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# ON PLATEAU'S PROBLEM FOR MINIMAL SURFACES OF HIGHER GENUS IN $\mathbf{R}^3$

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The classical solution of the Plateau problem by Radó [**10**] and Douglas [**3**] shows that any rectifiable Jordan curve in $\mathbf{R}^3$ is spanned by a minimal surface of disc type. Under what conditions a minimal surface of a given higher genus exists, spanning a given Jordan curve in a Riemannian manifold $N$, seems to be a much more difficult problem. For compact minimal surfaces without boundary and in case $N$ has sufficient topological complexity, the "incompressibility" method of Schoen and Yau gives a sufficient condition for existence.

In [**4**] Douglas did develop a method to treat the problem of when a *given* contour is spanned by a surface of genus $\mathfrak{p}$. Douglas' condition, however, seems quite difficult to verify in concrete cases. In this paper we will give simple geometric and topological sufficient conditions.

THEOREM. *Let $N$ be a solid torus of class $C^3$ and genus $\mathfrak{g}$ in $\mathbf{R}^3$ whose boundary has nonnegative inward mean curvature, and let $\gamma \in \Pi_1(N)$ denote the homotopy class of a rectifiable Jordan curve $\Gamma$ in $N$.*

(a) *If $\mathfrak{g} = 2\mathfrak{p}$ and $\gamma = a_1 a_2 a_1^{-1} a_2^{-1} \cdots a_{2\mathfrak{p}-1} a_{2\mathfrak{p}} a_{2\mathfrak{p}-1}^{-1} a_{2\mathfrak{p}}^{-1}$ where $a_1, \ldots, a_{2\mathfrak{p}}$ is a basis for $\Pi_1(N)$ then $\Gamma$ bounds an immersed oriented minimal surface of genus $\mathfrak{p}$.*

(b) *If $\mathfrak{g} = 1$ and $\gamma = 2\alpha$ for some $\alpha \neq 0$ in $\Pi_1(N)$ then $\Gamma$ bounds an immersed minimal surface of Möbius type.*

We sketch the proof of part (a).

Let $\Gamma$ be a rectifiable contour in a solid $2\mathfrak{p}$ torus $N \subset \mathbf{R}^3$, $M$ a surface of genus $\mathfrak{p}$ with $\partial M \cong S^1$ the unit circle. Further let $\mathcal{N}_\Gamma = \{u \colon M \to N | u \colon M \to \Gamma$ monotonically, $u \in H^1(M, \mathbf{R}^3) \cap C(M, \mathbf{R}^3)\}$. Denote by $\mathcal{M}$ the $C^\infty$ Riemannian metrics $g$ on the Schottky double $\hat{M}$ of $M$ such that the natural involution $T \colon \hat{M} \hookleftarrow$ is an isometry for $g$. Dirichlet's functional

$$E \colon \mathcal{M} \times \mathcal{N}_\Gamma \to R$$

is defined by

$$E(g,u) = \frac{1}{2} \sum_{i=1}^{3} \int_M g(x)(\nabla_g u^i, \nabla_g u^i)\, d\mu_g.$$

Let $P$ be the space of all $C^\infty$ positive functions on $\hat{M}$ which are symmetric and $D_0$ those $C^\infty$ diffeomorphisms which fix $\partial M \subset \hat{M}$ (as a set) and are homotopic to the identity. The Teichmüller space for $M$ is then defined to be $\mathcal{T} = (\mathcal{M}/P)/D_0$, a finite-dimensional $C^\infty$ manifold of dimension

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$-3\chi(M)$, $\chi(M)$ the Euler characteristic of $M$. The conformal invariance of Dirichlet's functional guarantees that $E$ is well defined as a map

$$E\colon \mathcal{T} \times \mathcal{N}_\Gamma \to \mathbf{R}.$$

Let $D/D_0 = \Upsilon$ be the modular group of $M$. Then the Riemann space $\mathcal{R}$ of moduli is defined as $\mathcal{R} = \mathcal{T}/\Upsilon$.

For $u \in \mathcal{N}_\Gamma$ consider the introduced map $u_*\colon \pi_1(M) \to \pi_1(N)$. Now $\pi_1(M) \cong \pi_1(N) \cong$ (free group on $2\mathfrak{p}$ generators). It is clear that $[\partial M]$ equals the commutator of the basis of $\pi_1(M)$ obtained from the standard polygonal model of $M$. By hypothesis $u_*$ then takes the commutator to the commutator. By Zieschang's [**11**] generalization of a classical result of Dehn (unpublished) and Nielsen [**9**], $u_*$ must be an isomorphism.

Let $([g_n], u_n) \in \mathcal{T} \times \mathcal{N}_\Gamma$ denote a minimizing sequence. Then since $(u_n)_*$ is an isomorphism, $u_n$ satisfies the Douglas-Courant nondegeneracy condition. Using the Mumford compactness result for the moduli space and a clever idea of Schoen and Yau [**11**] one can show that the class of $g_n$ in $\mathcal{R}$ has a convergent subsequence. This means that there exists a sequence $f_n \in D$ such that the pull back $f_n^*(g_n)$ converges in $\mathcal{M}$. The Courant-Lebesgue Lemma and nondegeneracy show that $u_n$ has a convergent subsequence. Lower semicontinuity of $E$ then guarantees the existence of a minimum $([\bar{g}], \bar{u}) \in \mathcal{T} \times \mathcal{N}_\Gamma$ for Dirichlet's functional. One must now show that $([\bar{g}], \bar{u})$ represents a minimal surface. This is not straightforward since we are minimizing Dirichlet's functional subject to an obstacle restraint. Nevertheless this can be done using a regularity result for variational inequalities of the first author [**12**] and a suitable version of a maximal principle which shows that $\bar{u}$ maps $\overset{\circ}{M}$ into $\overset{\circ}{N}$. The results of Gulliver, Osserman, Royden [**5, 6**] now imply that the resulting minimal surface is immersed.

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