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KARL H. HOFMANN

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The analysis of linear partial differential operators. III, by Lars Hörmander, Grundlehren der mathematischen Wissenschaften, Vol. 274, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985, viii + 522 pp., $48.50. ISBN 3-540-13828-5

The analysis of linear partial differential operators. IV, by Lars Hörmander, Grundlehren der mathematischen Wissenschaften, Vol. 275, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985, vii + 351 pp., $45.00. ISBN 3-540-13829-3

These two volumes complete L. Hörmander's treatise on linear partial differential equations. They constitute the most complete and up-to-date account of this subject, by the author who has dominated it and made the most significant contributions in the last decades.

The subject of the book is the study, in general, of linear partial differential equations (the book does not deal with nonlinear problems, except some which are used as tools, in particular in symplectic geometry, e.g., for the construction of normal forms).

The first two volumes of the treatise mainly dealt with L. Schwartz's distribution theory, Fourier transformation, and differential operators with constant coefficients or perturbations of these. The author himself describes them as a—rather thorough and far-reaching—updating of his book of 1963. Let us note, among the many new topics described in these books, the Malgrange preparation theorem, the method of the stationary phase and oscillatory integrals, and the definition of the wavefront set, which are important tools for the last two volumes.

The two last volumes are more specifically a treatise on microlocal analysis and its applications. They arrive at a time when microlocal analysis, after twenty years, can be considered as mature and has given many fruits, and yet is still perfectly alive and promising.

It is hardly possible in a review that must remain of reasonable length to give a complete description of the contents of these two volumes. Their fourteen chapters occupy over 870 pages; each of these could well be a book by itself and deserves a review in its own right. So I will just try to explain what the book is about, and list the questions dealt with.

On $\mathbf{R}^n$, or in a set of local coordinates, a linear partial differential equation can be written

$$P\left(x, \frac{\partial}{\partial x}\right)u = \sum_{|\alpha|<m} a_\alpha(x)\frac{\partial^\alpha u}{\partial x^\alpha} = v, \tag{1}$$

where the a_α are smooth functions, e.g., constants when P has constant coefficients. Let us recall the effect on an exponential, with linear exponent $x \cdot \xi$:

$$P\left(x, \frac{\partial}{\partial x}\right)e^{x\cdot\xi} = P(x,\xi)e^{x\cdot\xi}. \tag{2}$$

The importance of the Fourier transformation has been known for a long time, particularly for equations with constant coefficients such as the linearized heat equation; also the importance of some remarkable integral formulas yielding solutions of (1). The point of the Fourier transformation is that it diagonalizes differential operators with constant coefficients: if u is written as a superposition of exponentials

$$u = (2\pi)^{-n}\int e^{ix\cdot\xi}\hat{u}(\xi)\, d\xi \tag{3}$$

and P has constant coefficients, $v = Pu$, then $\hat{v}(\xi) = P(\xi)\hat{u}(\xi)$.

For equations with constant coefficients one also gets integral solution formulas of the convolution type

$$u(x) = \int a(x-y)v(y)\, dy, \tag{4}$$

where in good cases the function a is explicitly known (e.g., for the Laplace or wave or heat equation).

Of course, there are difficulties in the methods suggested above. For instance, when solving (1) by Fourier transformation, the first obstacle is that $P(\xi)^{-1}\hat{v}$ is usually not integrable. Also, in many formulas such as (4) the function $a(x-y)$ is not locally integrable (e.g., for the wave equation in high dimension). It is therefore necessary to modify the definition of the integral so that it will include integrals such as (3) or (4) in less obvious cases. This is one of the main purposes of distribution theory as initiated by J. Hadamard and founded by L. Schwartz. Distribution theory is described in the first volume of L. Hörmander's treatise,

Starting from explicit formulas, perturbation methods, e.g., energy or potential inequalities, may often still give a good description of the solution of (1), for equations which do not differ too much from equations with constant coefficients (e.g., fortunately, many of the linear equations which are used in physics). However a second difficulty in P.D.E. theory is that the methods suggested above, which are adequate for equations with constant coefficients, sometimes fail completely for other equations. In fact equations with genuinely nonconstant coefficients can behave in a quite different manner. For instance the H. Lewy equation

$$\frac{\partial u}{\partial \bar{z}} + z\frac{\partial u}{\partial t} = v \quad \text{on } \mathbf{C} \times \mathbf{R}$$

($\sim$ tangential Cauchy-Riemann equation on the sphere of $\mathbf{C}^2$) does not have local solutions if the right side $v \in C^\infty$ is not well chosen. This phenomenon never happens for equations with constant coefficients.

