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GENERALIZATIONS OF THE NEUMANN SYSTEM

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0. Introduction. The following observation, due to E. Trubowitz [**7**], illustrates an intimate relationship between spectral theory and Hamiltonian mechanics in the presence of constraints. Let $q(s)$ be a real periodic function such that Hill's operator,

$$L = \left(\frac{d}{ds}\right)^2 - q(s),$$

has only a finite number g of simple eigenvalues. There exist $g+1$ periodic eigenfunctions $x_0, \ldots, x_g$ and corresponding eigenvalues $a_0, \ldots, a_g$ of L such that

$$1 = \sum_{r=0}^{g} x_r^2 \quad \text{and} \quad q = -\sum_{r=0}^{g}(a_r x_r^2 + y_r^2),$$

where $y_r = dx_r/ds$. The equations $Lx_r = a_r x_r$ $(r = 0, \ldots, g)$ are equivalent to the classical Neumann system [**7**].

H. Flaschka [**3**] obtained similar results from a different point of view. His approach is based on the articles [**2** and **5**] of I. V. Cherednik and I. M. Krichever. The familiar Lax pairs, the constants of motion and the quadrics of the Neumann system emerge as consequences of the Riemann-Roch Theorem.

The purpose of our work is to apply Flaschka's techniques to operators of order $n \geq 2$. We will be defining higher Neumann systems whose theory is closely tied to the spectral theory of linear differential operators of order n. C. Tomei [**9**], using scattering theory, obtained some of our $n = 3$ formulas.

Preliminaries.

(1.1) RIEMANN SURFACE. Let R be a Riemann surface of genus g_R with a point ∞ and a rational function whose divisor of poles $(\lambda)_\infty$ is n^∞. We set $\kappa = \lambda^{1/n}$. Then κ^{-1} is a local parameter vanishing at ∞. Let W be the set of Weierstrass gap numbers of ∞.

(1.2) ALGEBRAIC CURVES. We assume that R admits a second rational function z with the following 3 properties. There exists an integer $N \geq 0$ and an integer $l \in \{1, 2, \ldots, n-1\}$ relatively prime to n such that

$$z = \lambda^{-N}\kappa^{-l}(z_0 + z_1\kappa^{-1} + \cdots), \qquad z_0 = 1, \text{ at } \infty.$$

Let $(z)_\infty = (0) + \cdots + (m)$, $(r) \in R$, be the divisor of poles of z. Let $a_r = \lambda(r)$. We assume that each (r) is a simple pole and $a_r \neq a_s$ whenever $s \neq r$. We

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assume that the genus g_R is related to m, n and l by the following important formula, $g_R = \frac{1}{2}(n-1)(2(m+1)-nN-(l+1))$. It is known that two rational functions on a Riemann surface satisfy a polynomial equation. Since that equation, it turns out, follows from the Baker function theory below, we need not discuss the existence of Riemann surfaces with the properties above.

Since n and l are relatively prime, there exist $r_j,\ s_j \in \mathbf{Z}$ such that $\lambda^{r_j} z^{s_j}$ has a pole of order j at ∞. Let $t = (t_j | j \in W)$ be a vector of g_R complex "time" parameters. Let $\theta = \sum_{j\in W} t_j \lambda^{r_j} z^{s_j}$.

(1.3) BAKER FUNCTIONS. Let δ be a positive nonspecial divisor of degree g_R that does not meet ∞ and satisfies $L(\delta - \infty) = \{0\}$. It is known that there exists a unique function $\psi = \psi_\delta(t,p)$, called the Baker function of δ, with the following two properties. ψ is meromorphic in $R - \infty$ and any pole of ψ lies in δ. Near ∞, ψ is given by $\psi e^{-\theta} = 1 + \xi_1(t)\kappa^{-1} + \xi_2(t)\kappa^{-2} + \cdots$, where the ξ_j are functions analytic on an open subset of $\mathbf{C}^{g_R}$ containing $t = 0$.

(1.4) DUAL BAKER FUNCTION. By the Riemann-Roch Theorem there exists a unique abelian differential Ω and a positive nonspecial divisor δ' of degree g_R such that $(\Omega) = \delta + \delta' - 2^\infty$ and $\Omega = -\kappa^2(1+O(\kappa^{-2}))\, d\kappa^{-1}$ at ∞. Let $\phi = \psi_{\delta'}(-t,p)$. We will refer to ϕ as the Baker function dual to ψ and δ' will be called the dual divisor [**2**].

