small and the symbols insufficiently varied. The first edition is much easier to read; but the present one is even more worth reading. It gives a very good account of its subject, and its title is well deserved.

REFERENCES

1. R. Engelking, *Dimension theory*, North-Holland, 1978.
2. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton, 1941.
3. K. Kuratowski, *Topology* I, II, Academic Press, 1966; 1968.
4. K. Menger, *Dimensionstheorie*, Teubner, 1928.
5. K. Nagami, *Dimension theory*, Academic Press, 1970.
6. J. Nagata, *Modern dimension theory*, first ed., North-Holland, 1965.
7. A. R. Pears, *Dimension theory of general spaces*, Cambridge, 1975.
8. P. Roy, *Failure of equivalence of dimension concepts for metric spaces*, Bull. Amer. Math. Soc. **68** (1962), 602–613.

A. H. STONE

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 14, Number 1, January 1986
©1986 American Mathematical Society
0273-0979/86 $1.00 + $.25 per page

Gravitational curvature, an introduction to Einstein's theory, by Theodore Frankel, W. H. Freeman and Co., San Francisco, California, 1979, xviii + 172 pp., $8.95. ISBN 0-7167-1062-5

General relativity, an introduction to the theory of the gravitational field, by Hans Stephani, (edited by John Stewart; translated from German by Martin Pollock and John Stewart) Cambridge Univ. Press, New York, New York, 1982, xvi + 298 pp., $49.50. ISBN 0-521-24008-5

General relativity, by Robert M. Wald, University of Chicago Press, Chicago, Illinois, 1984, xiii + 491 pp., $50.00 HB; $30.00 PB. ISBN 0-266-87033-2

One hundred years ago there appeared in New York a book by William K. Clifford [7] containing the following passages:

(i) Our space is perhaps really possessed of a curvature varying from point to point, which we fail to appreciate because we are acquainted with only a small portion of space...

(ii) Our space may be really same (of equal curvature), but its degree of curvature may change as a whole with the time...

(iii) We may conceive our space to have everywhere a nearly uniform curvature, but that slight variations of the curvature may occur from point to point, and themselves vary with the time... We might even go so far as to assign to this variation of curvature of space 'what really happens in that phenomenon which we term the motion of matter'.

It is impressive and moving to read this intuitive description of the fundamental ideas of the theory of general relativity written over thirty years before Albert Einstein gave the theory its final form. The subtle relations between physics and geometry have always influenced the heartbeat of mathematics; a recent case is the discovery of exotic differentiable structures on $\mathbf{R}^4$ [**9**].

The theory of general relativity has been, from its very beginning, a source of inspiration and the subject of interest to many mathematicians. David Hilbert and Hermann Weyl became fascinated with the idea of geometrizing all of fundamental physics. Weyl proposed to unify gravitation with electromagnetism in terms of a conformal geometry and a connection; he referred to conformal changes of the metric tensor as "gauge transformations". With the advent of quantum theory, Weyl's gauge transformations acquired a new interpretation: they now correspond to changes of sections of a principal $U(1)$-bundle over spacetime. The information about electromagnetism is encoded in the connection form on the bundle. Elie Cartan [**4**] analyzed the geometrical structure of Newtonian and relativistic gravitation. In the course of this analysis, he introduced torsion, a new differential-geometric concept, and conjectured, several years before the discovery of spin, that torsion is related to the intrinsic angular momentum of matter. These ideas have been rediscovered by physicists [**20, 37**] and, recently, absorbed into the "theory of supergravity" [**8, 11**]. In their work on physical problems, Weyl and Cartan were led to generalize the notion of a linear connection, as introduced by Tullio Levi-Civita. In fact, Cartan's article [**4**] contains an intuitive description of the general notions of fiber bundles and connections. Chen Ning Yang and Robert L. Mills [**42**] generalized the concept of an electromagnetic field by considering gauge transformations with values in a non-Abelian Lie group. It soon became clear that their "gauge fields" are connections on principal bundles. Gauge theories are now considered to provide the most promising framework for a unified description of fundamental forces.