Microlocal analysis, and the related use of symplectic geometry in P.D.E. theory, has permitted great progress in the understanding of operators with nonconstant coefficients. It is the description of this method and of its most spectacular applications which is the object of Volumes III and IV. Here is a naive description of what it is about.

The first observation, which leads to the construction of pseudodifferential operators and of the wave-front set, is that although the Fourier transformation no longer diagonalizes differential operators with nonconstant coefficients, it still diagonalizes them approximately in a suitable sense. This can be seen when one looks at the effect of a differential operator $P = \sum a_\alpha(x)(\partial/\partial x)^\alpha$ on the singularity of a distribution, or on an asymptotic expansion of the form

$$e^{it\varphi}a \sim e^{it\varphi} \sum_{k<k_0} a_k(x)t^k \qquad (t \to \infty). \tag{5}$$

In fact

$$P_\varphi = e^{-it\varphi}Pe^{it\varphi} = \sum t^j P_j\left(x, \frac{\partial}{\partial x}\right)$$

is a polynomial in t (and the derivatives of φ), whose coefficients are differential operators. The leading term of P_φ is a scalar operator

$$P_\varphi = t^m\sigma_P(d\varphi) + O(t^{m-1}), \tag{6}$$

where $\sigma_P(d\varphi)$ is by definition the symbol of P evaluated at the cotangent vector $d\varphi$. One has

$$(7) \qquad P(e^{it\varphi}a) = e^{it\varphi}P_\varphi(a) = e^{it\varphi}\big(t^m\sigma_P + O(t^{m-1})\big)\cdot a;$$

hence the effect of P on such an asymptotic expansion is approximately scalar, governed by the leading term. For instance the action of P is invertible if the leading term $\sigma_P(d\varphi)$ is not zero. The inverse is then given by a formula which belongs typically to pseudodifferential calculus. Let us further note that the effect of P_φ on asymptotic expansions is described by local formulas: in fact P_φ only depends on the Taylor expansion at the point $d\varphi$ of the total symbol $p(x,\xi) = \sum a_\alpha(x)(i\xi)^\alpha$ (and on the Taylor expansion of φ). The same is true for the inverse of P_φ when $\sigma_P(d\varphi) \neq 0$: this is an asymptotic differential operator of the form $\sum t^{-m-k}Q_k(x, \partial/\partial x)$, where Q_k is a differential operator of order $\leqslant k$ whose coefficients are polynomials of the Taylor coefficients of the total symbol of P (and of φ) at the point $d\varphi$.

The action of P on singularities of distributions can be localized in a similar manner. For this purpose one defines the wave-front set: if f is a distribution, one says that a covector (x_0, ξ_0) does not belong to the wave-front set of f (or that f is smooth at (x_0, ξ_0)) if f can be represented near x_0 by a Fourier integral

$$f = \int e^{ix\cdot\xi}a(\xi)\,d\xi \quad (\text{for } x \text{ close to } x_0),$$

where a vanishes in some conical neighborhood of the half-line $\mathbf{R}_+\xi_0$. Thus defined, the wave-front set does not depend on a choice of local coordinates. It decreases if one applies a differential operator, or more generally a pseudodifferential operator, so that again the effect of P on singularities, i.e. distributions mod those which are C^∞ at (x_0, ξ_0) depends only on the restriction of the total symbol of P in small conical neighborhoods of (x_0, ξ_0).

Microlocal analysis is an intelligent and organized exploitation of such remarks. The first step is the construction of pseudodifferential operators. These are operators of the form

$$(8) \qquad u \mapsto Au(x) = (2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}a(x,\xi)u(y)\,dy\,d\xi,$$

where $\langle x-y, \xi\rangle$ is the scalar product, and a (the total symbol) is a function on $\mathbf{R}^{2n}$ whose derivatives of high order decay suitably when $\xi \to \infty$. Differential operators correspond to the case where $a(x,\xi)$ is a polynomial in ξ. The next simplest case is the case where a has an asymptotic expansion (similar to (5)) in homogeneous functions with respect to ξ.

$$(9) \quad a(x,\xi) \sim \sum a_{m-k}(x,\xi), \quad \text{with } a_{m-k}(x,\lambda\xi) = \lambda^{m-k}a_{m-k}(x,\xi).$$

In fact the class of functions (total symbols) $a(x,\xi)$ for which the operator $a(x,D)$ defined as in (8) deserves the name "pseudodifferential operator" is wide. It is important in limit cases to prove positivity or L^2 continuity results. In limit cases it is sometimes technically more convenient to replace the function $a(x,\xi)$ in formula (8) by the more symmetric $a((x+y)/2,\xi)$ ("Weyl calculus"). The study of pseudodifferential operators, including the refined study of positivity, is done in Chapter 18.