(1.5) NEUMANN SYSTEMS. There exists a linear differential operator L of order n in d/dt_1 and, for each $j \in W$, a linear differential operator $\tilde{L}_j$ of order j in d/dt_1 such that

$$(1.5.1.1) \qquad L(t)\psi(t,p) = \lambda(p)\psi(t,p) \quad \text{and} \quad \tilde{L}_j(t)\psi(t,p) = \frac{\partial \psi}{\partial t_j}(t,p).$$

Let L^* be the formal real adjoint of L (for instance, $(qD^j)^* = (-1)^j D^j q$). The article [**2**] contains a clever proof of the following formulas:

$$(1.5.1.2) \qquad L(t)^*\phi(t,p) = \lambda(p)\phi(t,p) \quad \text{and} \quad \tilde{L}_j(t)^*\phi(t,p) = \frac{\partial t}{\partial t_j}(t,p).$$

We are now in position to define the main object of our analysis. Let $\rho_r = \operatorname{Res}_{(r)}(z\Omega)$ and choose constants $\alpha_r,\ \beta_r \in \mathbf{C}^*$ such that $\rho_r = \alpha_r\beta_r$. We evaluate the Baker functions ψ and ϕ over the poles of z to make the following definitions:

$$(1.5.2) \qquad x_1^r(t) = \alpha_r\psi(t,r) \quad \text{and} \quad u_n^r(t) = \beta_r\phi(t,r), \qquad r = 0, \ldots, m.$$

Let $\mathbf{m} \in \mathbf{C}^{2n(m+1)}$ be the point whose coordinates are x_1^r, u_n^r and their first $n-1$ derivatives with respect to t_1. We are concerned with the equations obtained from (1.5.1) by setting $p = (r)$, $r = 0, \ldots, m$.

(1.6) SOLITON EQUATIONS. The integrability condition of the simultaneous linear equations (1.5.1) is the partial differential equation

$$(*) \qquad \partial L/\partial t_j = [\tilde{L}_j, L], \qquad j \in W.$$

The Lax equation usually suggests that certain spectral data associated to L are preserved in time. In the present setup it is the Riemann surface R that is preserved. Two of the equations $(*)$ are important in their applications to soliton mathematics. If $n = 2$ and $j = 3$, $(*)$ is the Korteweg-de Vries equation. If $n = 3$ and $j = 2$, $(*)$ is the Boussinesq equation in the form of a system of equations.

Results.

(2.1) SYMPLECTIC MANIFOLD AND TRACE FORMULAS. The differential $\tilde{\eta} = \psi_j^{(i)} \phi_{j'}^{(i')} \Omega$ is meromorphic because the exponents of ψ and ϕ at ∞ cancel. The meromorphic differential $\eta = \lambda^k z \tilde{\eta}$ has simple poles in $(z)_\infty$ and it may have a pole at ∞. Let $C_\eta = \sum_{p \in R} \mathrm{Res}_p(\eta)$. The classical formula $\sum_{p \in R - \infty} \mathrm{Res}_p(\eta) = -\mathrm{Res}_\infty(\eta)$ expresses $\mathrm{Res}_\infty(\eta)$ in terms of $\mathbf{m}$. If $\mathrm{Res}_\infty(\eta)$ is constant (in t) the equation $C_\eta = 0$ defines a hypersurface in $\mathbf{C}^{2n(m+1)}$. The functions C_η with $\mathrm{Res}_\infty(\eta)$ constant are called constraints.

(2.1.1) THEOREM. *The algebraic subset M of $\mathbf{C}^{2n(m+1)}$ defined in terms of the quadratic constraints $C_\eta = 0$ is a symplectic manifold. The dimension of M is given by* $\dim(M) = 2g_R + 2(m+1)$.

(2.1.2) THEOREM. *The coefficients of L are expressible in terms of the point $\mathbf{m}$ associated with the Baker function and the poles of z.*

It follows then that the equations (1.5.1) with $p = (r)$, $r = 0, \ldots, m$, define g_R autonomous vector fields X_j^*, $j \in W$, on M. The $(n = 2)$ vector field X_1^* is a generalization of the Neumann system [**3** and **4**].