The first two decades of the development of general relativity theory were devoted to a derivation of its simplest physical consequences concerning the propagation of light in gravitational fields and the corrections to the Newtonian theory of planetary motion. A. Friedmann [**12**] discovered cosmological models of the Universe, characterized by a curvature constant in space and varying with time, as presaged by Clifford. Most of the work done in that period involved, as far as mathematics are concerned, only local differential geometry and simple approximation methods. Probably the first indication that finer mathematical tools may be needed in general relativity occurred in connection with the study of gravitational waves by Albert Einstein and Nathan Rosen [**10**]. Somewhat to their surprise, they found "singularities" in the solutions representing plane gravitational waves. There are no analogous singularities in plane electromagnetic waves. Only in the 1950s, thanks to the work of Hermann Bondi, Felix Pirani and Ivor Robinson [**2**] had it become clear that these singularities are connected with a particular choice of local coordinates. Physicists began to realize the need of a precise notion of spacetime as a differentiable manifold with additional structures satisfying suitable regularity assumptions. The monograph by André Lichnerowicz [**23**] played in this respect an essential role: it contained the first formulation of the foundation of general relativity in the language of the modern theory of differentiable manifolds. Differential topology made an impact on relativity through the work of Roger Penrose, Stephen Hawking and Robert Geroch [**13**, **18**, **28**, **30**]: to the surprise of many physicists, they showed that real singularities occur in generic models of spacetime as a result of gravitational collapse accompanying the formation of black holes and the large-scale evolution of the Universe (big-bang). Global, geometric ideas have been essential for the analysis of the notion of a black hole and the study of its properties.

Kurt Gödel [**14**] and Abraham Taub [**38**] introduced methods of Lie group theory into cosmology. The rotating Gödel universe shattered several preconceived ideas among relativists: in this model, the local definition of inertial frames does not agree with the global one, based on observations of distant galaxies. There are also there closed time-like lines, and, therefore no Cauchy hypersurfaces. Gödel's idea to construct the metric of a spatially homogeneous Universe from the Maurer-Cartan forms of a suitable Lie group has been used in a systematic study of cosmological models; cf. the work of Istvan Ozsvàth and Engelbert Schücking [**27**] as well as numerous references listed in [**21**].

During the last thirty years, there has been enormous progress in general relativity. It was stimulated by discoveries in astrophysics and the development of new experimental methods; it was made possible by the application of new mathematical ideas and tools. Relativistic gravitation has strengthened its—previously rather weak—links to the rest of physics. For a theoretician, the following areas of recent research are especially interesting: gravitational waves and radiation; exact solutions of Einstein's field equations; causal structure and the Cauchy problem; endeavors to establish a quantum theory of gravitation; black hole physics; creation of particles near black holes and their thermodynamics; proofs of positivity of the total energy of an isolated gravitating system; inflationary cosmology; "supergravity" and renewed attempts to unify gravitation with other elementary forces (generalized Kaluza-Klein theories and models based on "strings"). Many of these topics are well summarized in the book by Robert Wald.

To give the reader a feeling for the type of mathematics that has been developed in the course of work on general-relativistic problems, I shall briefly describe what is now called the *optical geometry* [**39**]. It arose in connection with the research on gravitational waves and exact solutions; it is a close relative—perhaps a stepchild—of Roger Penrose's twistor program [**32**]. The origins of optical geometry may be traced back to Harry Bateman [**1**] and Elie Cartan [**3**]. This geometry has been developed by Ivor Robinson [**34**], Joshua Goldberg and Ray Sachs [**15**], Roy Kerr and Alfred Schild [**19**], E. Ted Newman and Roger Penrose [**25**], and many others [**21**]. Hans Stephani's monograph contains a good account of the most important achievements in this field.

Consider—as a model of spacetime—a four-dimensional oriented differentiable manifold M, of class C^∞ or C^ω, together with a pair $(\mathscr{K}, \mathscr{L})$ of real line bundles such that $\mathscr{K} \subset TM$, $\mathscr{L} \subset T^*M$ and, if $\mathscr{K}_x$ and $\mathscr{L}_x$ denote, respectively, the fibers of $\mathscr{K}$ and $\mathscr{L}$ over $x \in M$, then

$$u \lrcorner \alpha = 0 \qquad \text{for any } u \in \mathscr{K}_x, \qquad \alpha \in \mathscr{L}_x \quad \text{and} \quad x \in M.$$

The bundle $\mathscr{K}$ defines on M a one-dimensional foliation, i.e., a congruence of curves. A section k of $\mathscr{K} \to M$ is a vector field on M; let $(\phi_t(k))$, $t \in \mathbf{R}$, denote the flow generated by k. A section λ of $\mathscr{L} \to M$ is a field of one-forms on M; for any such sections $k \lrcorner \lambda = 0$. A metric tensor g on M is said to be adapted to $(\mathscr{K}, \mathscr{L})$ if, for any such sections k and λ, one has $g(k) \wedge \lambda = 0$, where $g(k)$ is the one-form characterized by $l \lrcorner g(k) = g(k, l)$ for any vector field l. The elements of the bundles $\mathscr{K}$ and $\mathscr{L}$ have vanishing squares with respect to any adapted metric tensor; for this reason they are said to be "null". If $\mathscr{L}$ is invariant with respect to the flow $(\phi_t(k))$, then it is also invariant with respect to $(\phi_t(\rho k))$, where ρ is any function on M. It is meaningful, therefore, to define $\mathscr{L}$ as being invariant with respect to $\mathscr{K}$ if, for any sections k and λ, one has

