduction. Let $N, k \geq 1$ be integers and suppose $\varepsilon\colon (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ is a Dirichlet character. Set

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \,\middle|\, c \equiv 0 \bmod N \right\}.$$

A holomorphic function f on the Poincaré upper half plane is called a cusp form of level N, weight k, and Nebentypus ε (or briefly a cusp form of type (N, k, ε)) if f has a zero at each cusp and

$$f\left(\frac{az+b}{cz+d}\right) = \varepsilon(d)(cz+d)^k f(z) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

The space of all such cusp forms is denoted $S_k(\Gamma_0(N), \varepsilon)$; this space is acted upon by the Hecke operators $\{T_n, n \geq 1\}$. A function $f \in S_k(\Gamma_0(N), \varepsilon)$ has a Fourier expansion

$$f = \sum_{n \geq 1} a_n(f) q^n = \sum_{n \geq 1} a_n q^n, \qquad q = e^{2\pi i z}.$$

It is said to be normalized if $a_1(f) = 1$. A normalized $f \in S_k(\Gamma_0(N), \varepsilon)$ which is an eigenfunction of all the Hecke operators has $a_n = a_n(f)$ lying in the ring of integers $\mathcal{O} = \mathcal{O}_f$ of the algebraic number field $K_f = \mathbf{Q}(a_n)_{n \geq 1}$.

Consider now a continuous irreducible representation

$$\rho\colon \operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\overline{\mathbf{F}}_l)$$

which is odd, i.e. $\det(\rho(c)) = -1$ where c is complex conjugation. Let $f \in S_k(\Gamma_0(N), \varepsilon)$ be a normalized eigenfunction of the Hecke operators and λ be a prime above l in the extension of K_f generated by the values of ε. Suppose that for each prime p unramified in ρ

$$\operatorname{Tr}(\rho(\operatorname{Frob}_p)) = a_p(f) \mod \lambda$$
$$\det(\rho(\operatorname{Frob}_p)) = \varepsilon(p) p^{k-1} \mod \lambda$$

where $\operatorname{Frob}_p \in \operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is a Frobenius element at p. Then we say that ρ arises from f. In **[Se 2]**, Serre conjectures that every such ρ arises from some cusp form f. He furthermore gives a procedure for determining from ρ the type (N, k, ε) of a modular form which gives rise to it. A second conjecture found in **[Se 1]**, implied by the conjecture above, asserts that in certain cases if such a ρ arises from a modular form then it arises from one of the predicted level. This second conjecture was isolated in the case of weight two and dubbed "Epsilon" because it was sufficient due

Received by the editors February 1, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11F33, 14G25.

to the construction of G. Frey to show that the Taniyama-Weil Conjecture implies Fermat's Last Theorem. K. Ribet subsequently proved "Epsilon," cf. **[Ri]**. His proof is based on the jacobians of Shimura and modular curves.

Both of the conjectures above can be viewed as first steps in articulating a "Langlands philosophy mod p" for the special case of the group $GL(2,\mathbf{Q})$. As such, a context other than jacobians must be found so that the settings of higher weight, other groups, etc. may be studied. It is the purpose of this note to outline the proof of a weight k analogue of Conjecture "Epsilon." Perhaps the formulations offered here will also be of use in considering more general groups.

THEOREM 1. *Let $k \geq 4$ be an even integer and p, l be distinct prime numbers, $l > k$. Suppose the irreducible representation ρ:* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\overline{\mathbf{F}}_l)$ *arises from $S_k(\Gamma_0(pM))$ with $(pl, M) = 1$. Assume ρ is unramified at p. Then ρ also arises from $S_k(\Gamma_0(M))$.*

REMARKS. (1) We can treat odd weight greater than 1 and Nebentypus of conductor prime to p, but for simplicity this is not considered here.

(2) The restriction $l > k$ may be inessential to our proof, but we definitely need $l > 2$.

(3) Detailed proofs shall be forthcoming.

To prove Theorem 1, we follow closely the leitfaden of Ribet's proof **[Ri]** in the case of weight 2, showing that all ingredients may be supplied for weight k. This work was partially supported by the U.S.-Israel Binational Science Foundation, and the first author was also partially supported by the National Science Foundation. It is a pleasure to thank G. Faltings and K. Ribet for their generous help and interest.

