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L^p ESTIMATES FOR MAXIMAL FUNCTIONS AND HILBERT TRANSFORMS ALONG FLAT CONVEX CURVES IN $\mathbf{R}^2$

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1. Introduction and statement of results. Let $\Gamma: \mathbf{R} \to \mathbf{R}^n$ be a curve in $\mathbf{R}^n$ with $\Gamma(0) = 0$. For suitable test functions f, let $H_\Gamma f(x) = p.v. \int_{-a}^{a} f(x - \Gamma(t))t^{-1}\, dt$ and $M_\Gamma f(x) = \sup_{0<r\leq 1} |r^{-1} \int_0^r f(x - \Gamma(t))\, dt|$. H_Γ and M_Γ are called the Hilbert transform and maximal function along Γ, respectively. There has been considerable interest in estimates of the form $||H_\Gamma f||_p \leq C||f||_p$ and $||M_\Gamma f||_p \leq C||f||_p$ where $||\cdot||_p$ denotes the norm in $L^p(\mathbf{R}^n)$.

If Γ has some curvature at the origin, in a weak sense, then the above L^p estimates for H_Γ and M_Γ have been proved for $1 < p < \infty$ and $1 < p \leq \infty$ respectively, via techniques developed by Nagel, Riviere, Stein, and Wainger; see the survey [**SW**] and the references given there. More recently there has been interest in the case when Γ is flat to infinite order at $t = 0$. In particular if $\Gamma(t) = (t, \gamma(t))$ is a curve in $\mathbf{R}^2$ for which γ is convex for $t > 0$ and either even or odd, then a necessary and sufficient condition for H_Γ to be bounded on L^2 has been obtained in [**NVWW1**]. The condition for odd γ has also turned out to imply the L^2 boundedness of M_Γ [**NVWW2**]. There has also been progress in the study of L^p boundedness for $p \neq 2$ [**NW, CNVWW, C**].

In the present paper we consider (locally) C^1 curves $\Gamma(t) = (t, \gamma(t))$ in $\mathbf{R}^2$ defined for $t \geq 0$, with $\gamma'(0) = \gamma(0) = 0$, convex and increasing. To discuss the Hilbert transform $\Gamma(t)$ must be defined for $t < 0$; we define $\Gamma_e(t) = (t, \gamma(-t))$ and $\Gamma_0(t) = (t, -\gamma(-t))$ for $t < 0$. Curvature hypotheses are replaced by the much weaker "doubling property"

(1.1) there exists $\lambda > 1$ with $\gamma'(\lambda t) \geq 2\gamma'(t)$ for all $t > 0$.

We shall prove

THEOREM. *Let $\Gamma, \Gamma_e, \Gamma_0$ be as above and satisfy* (1.1). *Then* $||M_\Gamma f||_p \leq C||f||_p$ *for* $1 < p \leq \infty$, *and* $||H_{\Gamma_e} f||_p + ||H_{\Gamma_0} f||_p \leq C||f||_p$ *for* $1 < p < \infty$. *More precisely, the latter assertion is that the operators H_Γ, initially defined only for test functions, extend to bounded operators on L^p.*

By combining this theorem with the necessary condition for L^2 boundedness of H_{Γ_e} in [**NVWW1**], we obtain the following

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COROLLARY. *For all curves Γ_e as above, and for all p, $1 < p < \infty$, a necessary and sufficient condition for the boundedness of H_{Γ_e} on L^p is* (1,1).

(In fact, we can see that H_{Γ_e} is not even of weak type (p,p) for any p, unless (1.1) holds: for $0 < a < A$, let S be the quadrilateral with vertices at $(\pm a, 0)$, $(-2A, \gamma'(A)(-2A-a))$, $(-2A, \gamma'(a)(-2A+a))$; let T have vertices at $(0,0), (a,0)$ $(-A, -A\gamma'(A))$, $(a-A, -A\gamma'(A))$; then $H_{\Gamma_e}(\chi_S) > \log(A/a)$ on T, since Γ_e is even and convex. But, denying (1.1) implies that $|S|/|T|$ can be bounded while $A/a \to \infty$.)