$$\lambda \wedge L_k \lambda = 0,$$

where $L_k \lambda$ denotes the Lie derivative of λ in the direction of k. It is easy to prove that the invariance of $\mathscr{L}$ with respect to $\mathscr{K}$ is equivalent to the statement: The congruence of curves defined by $\mathscr{K}$ consists of *null geodesics* relative to the Levi-Civita connection corresponding to any metric tensor adapted to $(\mathscr{K}, \mathscr{L})$. An electromagnetic field is described by a two-form F on M. Such a two-form is said to be adapted to $(\mathscr{K}, \mathscr{L})$ if

$$k \lrcorner F = 0 \quad \text{and} \quad \lambda \wedge F = 0.$$

A part of Maxwell's equations is contained in $dF = 0$. Writing the other part requires the introduction of the Hodge dual of F. Let $*_g F$ denote the dual of F relative to the metric tensor g. If both g and F are adapted to $(\mathscr{K}, \mathscr{L})$, then so is $*_g F$.

Let $\mathscr{A}$ denote the set of all Lorentzian metrics on M, adapted to $(\mathscr{K}, \mathscr{L})$. If $F \neq 0$ is adapted and $g, g' \in \mathscr{A}$, then $*_g F = *_{g'} F$ defines an equivalence relation R in $\mathscr{A}$. This equivalence relation does not depend on F. An optical geometry on M consists of the pair $(\mathscr{K}, \mathscr{L})$ together with an element $\mathscr{B}$ of $\mathscr{A}/R$. It is easy to see that if $g \in \mathscr{B} \subset \mathscr{A}$ then $g' \in \mathscr{B}$ if there is a nonvanishing function ρ on M, and a one-form ζ such that

$$g' = \rho g + \zeta \otimes \lambda + \lambda \otimes \zeta,$$

where $\lambda \neq 0$ is a section of $\mathscr{L} \to M$. Let M and M' be two 4-manifolds with optical geometrics $(\mathscr{K}, \mathscr{L}, \mathscr{B})$ and $(\mathscr{K}', \mathscr{L}', \mathscr{B}')$, respectively. A diffeomorphism f: $M \to M'$ is an isomorphism of optical geometries if $f^*\mathscr{B}' = \mathscr{B}$, $f^*\mathscr{L}' = \mathscr{L}$ and $f_*\mathscr{K} = \mathscr{K}'$. According to I. Robinson [**34**], an optical geometry on M admits a nonzero, adapted solution of Maxwell's equations if the flow $(\phi_t(k))$ consists of optical automorphisms for any section k of $\mathscr{K} \to M$. In the physicist's terminology, the congruence of null geodesics generated by $\mathscr{K}$ is then said to be without shear; by a convenient abuse of language one also says that the optical geometry is *shear-free*. Optical isomorphisms can be used to transform one adapted solution of Maxwell's—or Yang-Mills'—equations into another.

An alternative definition of optical geometry is as follows: consider the bundle $\ker \mathscr{L} \to TM$. The quotient $\mathscr{H} = (\ker \mathscr{L})/\mathscr{K}$ is a real plane bundle and $\mathscr{B}$ defines on it a conformal, positive-definite structure. If, moreover, $\mathscr{H}$ is oriented, then it becomes a complex line bundle over M. Conversely, if $(\mathscr{K}, \mathscr{L})$ is as above and $\mathscr{H} = (\ker \mathscr{L})/\mathscr{K}$ is a complex line bundle, then this structure determines an optical geometry.

An optical geometry can be equivalently defined as a G-structure on M, where G is a 9-dimensional subgroup of $GL(4, \mathbf{R})$. This G-structure is locally flat if, and only if, the bundle $\ker \mathscr{L}$ is integrable, i.e., $\lambda \wedge d\lambda = 0$, and the optical geometry is shear-free [**35**].