The other essential tools of microlocal analysis are the Lagrangian distributions and Fourier integral operators (Chapter 25), and some symplectic geometry, e.g., normal forms for functions, quadratic forms, real or complex hypersurfaces or pairs of hypersurfaces (Chapter 21). The role of symplectic geometry appears immediately in the symbolic calculus of pseudodifferential operators; the symbol of a product $P \circ Q$ is the product $\sigma(PQ) = \sigma(P)\sigma(Q)$. Thus if P, Q are respectively of order m, m', the bracket $[P, Q] = PQ - QP$ is of order $m + m' - 1$ at most. Its symbol is

$$\sigma([P,Q]) = -i\{\sigma(P), \sigma(Q)\}, \tag{10}$$

where $\{f, g\}$ is the Poisson bracket associated with the canonical symplectic form $\Sigma\, d\xi_j\, dx_j$ on the cotangent bundle (in local coordinates

$$\{f, g\} = \sum \frac{\partial f}{\partial \xi_j}; \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}; \frac{\partial g}{\partial f_j}\Bigg).$$

A Lagrangian distribution is a distribution u that can be expressed locally as a "Fourier integral"

$$u = \int e^{i\varphi(x,\theta)} a(x, \theta)\, d\theta \tag{11}$$

where the phase function φ is real (or with positive imaginary part), homogeneous of degree 1 in θ, and the amplitude (symbol) a is of the same type as those which enter in the definition of pseudodifferential operators. Such a distribution is associated with a Lagrangian submanifold Λ of the cotangent bundle: the image of the critical locus of $\varphi(\partial\varphi/\partial\theta = 0)$ by the differential map $(x, \theta) \mapsto (x, d_X\varphi)$ (this is assumed to be an immersion). The leading term of a may be reinterpreted intrinsically as a section of the Maslov line bundle of Λ, which has a geometrical definition.

A generic example of a Lagrangian submanifold of T^*K is the conormal bundle of a hypersurface $f = 0$ of a manifold X, i.e., the set of covectors $(x, \xi) \in T^*X$ with $f(x) = 0$, ξ parallel to df. The associated Lagrangian distributions have the characteristic form

$$u = af^{-N} + b \operatorname{Log} f \quad (N \text{ an integer}), \text{ or} \quad af^s \quad (s \notin Z), \tag{12}$$

where a and b are smooth functions on X.

Many distributions which appear in integral solution formulas for partial differential equations are Lagrangian.

Fourier integral operators are operators defined by a formula

$$Au(x) = \int T(x, y) u(y)\, dy, \tag{13}$$

where T is a Lagrangian distribution. Many interesting operators are of this type, e.g., $\exp\, it\sqrt{-\Delta}$, where Δ is the Laplace operator on a Riemannian manifold. These operators also enable us to make arbitrary homogeneous canonical changes of coordinates in the cotangent bundle, thus reducing the local study of differential operators to much simpler forms than permitted by using just changes of coordinates.

We end with a short description of the contents of the two books. Volume III begins with a chapter on second-order differential equations (regularity of the solutions, existence of local solutions, unique continuation of solutions, ...) which does not use microlocal techniques at all, and serves as a reminder that older techniques are also quite efficient.

As was mentioned above, the books contain three chapters on the foundations: pseudodifferential operators (Chapter 17) and sharp L^2 inequalities for these: symplectic geometry (Chapter 21), Lagrangian distributions and Fourier integral operators (Chapter 25).

The rest of the book is devoted to applications, mostly those which are important or striking, use microlocal analysis in a convincing manner, and have now reached a reasonably mature form.

Some topics are now older, such as Calderón's theorem on the uniqueness of the solution of the Cauchy problem, which is described and generalized in Chapter 28. This theorem was proved in 1958 and uses in an essential manner singular integral operators (the earlier name of pseudodifferential operators), which appear as first-order factors of the top-order part of the given operator.