In previous work proofs of L^p estimates of the type under discussion here have depended upon favorable decay estimates for Fourier transforms of certain measures supported on the curve Γ. In limiting cases in which Γ consists of an infinite sequence of line segments tending to the origin such estimates fail to hold, yet (1.1) may be satisfied. The principal innovation here is a Littlewood-Paley argument based on a decomposition of the Fourier transform plane into lacunary sectors as in [**NSW**]. A preliminary result based on this technique was proved in [**CNVWW**]. A similar idea was also previously used in [**NSW**] in studying the "lacunary" maximal function. Subsequently [**DRdF**] showed how old results, for cases in which favorable decay estimates do hold, could be proved by clever applications of classical Littlewood-Paley decompositions. A combination of these ideas leads to the proof of the theorem in this paper.

2. A Paley-Littlewood decomposition. Now we describe a Paley-Littlewood decomposition. Let $\alpha_k = \gamma'(\lambda^k)$. Then by using the Marcinkiewicz multiplier theorem, (1.1), duality, and standard techniques, we can find multiplier operators P_k defined by $(P_k f)\widehat{\ }(\xi,\eta) = \Phi_k(\xi,\eta)\cdot \hat{f}(\xi,\eta)$ such that

$$\sum_k P_k = \text{identity};$$

$$\operatorname{supp}\Phi_k \subseteq \{(\xi,\eta)\colon \alpha_{k-2} < |\xi/\eta| < \alpha_{k+1}\};$$

$$\left\|\left(\sum_k |P_k f|^2\right)^{1/2}\right\|_p \le C_p\|f\|_p, \qquad 1 < p < \infty;$$

and

$$\left\|\sum_k P_k f_k\right\|_p \le C_p\left\|\left(\sum_k |P_k f_k|^2\right)^{1/2}\right\|_p, \qquad 1 < p < \infty.$$

3. The proof of $\|M_\Gamma f\|_p \le C\|f\|_p$ for $1 < p \le \infty$. We may assume $\lambda \ge 2$. For each integer k let I_k be the interval $[\lambda^{k-1}, \lambda^k]$. Define measures μ_k by their action on test functions ϕ: $\mu_k(\phi) = |I_k|^{-1}\int_{I_k}\phi(t,\gamma(t))\,dt$. Then

$$(\mu_k)\widehat{\ }(\xi,\eta) = |I_k|^{-1}\int_{I_k}\exp(i\xi t + i\eta\gamma(t))\,dt.$$

The L^p boundedness of M_Γ is equivalent to

$$\|\sup_k|\mu_k * f|\,\|_p \le C\|f\|_p, \qquad 1 < p \le \infty. \tag{3.1}$$

The proof of 3.1 will be by a bootstrapping argument similar to that of [**NSW**]. We prove the following two lemmas:

LEMMA 1. *$\|\sup_k|\mu_k * f|\|_2 \le C\|f\|_2$. Moreover, if there exists $r < 2$ and $C < \infty$ with*

$$\left\|\left(\sum_k |\mu_k * f_k|^2\right)^{1/2}\right\|_r \le C\left\|\left(\sum_k |f_k|^2\right)^{1/2}\right\|_r \tag{3.2}$$

for all sequences f_k, then for each $r < p \le 2$ there exists $C_p < \infty$ such that $\|\sup_k|\mu_k * f|\|_p \le C_p\|f\|_p$.

LEMMA 2. *If $\|\sup_k|\mu_k * f|\|_p \le C_p\|f\|_p$ for some p, $1 < p \le 2$, then $\|(\sum_k|\mu_k * f_k|^2)^{1/2}\|_r \le C\|(\sum_k|f_k|^2)^{1/2}\|_r$ for all r with $r^{-1} < (1+p^{-1})/2$.*

3.1 follows by applying Lemmas 1 and 2 infinitely often as in [**NSW**]. The proof of Lemma 2 is the same as the proof of Lemma 3 of [**NSW**].

