BOOK REVIEWS

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Stone spaces, by Peter T. Johnstone, Cambridge Studies in Advanced Mathematics, Vol. 3, Cambridge University Press, New York, 1983, xxi + 370 pp., $59.50. ISBN 0-5212-3983-5

This book is a bid to create Stone Spaces, much as Genghis Khan created Mongolia. Not, of course, from the void. Stone theory has been driven by the dialectic between spaces and rings or lattices. It begins in Boolean algebras, which are rings and lattices at once. It is their dual objects, the totally disconnected compact Hausdorff spaces, that are to be called Stone spaces.

(Better term, if the only obstruction were the usual name, Boolean spaces; but what is to be done with the established stonian and hyperstonian spaces?)

In this area, natural and nonnatural constructions combine in fairly complex ways. First half of a notable illustration: In Boolean algebras as in abelian groups there are injective envelopes (in the now standard sense), given by MacNeille completion, and practically identified by Sikorski [**20**] as injective envelopes. Hence, by mere duality, Boolean (or Stone) spaces have projective covers. But Gleason found [**6**] that all compact Hausdorff spaces have projective covers: the same (Boolean) projectives, and constructed by something very like MacNeille completion. "The same projectives" is easily explained (and soon was [**18**]): the compact Hausdorff spaces form what is now familiar as the Eilenberg-Moore category of a monad, and projectives are just retracts of free spaces. Boolean spaces live on the same monad and form the smallest quasivariety containing its free spaces. Explaining "something very like" MacNeille completion takes us much further. The Gleason cover of compact Hausdorff X is reached by way of (1) the lattice of open sets ΩX, and (2) the quotient lattice $(\Omega X)_{\neg\neg}$ in which elements u, v are identified if $t \wedge u = \varnothing \Leftrightarrow t \wedge v = \varnothing$, by observing that $(\Omega X)_{\neg\neg}$ is (though not a topology) a Boolean algebra, and taking its dual. Since the crucial station (2) is not a topology, traveling through it requires keeping an eye on three categories at once, or inventing (as in fact Ehresmann did [**3**] before Gleason's theorem) the category of pointless topological spaces or *locales*, and describing the Gleason cover of the spatial locale X as the (spatial) compact Hausdorff reflection of the (nonspatial) smallest dense sublocale $D(X)$. Obviously this is not a reduction to something simpler, but it is an explanation of another frequently useful type, surrounding a beautiful construction with a theory.

Gleason's theorem might be called the Kashgar of Stoneland; it leads to further developments in topological spaces and in toposes. But the second half of the present illustration runs to the junction with seemingly unrelated work in functional analysis by Phillips **[17]**, Nachbin **[16]**, Goodner **[7]**, Kelley **[15]** and Hasumi **[9]**, who showed that in real or complex Banach spaces—the metric category with linear maps of norm at most 1—the injectives are just the (real or complex) spaces $C(X)$ where X is stonian, i.e. (after Gleason) projective compact Hausdorff. There is no adequate duality here. There is a sort of homological sufficiency: free Banach spaces exist **[19]**, though they don't help, and there are more-or-less cofree ones, the l_∞ spaces, which in the different setting of Kakutani's abstract M-spaces can play the role of the free compact spaces. But in mere metric Banach spaces there are too many spaces (not hopelessly: all are embeddable in l_∞ spaces) and, worse, too many morphisms. So H. B. Cohen's initial construction of injective envelopes **[1]** used more Krein-Milman than Stone-Gleason. However, later proofs by Kaufman **[14]** and this reviewer **[11]** used more and more general theory, and reduced the ad hoc element to plane geometry. Plane geometry probably has little to gain from categorical treatment.

Likening this country to Inner Asia is more than an attention-getting stunt. There are similar difficulties in estabishing boundaries. Johnstone's book does not go into Banach spaces, just a quick raid on Banach lattices. (C^*-algebras are another, and a more central, province where Stone has trod.) Neither does it go into toposes. But if this is the generation of Genghis, it certainly has its eye on Kitai (I mean toposes). The treatment is informally (and variably) constructive. The axiom of choice is never used without comment; beyond that, some pains are taken to use constructive arguments, but not unlimited pains. Further, though toposes do not occur, locales do, from p. 39. The basic idea of a real-valued function in Johnstone's spaces is a morphism into the locale R "of real numbers"; classical logic (though not choice) is needed **[4]** to show that R has sufficiently many points.