Assume now that the set of all curves generated by $\mathscr{K}$ determines a fibration of M so that the quotient space $M/\mathscr{K}$ is a 3-manifold N. If the optical geometry on M is shear-free, then N inherits from M the structure of a CR-manifold [**6**, **31**, **41**]. In particular, the complex line bundle $\mathscr{H}$ descends to a subbundle of TN. The simple case of $N = S_3 \subset \mathbf{C}^2$ corresponds to an optical geometry studied by I. Robinson [**29**]. In this case, the complex line bundle $\mathscr{H} \to S_3$ coincides with the pull-back of the tangent bundle to $\mathbf{CP}_1$ by the Hopf map $S_3 \to \mathbf{CP}_1$. The CR-structure on S_3 defines a shear-free and nonintegrable ("twisting") optical geometry on the compactified Minkowski space $M = S_1 \times S_3$. A conformal embedding of Minkowski space $\mathbf{R}^4$ into M can be used to pull this geometry back to $\mathbf{R}^4$. This Robinson geometry is induced on $\mathbf{R}^4$ by

$$k = \partial/\partial v, \qquad \lambda = du + x\,dy - y\,dx$$

and

$$g = \lambda \otimes dv + dv \otimes \lambda - (v^2 + 1)(dx \otimes dx + dy \otimes dy),$$

where (x, y, u, v) are coordinates. The two-form $F = \operatorname{Re} \Phi$, where

$$\Phi = f\lambda \wedge dz, \qquad z = x + iy \quad \text{and} \quad f\colon \mathbf{R}^4 \to \mathbf{C},$$

is adapted to the optical geometry under consideration. The Maxwell equations for F read $d\Phi = 0$ and are equivalent to $\partial f/\partial v = 0$ and the Lewy equation [**22**], $z(\partial f/\partial u) + 2i(\partial f/\partial \bar{z}) = 0$. The local geometry of this electromagnetic field is similar to that of plane waves, but, in the present case, the amplitude f is an analytic function of u, whereas plane waves may very well be merely smooth. The correlation between analyticity and twist follows also from the general theory of foliations: according to André Haefliger [**16**] and Sergei P. Novikov [**26**], a compact manifold with a finite fundamental group admits no analytic foliation of codimension one. Therefore, an optical geometry on $\mathbf{R}^4$ which extends to a real-analytic optical geometry on the compactified Minkowski space is necessarily nonintegrable [**35**].

Several excellent textbooks and monographs on general relativity, and associated topics, have appeared in recent years [**5**, **17**, **21**, **24**, **33**, **40**]. The book by Theodore Frankel is a short introduction to the subject, with emphasis on intrinsic methods and the geometrical significance of Einstein's equations. There is a fairly detailed presentation of cosmology and of spherically symmetric gravitational fields, but black holes, gravitational collapse and waves are just mentioned. There is no reference to the Kerr metric, gravitational radiation or quantization. The other two books under review give a more comprehensive treatment of the subject. They both use—with skill—the traditional tensor calculus and local differential geometry. However, Wald introduces, in an Appendix, differential forms, the de Rham cohomology and states the theorems of Frobenius and Stokes. When confronted with the problem of defining spinor fields over a Riemannian space, he recognizes the need to introduce the notion of a fiber bundle. Besides a nice presentation of the standard material, Stephani's book contains a valuable chapter on exact solutions of Einstein's equations. Wald's volume is the last, among the three, to have been written. It is the most up-to-date and gives a thorough introduction to the theory of general relativity. It contains much material that is otherwise not available in book form. It is intended to serve as a text for graduate students and a reference book for researches; it may be recommended as such.

References

1. H. Bateman, *The transformation of coordinates which can be used to transform one physical problem into another*, Proc. London Math. Soc. **8** (1910), 469–488.

2. H. Bondi, F. A. E. Pirani and I. Robinson, *Gravitational waves in general relativity* III. *Exact plane waves*, Proc. Roy. Soc. London **A251** (1959), 519–533.

3. E. Cartan, *Sur les espaces conformes généralisés et l'Univers optique*, C. R. Acad. Sci. Paris **174** (1922), 857–859.

4. ______, *Sur les variétés à connexion affine et la théorie de la relativité généralisée* I; I (suite); II, Ann. Sci. Ecole Norm. Sup. **40** (1923), 325–412; **41** (1924), 1–25; **42** (1925), 17–88. English translation by A. Ashtekar and A. Magnon-Ashtekar, Bibliopolis, Naples, 1985.

5. S. Chandrasekhar, *The mathematical theory of black holes*, Clarendon Press, Oxford, 1983.

6. S. S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.

7. W. K. Clifford, *The common sense of the exact sciences*, D. Appleton and Co., New York, 1885.

8. S. Deser and B. Zumino, *Consistent supergravity*, Phys. Lett. **62B** (1976), 335–337.

9. S. K. Donaldson, *Self-dual connections and the topology of smooth* 4-*manifolds*, Bull. Amer. Math. Soc. (N.S.) **8** (1983), 81–83.