To prove Lemma 1 we compare μ_k to σ_k where $\sigma_k = \mu_k*[(\phi_k-\delta)\otimes(\psi_k-\delta)]$. Here $\phi(t)$, $\psi(t)$ are nonnegative C^∞ functions on $\mathbf{R}$ with support in $[-1,1]$ and $\int\phi = \int\psi = 1$; $\phi_k(t) = \lambda^{-k}\phi(\lambda^{-k}t)$, and

$$\psi_k(t) = [\gamma(\lambda^{k+1})]^{-1}\psi[(\gamma(\lambda^{k+1}))^{-1}t].$$

δ is the dirac point mass at the origin. The meaning of $(\phi_k-\delta)\otimes(\psi_k-\delta)$ is that $\phi_k-\delta$ acts on the first variable and $\psi_k-\delta$ on the second. We set $\nu_k = \mu_k - \sigma_k$. Notice that

$$\nu_k = \mu_k * (\phi_k\otimes\delta) + \mu_k * (\delta\otimes\psi_k) - \mu_k * (\phi_k\otimes\psi_k)$$

is a sum of smoothed out μ_k. One can show $\sup_k|\nu_k * f|(x,y) \le CM_sf(x,y)$ where M_s is the usual strong maximal function. Thus,

$$\|\sup_k|\nu_k * f|\|_p \le C_p\|f\|_p, \tag{3.3}$$

$$\left\|\left(\sum_k |\nu_k * f_k|^2\right)^{1/2}\right\|_p \le C_p\left\|\left(\sum_k |f_k|^2\right)^{1/2}\right\|_p \tag{3.4}$$

both hold for $1 < p \le \infty$; see [**FS**].

To prove Lemma 1 it suffices to bound $\sup_k|\sigma_k * f|$, in view of 3.3. But (letting P_k be as in §2)

$$\begin{aligned}\sup_k|\sigma_k * f| &= \sup_k\left|\sum_j \sigma_k * P_{j+k}f\right| \\ &\le \sum_j \sup_k|\sigma_k * P_{j+k}f| \\ &\le \sum_j\left(\sum_k|\sigma_k * P_{j+k}f|^2\right)^{1/2} \equiv \sum_j G_jf.\end{aligned}$$

We show

$$\|G_j f\|_p \le C\|f\|_p, \qquad r < p \le 2; \tag{3.5}$$

$$\|G_j f\|_2 \le C \cdot 2^{-|j|/2}\|f\|_2. \tag{3.6}$$

3.5 and 3.6 imply the conclusion of Lemma 1 by a standard interpolation argument. 3.5 follows from §2, 3.2, and 3.4, 3.6 follows from the following estimates on $\hat{\sigma}_k(\xi,\eta)$: $|\hat{\sigma}_k(\xi,\eta)| \le C\lambda^k|\xi|$, $|\hat{\sigma}_k(\xi,\eta)| \le C\gamma(\lambda^{k+1})|\eta|$, and $|\hat{\sigma}_k(\xi,\eta)| \le |I_k|^{-1}\max_{t\in I_k}|\xi+\eta\gamma'(t)|^{-1}$.

4. The proof of $\|H_\Gamma f\|_p \le C_p\|f\|_p$, $1 < p < \infty$. The proof is similar to the proof in §3. The analogue of the operation $f \to \sigma_k * f$ is

$$L_k f = H_k\{[(\phi_k - \delta)\otimes(\psi_k - \delta)] * f\},$$

where $H_k g(x,y) = \int_{|t|\in I_k} g(x-t, y-\gamma(t))t^{-1}\,dt$. Then we must show

$$\left\|\sum_k P_{j+k}L_k f\right\|_p \le C\|f\|_p \quad \text{and} \quad \left\|\sum_k P_{j+k}L_k f\right\|_2 \le C\cdot 2^{-|j|/2}\|f\|_2.$$

The latter follows from simple Fourier transform estimates. For the former,

$$\begin{aligned}
\left\|\sum_k P_{j+k}L_k f\right\|_p &\le C\left\|\left(\sum_k |P_{j+k}L_k f|^2\right)^{1/2}\right\|_p \\
&= C\left\|\left\{\sum_k |\mu_k * [(\phi_k-\delta)\otimes(\psi_k-\delta)] * P_{j+k}f|^2\right\}^{1/2}\right\|_p \\
&\le C\left\|\left\{\sum_k |[(\phi_k-\delta)\otimes(\psi_k-\delta)] * P_{j+k}f|^2\right\}^{1/2}\right\|_p \\
&\le C\left\|\left\{\sum_k |P_{j+k}f|^2\right\}^{1/2}\right\|_p \le C\|f\|_p,
\end{aligned}$$

by §2, Lemmas 1 and 2, and [**FS**].

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