the very beginnings of the subject through ideals, class numbers etc., and ends with applications to Mersenne primes and diophantine equations. This is achieved with no sacrifice of lucidity.

What is true of this chapter holds to a greater or lesser extent for most chapters. The interested mathematician may approach the material with minimal prior knowledge. The language is classical and the reader will not be impeded by the necessity of having a large mathematical vocabulary. On the other hand, the reader will be amply rewarded with beautiful results of considerable depth and can come away with a sense of satisfaction.

In one of his letters to Sophie Germain, Gauss, referring to number theory, wrote that "the enchanting charms of this sublime science are not revealed except to those who have the courage to delve deeply into them." This book provides an admirable vehicle for so delving.

RAYMOND G. AYOUB

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0273-0979/84 $1.00 + $.25 per page

The analysis of linear partial differential operators. I, by Lars Hörmander, Grundlehren der mathematischen Wissenschaften, Vol. 256, Springer-Verlag, Berlin, 1983, ix + 391 pp., $39.00. ISBN 3-5401-2104-8

The analysis of linear partial differential operators. II, by Lars Hörmander, Grundlehren der mathematischen Wissenschaften, Vol. 257, Springer-Verlag, Berlin, 1983, viii + 389 pp., $ 49.50. ISBN 3-5401-2139-0

Since the second World War the theory of linear partial differential equations has undergone two major revolutions. The first was the advent, in the late forties, of a formalized theory of "generalized functions". Its starting point was the use of test-functions. The idea was not entirely new; it had been introduced earlier in the theory of Radon measures (in particular, on locally compact groups [**Weil 1940**]) and had something to do with the old quantum mechanics: one could not always assign a value at a point to certain "functions", such as Dirac's, but one could "test" them on suitable sets, or "against" suitable functions. In the most important case the test-functions are smooth (i.e., C^∞) and vanish identically off some compact set. The corresponding generalized functions were called "distributions" in [**Schwartz 1948**]. Distribution theory assimilated many ideas and discoveries of the preceding decades (by Heaviside, Hadamard, Sobolev, Bochner and others). To these it added new ones, of which the most successful were perhaps the now-called Schwartz spaces $\mathscr{S}$, $\mathscr{S}'$ and the theory of Fourier transform of tempered distributions—although again the link between slow (or tempered) growth, the Fourier transform and localization, and, beyond, causality, was not absolutely new, and certainly not to physicists. Schwartz gave a strong functional analysis slant to the theory, drawing much from, and enlarging upon, the book [**Banach 1932**]. But the enduring merit of distribution theory has been that the basic operations of analysis, differentiation and convolution (and smoothing), and the Fourier/Laplace transforms and their inversion, which demanded so much care in the classical framework, could now be carried out without qualms by obeying purely algebraic rules.

Distribution and PDE had been growing together for many years before the publication of [**Schwartz 1950**]: the purpose of Heaviside's calculus was to solve the differential equations of electrical networks; the realization by Hadamard that one could not avoid bringing in those special distributions, the "principal values", had originated in the context of the Cauchy problem for the wave equation; the Sobolev estimates and the Sobolev spaces provided the most precise measure of the regularity of the weak solutions to elliptic boundary problems. Now everything was woven together in a harmonious whole, and the tools were standardized. The neatest example of this is perhaps to be found in the statement and proof of the Friedrichs lemma.

Gradually distributions gained acceptance, and in the fifties they gave a renewed impetus to the study of linear PDE. They made the subject appealing to a number of then young analysists, of which the foremost among them were Ehrenpreis in the U.S., Malgrange in France, and, in Sweden, the author of the books under review. It is not easy, today, to appreciate how much of that appeal came precisely from the functional analysis "connection": it held (and to a remarkable extent, kept) the promise that a skillful blend of the Hahn-Banach theorem, or of the closed graph theorem, with some (but not too much) hard analysis would enable one to reap a harvest of beautiful theorems. There was also the belief that the much extended role of the Fourier transform would help achieve the age-old objective of symbolic calculus: to reduce the problems of solving partial differential equations to those of dividing functions, and now distributions, by polynomials—this at a time when algebraic and analytic geometry were making spectacular advances. Today the theorems of the early fifties may seem simple; but this is the usual illusion of hindsight in mathematics. Laurent Schwartz likes to recall a conversation he had with Marcel Riesz in Stockholm in 1949. He had mentioned that there might be a possibility of proving the existence of a fundamental solution for any linear PDE with constant coefficients. To which Riesz replied, "This is a goal for the next century!"

Actually, soon afterwards, a number of important results about general differential operators with constant coefficients were obtained: of course, the existence of fundamental solutions, also the equivalence between the global solvability of the equation $P(D)u = f$ in $C^\infty(\Omega)$ and the $P(D)$-convexity of the open set $\Omega \subset \mathbf{R}^n$ [**Malgrange 1953**]; the characterization of hypoellipticity [**Hörmander 1954**]; the description, by means of their Fourier/Laplace transforms, of *all* the solutions of overdetermined systems of linear PDE with constant coefficients in convex open sets—the so-called "fundamental principle" of Ehrenpreis (ca 1960).