10. A. Einstein and N. Rosen, *On gravitational waves*, J. Franklin Inst. **223** (1937), 43–54.

11. D. Z. Freedman, P. Van Nieuwenhuizen and S. Ferrara, *Progress toward a theory of supergravity*, Phys. Rev. **D13** (1976), 3214–18.

12. A. Friedmann, *Über die Krümmung des Raumes*, Z. Phys. **10** (1922), 377–386.

13. R. P. Geroch, *What is a singularity in general relativity*?, Ann. Physics **48** (1968), 526–540.

14. K. Gödel, *An example of a new type of cosmological solutions of Einstein's field equations of gravitation*, Rev. Modern Phys. **21** (1949), 447–450.

15. J. N. Goldberg and R. K. Sachs, *A theorem on Petrov types*, Acta Phys. Polon. **22** Suppl. (1962), 13–23.

16. A. Haefliger, *Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes*, Comment. Math. Helv. **32** (1958), 248–329.

17. S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge Univ. Press, Cambridge, 1973.

18. S. W. Hawking and R. Penrose, *The singularities of gravitational collapse and cosmology*, Proc. Roy. Soc. London **A314** (1970), 529–548.

19. R. P. Kerr and A. Schild, *A new class of vacuum solutions of the Einstein field equations*, Atti del Convegno sulla Relatività Generale: Probleme dell'energia e onde gravitazionali, G. Barbèra, Editore, Florence, 1965.

20. T. W. B. Kibble, *Lorentz invariance and the gravitational field*, J. Math. Phys. **2** (1961), 212–221.

21. D. Kramer, H. Stephani, M. MacCallum and E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge Univ. Press and Deutsch Verlag der Wissenschaften, Cambridge and Berlin, 1980.

22. H. Lewy, *An example of a smooth linear partial differential equation without solution*, Ann. of Math. (2) **66** (1957), 155–158.

23. A. Lichnerowicz, *Théories relativistes de la gravitation et de l'électromagnétisme*, Masson et Cie, Paris, 1955.

24. C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman and Co., San Francisco, 1973.

25. E. T. Newman and R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, J. Math. Phys. **3** (1962), 566–578; Errata: Ibid. **4** (1963), 998.

26. S. P. Novikov, *Topology of foliations*, Trudy Moskov. Mat. Obšč. **14** (1965), 248–278 = Trans. Moscow Math. Soc. (1965), 268–304.

27. I. Ozsvàth and E. Schücking, *The finite rotating universe*, Ann. Physics **55** (1969), 166–204.

28. R. Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. **14** (1965), 57–59.

29. ______, *Twistor algebra*, J. Math. Phys. **8** (1967), 345–366.

30. ______, *Techniques of differential topology in relativity*, SIAM, Philadelphia, 1972.

31. ______, *Physical space-time and nonrealizable* CR-*structures*, Bull. Amer. Math. Soc. (N.S.) **8** (1983), 427–448.

32. R. Penrose and M. A. H. MacCallum, *Twistor theory: An approach to the quantisation of fields and space-time*, Phys. Rep. **6** (1972), 241–316.

33. R. Penrose and W. Rindler, *Spinors and space-time*, Cambridge Univ. Press, Cambridge, vol. I: 1984, vol. II: to appear.

34. I. Robinson, *Null electromagnetic fields*, J. Math. Phys. **2** (1961), 290–291.

35. I. Robinson and A. Trautman, *Integrable optical geometry*, Lett. Math. Phys. **10** (1985), 179–182.

36. ______, *A generalization of Mariot's theorem on congruences of null geodesics*, Proc. Roy. Soc. London (to appear).

37. D. W. Sciama, *On the analogy between charge and spin in general relativity*, Recent developments in General Relativity, Pergamon Press and PWN, Oxford and Warsaw, 1962.

38. A. H. Taub, *Empty space-times admitting a three-parameter group of motions*, Ann. of Math. **53** (1951), 472–490.

39. A. Trautman, *Deformations of the Hodge map and optical geometry*, J. Geometry and Physics (Florence) **1** (1984), 85–95.

40. S. Weinberg, *Gravitation and cosmology*, J. Wiley, New York, 1972.

41. R. O. Wells, Jr., *The Cauchy-Riemann equations and differential geometry*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), 187–199.

42. C. N. Yang and R. L. Mills, *Conservation of isotopic spin and isotopic gauge invariance*, Phys. Rev. **96** (1954), 191–195.

ANDRZEJ TRAUTMAN