I. Sheaves on graphs attached to definite quaternion algebras and equivariant cohomology [Jo-Li]. Let Δ be the Bruhat-Tits tree for $SL_2(\mathbf{Q}_p)$ with vertices Ver Δ, and edges Ed Δ. Let $D/\mathbf{Q}$ be a quaternion algebra of discriminant $q\infty$, with q a prime, $q \neq p$. Fix an Eichler order $\mathscr{M} \subseteq D$ of conductor M, $(M, q) = 1$. Set $\overline{\Gamma} = \mathscr{M}[\frac{1}{p}]^\times$ and $\Gamma = \{\gamma \in \overline{\Gamma} | N_{D/\mathbf{Q}}(\gamma)$ has even p-adic valuation$\}$, viewed as discrete co-compact subgroups of $GL_2(\mathbf{Q}_p)$. They act on Δ and $Y = \Gamma\backslash\Delta$ is a finite graph. Let L be a $\mathbf{Z}[\frac{1}{p}]$-model of the representation Symm^{k-2} of $\mathscr{M}[\frac{1}{p}]^\times$; L is explicitly constructed in **[Jo-Li]**. For a prime $l \neq p$, set $L_l = L \otimes_{\mathbf{Z}[1/p]} \mathbf{Z}_l$. Regarding L as a constant sheaf over Δ and letting $\pi\colon \Delta \to \Gamma \backslash \Delta$ be the projection, set $\mathscr{L} = (\pi_* L)^\Gamma$ and $\mathscr{L}_l = \mathscr{L} \otimes_{\mathbf{Z}[1/p]} \mathbf{Z}_l$. Finally let $H^*_\Gamma(\Delta, -)$ denote Γ-equivariant cohomology as in **[Br]**.

We will now define the weight k analogues of the duals to the character groups of the Néron special fibers of jacobians of Shimura and modular curves. Let B be the indefinite rational quaternion algebra ramified at p and q. Denote by $V_B(M)$ the Shimura curve attached to an Eichler order of conductor M in B. Make the following definitions:

$$\hat{X}_q(X_0(qM))_l \stackrel{\text{def}}{=} C^0_{\overline{\Gamma}}(L_l) = \mathrm{Hom}_{\overline{\Gamma}}(\mathrm{Ver}\,\Delta, L_l)$$
$$C^0_{\Gamma}(L_l) = \mathrm{Hom}_{\Gamma}(\mathrm{Ver}\,\Delta, L_l)$$
$$\hat{X}_q(X_0(pqM))_l \stackrel{\text{def}}{=} C^1_{\Gamma}(L_l) = \mathrm{Hom}_{\Gamma}(\mathrm{Ed}\,\Delta, L_l)$$
$$\hat{X}_p(V_B(M))_l \stackrel{\text{def}}{=} H^1(Y, \mathscr{L}_l) = H^1_{\Gamma}(\Delta, L_l).$$

All of the above modules are equipped with natural inner products. Moreover Hecke operators constructed from D act. In addition, let $\alpha \in \mathscr{M}[\frac{1}{p}]^\times$ be an element of norm p. Then α induces an operator w_p on $\hat{X}_p(X_0(pqM))_l$ and $\hat{X}_{q/p}(V_B(M))_l$ with w_p^2 acting as multiplication by p^{k-2}. There is an isomorphism

$$\hat{X}_q(X_0(qM))_l^2 = C^0_{\overline{\Gamma}}(L_l) \oplus C^0_{\overline{\Gamma}}(L_l) \xrightarrow{\approx} C^0_{\Gamma}(L_l).$$

Theorem 2 (The Ribet exact sequence). *There is an exact sequence*

$$0 \to \hat{X}_q(X_0(qM))_l^2 \xrightarrow{i} \hat{X}_q(X_0(pqM))_l \to \hat{X}_p(V_B(M))_l \to 0$$

compatible with the action of Hecke (i.e., equivariant for Hecke operators $T(l)$, $l \neq p$, and known action for $T(p)$).

This Ribet Exact Sequence is simply the cellular resolution for equivariant cohomology

$$C^0_{\Gamma}(L_l) \xrightarrow{d} C^1_{\Gamma}(L_l) \to H^1_{\Gamma}(\Delta, L_l) \to 0.$$

There is a similar sequence of $\mathbf{Z}[\frac{1}{p}]$-modules when we replace L_l by L.