On a closely related subject, Chapter 23 also contains old results: Hadamard's theory of the hyperbolic Cauchy problem was published in 1932. Here it is generalized and improved. New phenomena linked with nonconstant coefficients are analyzed. For these the use of symplectic geometry and pseudodifferential operators is essential and makes it possible to exhibit well-posed Cauchy problems for operators which have multiple characteristic roots on the initial hypersurface, such as the Tricomi operator

$$\frac{\partial^2}{\partial x_n^2} - x_n \sum_1^{n-1} \frac{\partial^2}{\partial x_j^2},$$

in the half-space $x_n > 0$.

Two chapters (19, 20) are devoted to the theory of elliptic operators and the index formula for these. So far as analysis is concerned this is the simplest application of the theory of pseudodifferential operators. The index formula was described and proved for general elliptic pseudodifferential operators by Atiyah and Singer in 1963. Although pseudodifferential operators are not really indispensable in the proof, they are very practical, e.g., in making continuous deformations.

The other applications are more recent. The first occurrences of authentic Fourier integral operators took place in a work of L. Hörmander on the asymptotics of the spectral function of an elliptic operator and in a work of L. Nirenberg and F. Treves on the existence of local solutions of equations of principal type. These theories are described in the book, updated and completed (Chapters 29 and 26).

The theory of subelliptic estimates is closely related to the local existence theory for equations of principal type. Egorov's results, announced in 1975, were proved later by L. Hörmander (in 1979). They take their definitive form in Chapter 27.

Chapter 22 analyzes some cases of microhypoellipticity. A pseudodifferential operator P is microhypoelliptic if for any covector ξ and any distribution f, f

is C^∞ near ξ whenever $P(f)$ is so. In some cases this can be seen because one can construct directly a pseudodifferential inverse; for this the large classes of pseudodifferential operators described in Chapter 19 are a very sharp tool. In other cases one cannot construct directly a pseudodifferential inverse, but one can prove microhypoellipticity by arguments using both microlocal geometry and a priori estimates. Such is the case for operators "of Kolmogorov type": $X_0 + \sum X_i^2$, where the X_j are vector fields whose Lie algebra spans the whole space at every point.

Chapter 24 contains the theory of the mixed Cauchy-Dirichlet problem for second-order differential operators (i.e., the study of the evolution in time of the solution of the wave equation in a bounded domain, with some reflection condition on the boundary). The existence of solutions has been proved by techniques using energy inequalities. The precise study of the singularities of the solutions and of their propagation requires the full arsenal of microlocal analysis.

The last chapter (30) is on scattering theory for long-range potentials (short-range scattering is dealt with in Volume II). The typical example is the theory of $H = \sum \partial^2/\partial x_j^2 + V(x)$, where the potential V does not decay fast enough at infinity (e.g. $V = O(1/|x|)$). The aim is to intertwine the part of H with continuous spectrum with the Laplace operator $\sum \partial^2/\partial x_j^2$ (i.e., prove that nonbounded particles behave at infinity as free particles). One of the key ingredients of the theory is the construction of a distorted Fourier transformation adapted to H, i.e., of a family of approximate solutions of $H(f_\xi) = -|\xi|^2 f_\xi$ which behave at infinity asymptotically as $\exp(-ix \cdot \xi)$. The same ideas of microlocal analysis are used, now applied to asymptotic expansions when $x \to \infty$.

The lines above only give a very short idea of the contents of the book. I at least hope they will be motivation to read it. Each chapter of the book also contains an introduction, which describes with more details the contents and methods of the chapter, and a bibliographical and historical notice. The book also contains a very complete bibliography. It is a superb book, which must be present in every mathematical library, and an indispensable tool for all—young and old—interested in the theory of partial differential operators.

LOUIS BOUTET DE MONVEL

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Minimal surfaces and functions of bounded variation, by Enrico Giusti, Monographs in Mathematics, Vol. 80, Birkhäuser, Boston-Basel-Stuttgart, 1984, xii + 240 pp., $39.95. ISBN 0-8176-3153-4

Among all surfaces spanning a given boundary is there one of least area? Such problems have sometimes been called collectively *the problem of Plateau* in honor of a nineteenth-century physicist who wrote a treatise on equilibrium