The first seven chapters of Volume I of the books under review here are devoted to a detailed exposition of distribution theory. In my opinion it is the best now available in print. As in the earlier books of the same author, the economy of the argument is unequalled. All the theorems are there (among them the Schwartz kernel theorem), and they all have short, sometimes very short, yet always complete, proofs. The reasoning often follows the most clever shortcuts. The accent is on precision, reliability and brevity. This can make for arduous reading on the part of the inexperienced student, but it provides an ideal text to the mathematician who would like to teach a course or base a seminar on some of the topics discussed in the book. The description of "pure" distribution theory is interspersed with applications to PDE (e.g., §§3.3, 4.4, 6.2). These provide examples, both of the "things" one can do with distributions, and of the general properties of linear PDE, expanded on in the subsequent volume(s).

Volume II is devoted to PDE proper, and essentially to differential operators with constant coefficients. (Practically the only equations with variable coefficients that are touched upon in Volume II are those of equal strength, in Chapter XIII.) To some extent it is an updated version of certain chapters of the book [**Hörmander 1963**] covering fundamental solutions, inhomogeneous equations, hypoellipticity and the Cauchy problem. But it contains material that could not be found in the earlier book: a systematic approach (in Chapter XIII) to the phenomena of nonuniqueness in the Cauchy problem, which originated with the examples of Cohen and Plĭs in the late fifties; a chapter on scattering theory (Chapter XIV); one (Chapter XV) on global weighed estimates for the solutions of the inhomogeneous Cauchy-Riemann equations in $\mathbf{C}^n$, very much in the spirit of the author's approach to the analogous problem in strongly pseudoconvex domains (see Chapter IV in [**Hörmander 1966**]). The estimates in Chapter XV are aimed at the "cohomology with bounds" in $\mathbf{C}^n$, but the author restricts his attention to $(0,1)$-forms and refers to the monographs [**Ehrenpreis 1970**] and [**Palamodov 1970**] for the treatment of equations on $(0, q)$-forms and the important applications to over-determined systems of constant coefficients PDE in convex open sets. The last chapter of Volume II is devoted to convolution equations: approximation of solutions of homogeneous convolution equations by exponential-polynomial solutions, solvability of inhomogeneous convolution equations in convex open sets, hypoelliptic and hyperbolic convolution equations, thus presenting in practically definitive form work by Ehrenpreis, Malgrange, and Hörmander himself which dates back to the fifties. Volume II closes with an Appendix devoted to the study of the properties of polynomial equations in several variables that are needed in the preceding chapters. It is based on Puiseux expansion and the Seidenberg-Tarski theorem, of which a proof is given.

It is time now for this reviewer to go back to the last chapters of Volume I. For their contents partake of the second revolutionary phase of PDE theory and herald the new things to come, and Volume III, to appear soon. The second revolution arrived under the banners of pseudodifferential operators, microlocal analysis, Fourier integral operators and Sato's hyperfunctions. It sprang to life after Calderon's work on uniqueness in the Cauchy problem and the Atyah-Singer index theorem; it gained momentum thanks to the interest in subelliptic estimates, in local solvability and in propagation of singularities.

But this is another story, better left to the reviewer of Volume III. As a preparation to all this new "microlocal" world, Chapter VII of Volume I presents a detailed treatment of the stationary phase formula (and a proof of the Malgrange preparation theorem, eventually needed in the microlocal reduction to standard forms). Chapter VIII is entirely devoted to the wave-front set, which is the central notion of (the first) microlocalization. Chapter IX looks at the analytic wave-front set and introduces a definition of hyperfunctions in the spirit of Martineau.

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FRANCOIS TREVES

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Shape theory, by S. Mardešić and J. Segal, North-Holland, Amsterdam, The Netherlands, 1982, xv + 378 pp., $81.50, Dfl 175.00. ISBN 0-4448-6286-2

The appearance of a book on shape theory provides the reviewer with the opportunity of assessing where shape theory came from, and what of value is coming out of it.[1]

1. A little history: Čech homology 1928–1968. In the late twenties, there was point set topology and there was algebraic topology, but the correct relationship between the two subjects had not yet become clear. In those days, algebraic topology meant, in the main, the homology theory of simplicial complexes with integer coefficients. The topological invariance of this theory was more or less established, but the restriction of the theory to polyhedra appeared to the point set topologist to be arbitrary and ugly.

Then, in 1928, Alexandroff [2] discovered a theorem which, with hindsight, can be seen to express the proper relationship between the two subjects. The

[1] References refer to the bibliography of the book under review.