Higher weight analogues to the group of connected components of the Néron special fibers of jacobians of Shimura and modular curves can also be defined. For $r = p$ or q and C the curve $X_0(rM)$, $X_0(pqM)$ or $V_B(M)$, set

$$\Phi_r(C)_l = \hat{X}_r(C)_l / \hat{X}_r(C)_l^{\widehat{}}.$$

If C is an elliptic modular curve, then $\Phi_r(C)_l = 0$ (recall that $l \neq 2, 3$). For Shimura curves these groups admit the following description. Let $X_l = \hat{X}_p(V_B(M))_l$. Then

$$\Phi_p(V_B(M)) \approx (X_l/(p^{(k-2)/2}(1+p) + T(p))X_l)$$
$$\oplus (X_l/(p^{(k-2)/2}(1+p) - T(p))X_l).$$

Theorem 3. *$\Phi_p(V_B(M))_l$ is a module of fusion relative to the decomposition*

$$S_k(\Gamma_0(pqM))^{q-\mathrm{new}} = [S_k(\Gamma_0(qM))^{q-\mathrm{new}}]^2 \oplus [(S_k(\Gamma_0(qM))^{q-\mathrm{new}})^2]^\perp.$$

Corollary 4. *There are infinitely many primes $q \not\equiv 1 \mod l$ such that ρ also comes from $f' \in S_k(\Gamma_0(pqM))^{q-\mathrm{new}}$.*

This "raising the level" result is proved precisely as in [**Ri**].

II. Sheaf cohomology on modular and Shimura curves. The following results were proved by one of us a year ago. For a natural number N let $\phi\colon E \to Y_0(N)$ be the universal elliptic curve. Suppose $(l, N) = 1$ and let ϑ_l be the locally constant l-adic sheaf $\mathrm{Symm}^{k-2}(R^1\phi_*\mathbf{Z}_l)$ on $Y_0(N)$. On $V_B(M)$ with $(l, Mpq) = 1$ let ϑ_l be the sheaf $\mathrm{Symm}^{k-2}\mathscr{F}_l$ where $\mathscr{F}_l$ is defined as in **[Ca 1]**. Equivalently $\mathscr{F}_l$ may be defined by an idempotent splitting $\mathscr{F}_l \oplus \mathscr{F}_l \approx R^1\phi'_*\mathbf{Z}_l$, where $\phi'\colon A \to V_B(M)$ is the universal abelian surface. While E and A are defined only locally for the étale topology, ϑ_l and $\mathscr{F}_l$ make sense. On $Y_0(N)$ or $V_B(M)$, set $\overline{\vartheta}_l = \vartheta_l / l\vartheta_l$. Let r be p or q and let C be the curve $Y_0(rM)$, $Y_0(pqM)$ or $V_B(M)$. Accordingly set $\overline{C}$ equal to $X_0(rM)$, $X_0(pqM)$ or $V_B(M)$, respectively.

THEOREM 5. *Let $\Phi_r(\overline{C})[l](-1)$ be the (-1)-Tate-twist of the kernel of multiplication by l on $\Phi_r(\overline{C})_l$. Then there is a Hecke and $\mathrm{Gal}(\overline{\mathbf{F}}_r/\mathbf{F}_r)$-equivariant exact sequence*

$$0 \to H^1_{\mathrm{par}}(C \times \overline{\mathbf{F}}_r, \overline{\vartheta}_l) \to H^1_{\mathrm{par}}(C \times \overline{\mathbf{Q}}_r, \overline{\vartheta}_l)^I \to \Phi_r(\overline{C})_l[l](-1) \to 0,$$

where $I \subset \mathrm{Gal}(\overline{\mathbf{Q}}_r/\mathbf{Q}_r)$ is the inertia subgroup.

THEOREM 6. *There is an exact sequence of $\mathbf{T}$-modules*

$$0 \to \hat{X}_r(\overline{C})_l \otimes (\mathbf{Z}/l\mathbf{Z}) \to H^1_{\mathrm{par}}(C \times \overline{\mathbf{F}}_r, \overline{\vartheta}_l) \to (H^1_{\mathrm{par}})^{\mathrm{old}} \to 0$$

where $(H^1_{\mathrm{par}})^{\mathrm{old}}$ is 0 in the Shimura curve case and is isomorphic to two copies of $H^1_{\mathrm{par}}(Y_0(N/r), \overline{\vartheta}_l)$ when $C = Y_0(N)$. Again, some care must be taken regarding the action of T_r.

Define Frobenius at r to act on $\hat{X}_r(\overline{C})_l$ and $\Phi(\overline{C})_l$ via w_r if C is a Shimura curve and by $-w_r$ if C is an elliptic modular curve. Then the above short exact sequence is Frobenius-equivariant.