Marshall Stone's fourth or fifth major creation, beyond Boolean algebras, their representation and duality, compact Hausdorff reflection, and stonian spaces, was the representation and duality theory for distributive lattices. The dual objects are the *spectra*, spaces of homomorphisms onto $2 = \{0,1\}$ (provided we take lattices having 0 and 1) in the topology of pointwise convergence with respect to the Sierpinski topology on 2 ($\{1\}$ is open, not closed). One major road forward is into representation theory built on spectra. The elements $h\colon L \to 2$ are equally well and, more traditionally, described by their kernels $h^{-1}(0)$, the prime ideals. For commutative rings A one again has prime ideals I, this time without knowing in advance what A/I will be. There is still a pointwise topology (pointwise for the characteristic functions of ideals I: "hull-kernel topology"). The spaces $\operatorname{spec} A$ that arise are the same class as the lattice spectra $\operatorname{spec} L$: precisely the inverse limits of finite T_0 spaces **[10]**. Ring theorists usually call them *spectral* spaces, category and lattice theorists *coherent* spaces. Generally, lattice-based structures can be represented directly on $\operatorname{spec} L$; ring-based structures should have representations in sheaves on $\operatorname{spec} A$, and for the stalks one has to determine ranges of possibilities. The literature is a growing manual of techniques and solved cases. Samples: On the lattice side, continuous lattices (not necessarily distributive: just complete lattices in which each element x is the join of all u way below x, meaning every up-directed set with join $j \geqslant x$ has an element $e \geqslant u$) are precisely the isomorphs of subsets of powers of $I = [0, 1]$ closed under arbitrary meets and directed joins. (This is mostly by J. D. Lawson, but others contributed, see [5].) On the ring side, every commutative ring A is isomorphic with the ring of global sections of a sheaf of local rings [8]; it is a sheaf on a modified spectrum $\text{spec}_Z\, A$, and Grothendieck and Dieudonné proved not just that there is an isomorphic representation but that this is a natural representation [8].

As the reader has noticed, these typical representation theorems are special results; one has tools to work with and one may be able to find out how to do the job. On the other road, in concrete duality theory, one typically finds, not help on hard problems, but cut-and-dried solutions to classes of easy problems. At least that is the spirit, but before you dry your solution you must catch it. The most convincing illustration is the reviewer's unpublished theorem [13] to the effect that given a theory **T** of operations and relations of such form that all limits of models of **T** are models of **T** (the precise form is in [12]; the specialization to finitary logic in [2]), and given a suitable cogenerator R of the category $\mathscr{C}$ of models of **T**, (1) the contravariant functor Hom(, R) lifts to a duality of $\mathscr{C}$ with the category of models of a theory $\mathbf{T}^*$ of the same form, and (2) the occurrence of R in $\mathscr{C}$ induces an embedding of **T** in a certain theory $\mathbf{U}(|R|)$ (just the theory of Hom(, $|R|$) in the category $\mathscr{S}$ of sets), where $\mathbf{T}^*$ is the centralizer of **T**. Further, (3) the duality from $\mathscr{C} = \mathscr{S}^{\mathbf{T}}$ to $\mathscr{S}^{\mathbf{T}^*}$ and the duality back are both pointwise; the dual object of $X \in \text{Ob}\,\mathscr{C}$ is the set $\text{Hom}(X, R) \subset |R|^{|X|}$ with the $\mathbf{T}^*$-structure induced, on $|R|$ by $\mathbf{T}^* \subset \mathbf{U}(|R|)$, and on the power set by the product structure.

Johnstone's book is a very carefully composed first course in all this. The second half is three chapters that our last three illustrations illustrate. Chapter V, Representations of Rings, opens with "A crash course in sheaf theory" ("in conjunction with a first course in sheaf theory it should be sufficient to unlock the rest of the chapter", he says on p. ix), then "The Pierce spectrum", "The Zariski spectrum" (spec_Z above), "Ordered rings and real rings". Chapter VI, Profiniteness and Duality, gives not the nonconstructive master theorem above, but a good selection of more fully worked out special cases, usually with $|R|$ having just two elements. Chapter VII, Continuous Lattices, samples the *Compendium* [5], and (like all Johnstone's book) separates constructive from nonconstructive and relates these matters to locales, which don't occur in [5]. To reach this half, one pretty well has to master the first half on lattices, Heyting algebras, locales, Stone-Čech compatification and $C(X)$. If the author's class that covered "most of" the first half in a semester had not much background in $C(X)$ theory or in constructivism, then they did very well indeed.

For the reader with special purposes leading him in, or with ample time, the book should serve well. He(she) will probably be infuriated by some minor pecularities of the treatment. What galls the reviewer most is that, after carefully distinguishing the locales in Frm^{op} from the frames in Frm, Johnstone avoids the word "frame"; a locale may turn in your hand through 180° at any moment.

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Statistical estimation, asymptotic theory, by I. A. Ibragimov and R. Z. Has′minskii, Applications of Mathematics, vol. 16, Springer-Verlag, New York, 1981, vii + 403 pp., $42.00. ISBN 0-3879-0523-5

Contributions to a general asymptotic statistical theory, by J. Pfanzagl (with the assistance of W. Wefelmeyer), Lecture Notes in Statistics, vol. 13, Springer-Verlag, New York, 1982, vii + 315 pp., $16.80. ISBN 0-3879-0776-9

Asymptotic statistical theory is a body of limit or, better yet, approximation theorems used by statisticians to elude the intractability of all but the very simplest practical statistical problems and to obtain usable results. As such it is not a subject with a well-defined scope or natural boundaries. An early