(2.2) LAX EQUATIONS. One of the nicest results of Flaschka's work is a systematic derivation of the well-known Neumann-Lax pairs. The best explanation for the existence of the Neumann-Lax pairs comes from Krichever's theory of commutative rings of matrix differential operators. The divisor $\Delta' \stackrel{\text{def}}{=} \delta' + (z)_0 - \infty$ is nonspecial and its degree is $g_R + m$. Following [**4**] we call Δ' the augmented dual divisor. According to [**8**], there exists a vector function $\Phi = (\Phi^0, \ldots, \Phi^m)^T$ with the following two properties. Φ is meromorphic in $R - (z)_\infty$ and any pole in Φ lies in Δ'. Near (r), Φ^s is given by $\Phi^s e^{-\theta} = \alpha_r \delta_{r,s} + O(z^{-1})$. Let $(\ ;\ )$ be the bilinear form associated to L by the Lagrange identity, $d(f;g)/dt_1 = Lf \cdot g - f \cdot L^* g$. H. Flaschka discovered the $n = 2$ version of the very beautiful formula,

$$\Phi^r(t,p) = (x_1^r(t); \phi(t,p)) \frac{z^{-1}(p)}{\lambda(p) - a_r} e^{\theta(t,p)}. \tag{2.2.1}$$

According to Krichever there exists an $(m+1) \times (m+1)$ matrix $\mathbf{B}_j$ that depends polynomially on z such that $\Phi_{t_j} = \mathbf{B}_j \Phi$. Using Flaschka's formula (2.2.1) we are able to express $\mathbf{B}_j$ in terms of $\mathbf{m}$. The function λz^n belongs to the ring $H^0(R - (z)_\infty, O_R)$. Thus according to Krichever there exists an $(m+1) \times (m+1)$ matrix $\mathbf{L}$ that depends polynomially on z such that $\mathbf{L}\Phi = \lambda z^n \Phi$. The Lax equation $\mathbf{L}_{t_j} = [\mathbf{B}_j, L]$ is immediate. Our explicit formulas show that $\mathbf{L}$ is a rank n perturbation of the diagonal matrix az^n in that the range of $\mathbf{L} - az^n$ is spanned by $x_1, \ldots, x_n$. The $(n = 2)$ $\mathbf{L}$ and $\mathbf{B}_1$ generalize the Neumann-Lax pairs in [**1**, **3** and **4**].

We have $\Delta' - (\phi)_\infty \geq 0$ and therefore ϕe^θ belongs to the linear space of Baker functions spanned by the components of Φ. This observation led

Flaschka to the $n = 2$ version of the following formula:

$$\phi(t,p)e^{\theta} = \sum_{r=0}^{m} u_n^r(t)\Phi^r(t,p) = \langle u_n(t), \Phi(t,p)\rangle. \tag{2.2.2}$$

The formula has two applications. We use (2.2.2) to obtain explicit formulas for the operators $\tilde{L}_j$. Such formulas were one of Cherednik's objectives [**2**]. When Φ is eliminated from (2.2.2) by use of (2.2.1) we obtain the following result.

(2.2.5) THEOREM. *There exists an $n \times n$ matrix $Z = Z(\mathbf{m}, \lambda)$, rational in λ, whose spectrum is independent of t. The algebraic relationship* (1.2) *between λ and z is given by the characterization polynomial* $\det(Z - zI) = 0$.

(2.3) COMPLETE INTEGRABILITY. The $m+1$ Hamiltonians $(x_1^r; u_n^r)$, $r = 0, \ldots, m$, are rather trivial involutive constants of motion. A reduction of M by these Hamiltonians defines a symplectic manifold which, by (2.1.1), has dimension $2g_R$. We use the fact that the eigenvalues of $\mathbf{L}$ and Z are constants of the motion to construct a Hamiltonian H_j^* for each vector field X_j^*, $j \in W$.

(2.3.1) THEOREM. *The g_R Neumann vector fields X_j^* of* (2.1.4) *form a completely integrable Hamiltonian system.*

It is known that the level surface $M_C \stackrel{\text{def}}{=} \{\mathbf{m}^* \in M | H_j(\mathbf{m}) = c_j\}$ of a completely integrable system, if real and compact, is a torus. Our last result is concerned with the structure of these energy level sets.

(2.3.2) THEOREM. *The level surface of the reduced manifold is locally isomorphic to the Zariski-open subset, Jacobian-(theta divisor) of the Jacobian variety of the algebraic curve given by* $\det(Z(\lambda) - zI)$.

The idea in the proofs of (2.3.1,2) is an algebro-geometical version of the solitonic inverse scattering transform. Let M be one of the symplectic manifolds of Theorem (2.1.1). We assign to each point $\mathbf{m} \in M$ an algebraic curve C and a divisor $\delta = \delta_{\mathbf{m}}$ on C. The isomorphism of Theorem (2.3.2), called the divisor map, is given by

$$\mathbf{m} \in M \to (C, \delta) \to (\mathrm{Jac}(C), A(\delta))$$

where A is the Abel map. It contains a method for linearizing the equations of motion. The important ideas can be found in [**1** and **5**]. We apply McKean's pole conditions [**6**, p. 624] to make certain results, especially the description of δ, more explicit.

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