For the proof one uses the end of **[La]** and §11 of **[Ca 2]**.

III. Crystalline cohomology and Galois representations. Gerd Faltings has kindly explained to us how the relevant multiplicity 2 result can be deduced from applying his recent work on crystalline cohomology for open varieties with normal crossings compactifications **[Fa 1]** to the context studied in his previous work **[Fa 2]**. To set notation, let $N \geq 1$ be an integer, let $\mathbf{T}$ be the weight k Hecke algebra for $\Gamma_0(N)$ and let $m \subseteq \mathbf{T}$ be a non-Eisenstein maximal ideal of residual characteristic l. Denote the two dimensional representation attached to m by W and set $\overline{\vartheta}_l = \vartheta_l / l\vartheta_l$.

THEOREM 7 (FALTINGS). *Suppose that $(l, N) = 1$ and $l > \max(3, k)$. Then $H^1_{\mathrm{par}}(Y_0(N) \times \overline{\mathbf{Q}}, \overline{\vartheta}_l)[m]$ is two-dimensional over $\mathbf{T}/m$.*

PROOF (SKETCH). By **[Fa 1]**, the $\mathrm{Gal}(\overline{\mathbf{Q}}_l/\mathbf{Q}_l)$-representation

$$V = H^1_{\mathrm{par}}(Y_0(N) \times \overline{\mathbf{Q}}_l, \overline{\vartheta}_l)$$

is crystalline. Let $\mathbf{M}$ be the F-crystal corresponding to the dual of V. Then by imitating **[Fa 2]** one obtains on $\mathbf{M}$ a canonical Frobenius filtration

$$\mathbf{M} = F^0 \supset F^{k-1} \supset 0,$$

with $F^{k-1} \approx S_k(\Gamma_0(N)) \otimes \overline{\mathbf{F}}_l$. As in **[Ma]**, one proves that $F^0[m]$ is W-isotypical. Then multiplicity one for $S_k(\Gamma_0(N))$ implies that W occurs in $F^0[m]$ only once.

IV. The Proof of Theorem 1. Let $\mathbf{T}$ be the weight k Hecke algebra for $\Gamma_0(pM)$ and suppose $m \subset \mathbf{T}$ is the non-Eisenstein maximal ideal associated with ρ. Put $\mathbf{k} = \mathbf{T}/m$ and let W be the representation space of ρ. Hence $\dim_{\mathbf{k}} W = 2$.

THEOREM 8 (*After Mazur*). *Theorem* 1 *is true if* $p \not\equiv 1 \bmod l$.

PROOF. $W \subset H^1_{\mathrm{par}}(Y_0(pM) \times \overline{\mathbf{Q}}, \overline{\vartheta}_l)$ as a $\mathbf{T}[\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})]$-module since m is not Eisenstein. By Theorem 5, $W \subset H^1_{\mathrm{par}}(Y_0(pM) \times \overline{\mathbf{F}}_p, \overline{\vartheta}_l)$ as a $\mathbf{T}[\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)]$-module since ρ is unramified and $\Phi_p(X_0(pM))_l = 0$. If Theorem 1 is false, so that W has no intersection with

$$H^1_{\mathrm{par}}(Y_0(M) \times \overline{\mathbf{F}}_p, \overline{\vartheta}_l)^2,$$

then by Theorem 6 $W \subset \hat{X}_p(X_0(pM))_l \otimes \mathbf{Z}/l\mathbf{Z}$. Frob_p, w_p, and T_p coincide here up to sign, so in particular Frob_p is in $\mathbf{T}$ and acts on W via its image in $\mathbf{T}/m\mathbf{T} = \mathbf{k}$. Now $w_p^2 = \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)\gamma$ where $\gamma \in \Gamma$ and the scalar matrix $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ acts on Symm^{k-2} by p^{k-2}. Hence Frob_p acts on $W \subset \hat{X}_p(X_0(pM))_l \otimes \mathbf{Z}/l\mathbf{Z}$ by multiplication by $\pm p^{(k-2)/2}$ and the determinant of its action is p^{k-2}. On the other hand the determinant of Frob_p must be $\chi^{k-1}(\mathrm{Frob}\, p) = p^{k-1}$ where χ is the l-cyclotomic character. So $p^{k-2} \equiv p^{k-1} \pmod{l}$ and $p \equiv 1 \pmod{l}$.

This same argument together with multiplicity 2 (Theorem 7) gives (compare **[Ri]**).

THEOREM 9. *Suppose* $p \not\equiv 1 \bmod l$ *and* $X_l = \hat{X}_p(X_0(pM))_l$. *Then* $\dim_{\mathbf{k}} X_l/lX_l \leq 1$.

We are supposing W is unramified at p and comes from $S_k(\Gamma_0(pM))^{p-\mathrm{new}}$. By Corollary 4 there exists a prime $q > lM$ such that $q \not\equiv 1 \bmod l$, and such that W comes from $S_k(\Gamma_0(pqM))^{p-\mathrm{new},q-\mathrm{new}}$. Hence

$$W \subseteq H^1(V_B(M) \times \overline{\mathbf{Q}}_p, \overline{\vartheta}_l)^I.$$

Now apply Theorem 5: if W has a nonzero projection to $\Phi_p(V_B(M))_l$ then we are done since $\Phi_p(V_B(M))_l$ is a module of fusion between p-old and p-new forms by Theorem 3. By Theorem 6 we can suppose W lives in $\hat{X}_p(V_B(M))_l$. But now consider the Ribet Exact Sequence (Theorem 2):

$$0 \to \hat{X}_q(X_0(qM))_l^2 \to \hat{X}_q(X_0(pqM))_l \to \hat{X}_p(V_B(M))_l \to 0.$$

If m is in the support of $\hat{X}_q(X_0(qM))_l^2$, then we are done since by Theorem 8 we can remove the q from the level qM. So assume that m is not in the support of $\hat{X}_q(X_0(qM))_l^2$ and we obtain:

$$\begin{aligned} \mathrm{LHS} &\stackrel{\mathrm{def}}{=} \hat{X}_q(X_0(pqM))_l/m\hat{X}_q(X_0(pqM))_l \\ &\approx \hat{X}_p(V_B(M))_l/m\hat{X}_p(V_B(M))_l \stackrel{\mathrm{def}}{=} \mathrm{RHS}. \end{aligned}$$

W living in $\hat{X}_p(V_B(M))_l$ shows that $\dim_{\mathbf{k}}(\mathrm{RHS}) \geq 2$. But Theorem 9 shows that $\dim_{\mathbf{k}}(\mathrm{LHS}) \leq 1$, a contradiction.

REFERENCES

[Br] G. Bredon, *Equivariant cohomology theories*, Lecture Notes in Math., vol. 34, Springer-Verlag, Berlin and New York, 1967.

[Ca 1] H. Carayol, *Sur la mauvaise réduction des courbes de Shimura*, Compositio Math. **59** (1986), 151–230.

[Ca 2] ——, *Sur les représentations l-adiques associées aux formes modulaires de Hilbert*, Ann. Sci. École Norm. Sup. **19** (1986), 409–468.

[Fa 1] G. Faltings, *Crystalline cohomology and p-adic Galois-representations*, preprint Princeton University.

[Fa 2] ——, *Hodge-Tate structures and modular forms*, Math. Ann. **278** (1987), 133–149.

[Jo-Li] B. Jordan and R. Livné, *Integral Hecke structures arising from definite quaternion algebras* (in preparation).

[La] R. P. Langlands, *Modular forms and l-adic representations*, Antwerp II. Lecture Notes in Math., vol. 349, Springer-Verlag, Berlin and New York, 1973, pp. 361–500.

[Ma] B. Mazur, *Modular curves and the Eisenstein ideal*, Publ. Math. Inst. Hautes Études Sci. **47** (1978), 33–186.

[Ri] K. Ribet, *On modular representations of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *arising from modular forms*, preprint (M.S.R.I.).

[Se 1] J.-P. Serre, *Lettre à J.-F. Mestre*, Current Trends in Arithmetical Algebraic Geometry, Arcata 1985, Amer. Math. Soc., Providence, R.I., 1987, pp. 263–268.

[Se 2] ——, *Sur les représentations modulaires de degré* 2 *de* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Duke Math. J. **54** (1987), 179–230.

DEPARTMENT OF MATHEMATICS, BARUCH COLLEGE, CUNY, 17 LEXINGTON AVENUE, NEW YORK NEW YORK 10010

SCHOOL OF MATHEMATICS, BEVERLY AND RAYMOND SACKLER FACULTY OF EXACT SCIENCES, TEL-AVIV UNIVERSITY, RAMAT-AVIV 69978 ISRAEL