BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 10, Number 1, January 1984

0273-0979/84 $1.00 + $.25 per page

Classes unipotentes et sous-groupes de Borel, by Nicolas Spaltenstein, Lecture Notes in Math., vol. 946, Springer-Verlag, Berlin and New York, 1982, ix + 259 pp., $14.00. ISBN 3-5401-1585-4.

In the last fifteen years or so a great deal of work has been done on geometric problems related to unipotent elements of a semisimple or, more generally, reductive algebraic group G. If u is a unipotent element of G, one is interested, in particular, in geometric properties of $\mathfrak{B}_u$, the variety of fixed points of u for the action of G on the variety $\mathfrak{B}$ of Borel subgroups of G, and in other closely related varieties. The varieties $\mathfrak{B}_u$ are of considerable interest in themselves. For example, for u a "subregular" element of G, they are closely related to the Kleinian singularities of a normal algebraic surface; see [**4**] for details. However, the chief motivation for the study of these varieties comes from two (apparently) distinct areas of representation theory: (i) the Deligne-Lusztig theory of (complex) representations of the finite Chevalley group $G(\mathbf{F}_q)$ (here G is defined over the finite field $\mathbf{F}_q$); and (ii) the (infinite-dimensional) representation theory of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g} = \mathrm{Lie}(G)$ (here G is defined over the complex numbers). In the representation theory of $G(\mathbf{F}_q)$ and $U(\mathfrak{g})$, very deep and delicate properties of the varieties $\mathfrak{B}_u$ play a crucial technical role in a way that is only now beginning to be understood (by a few experts; the reviewer does not belong to this small group). For example, in some of the recent work on the subject the (Deligne-Goresky-Macpherson) intersection homology groups of $\mathfrak{B}_u$ play an important technical role; see [**2**] for more details.

The link between the representation theories of $G(\mathbf{F}_q)$ and $U(\mathfrak{g})$ and the varieties $\mathfrak{B}_u$ is the Springer theory of Weyl group representations [**1**]. Very briefly, this theory can be summarized as follows. If u is unipotent, there is a canonically defined representation of the Weyl group W of G on the (étale) cohomology groups $H^i(\mathfrak{B}_u)$. Let C be a set of representatives for the (finite) set of unipotent conjugacy classes of G. For $u \in C$, let $e(u) = \dim \mathfrak{B}_u$. Then for each irreducible character χ of W, there exists a unique $u \in C$ such that χ occurs in the W-module $H^{2e(u)}(\mathfrak{B}_u)$.

The main objective of Spaltenstein's book, *Classes unipotentes et sous-groupes de Borel*, is the study of the varieties $\mathfrak{B}_u$. Chapter II, which is more than half of the book, is devoted to the study of these varieties. The author does not develop the link with the representation theories of $G(\mathbf{F}_q)$ and $U(\mathfrak{g})$, nor with the Springer theory of Weyl group representations. However he does give a detailed and elegant treatment of geometric properties of $\mathfrak{B}_u$. In the other chapters he gives very detailed information on classification of unipotent conjugacy classes, closures of unipotent classes, induced unipotent classes and the "Richardson classes" associated to parabolic subgroups. He also discusses a duality theory for unipotent classes.

In most of the book the author considers the more general situation in which G is not necessarily connected, but the identity component G^0 is reductive. In this review we will confine our comments to the case of connected G.

In order to discuss the results in Chapter II, we need more notation. Let G be a (connected) reductive algebraic group over an algebraically closed field and let $\mathcal{B}$ denote the variety of Borel subgroups of G. Let $B \in \mathcal{B}$, let T be a maximal torus of B and let $W = N_G(T)/T$ be the Weyl group. The variety $\mathcal{U}$ of unipotent elements of G is connected and irreducible, and the set $\mathrm{Cl}(\mathcal{U})$ of unipotent conjugacy classes is finite. If $u \in \mathcal{U}$, then $\mathrm{Cl}(u)$ denotes the conjugacy class of u. Let C be a set of representatives for the unipotent classes in G.

For $u \in \mathcal{U}$, let $S(u)$ denote the set of irreducible components of $\mathcal{B}_u$ and let $A(u)$ denote the finite group $C_G(u)/C_G(u)^0$. Each element of $A(u)$ permutes the elements of $S(u)$. If $\sigma \in S(u)$, then we frequently denote by X_σ the corresponding closed irreducible subvariety of $\mathcal{B}_u$. If $\sigma, \tau \in S(u)$, then X_σ and X_τ are of the same dimension.

The Weyl group W enters the picture in the following manner. For each $w \in W$, let $\mathcal{O}(w)$ denote the G-orbit of $(B, {}^wB)$ on $\mathcal{B} \times \mathcal{B}$. It follows from the Bruhat decomposition of G that each G-orbit on $\mathcal{B} \times \mathcal{B}$ is equal to exactly one of the $\mathcal{O}(w)$'s. If $(\sigma, \tau) \in S(u) \times S(u)$, then there exists a unique $w \in W$ such that $\mathcal{O}(w) \cap (X_\sigma \times X_\tau)$ is dense in $X_\sigma \times X_\tau$; set $w = \varphi_u(\sigma, \tau)$. The map φ_u is constant on $A(u)$-orbits on $S(u) \times S(u)$ and thus induces a map of the set of orbits $(S(u) \times S(u))/A(u)$ to W. By a straightforward geometric argument, due essentially to Steinberg [3], it can be shown that the maps φ_u, $u \in C$, determine a bijection of $\bigcup_{u\in C}(S(u) \times S(u))/A(u)$ onto W. (For this result one needs minor restrictions on $\mathrm{char}(K)$.) If G is a classical group, then $\Sigma_{u\in C} |S(u)|$ is equal to the number of involutions in W, which is in turn equal to the sum of the degrees of the irreducible representations of W.

If $G = \mathrm{GL}_n(K)$, then the results are even sharper. In this case W is the symmetric group S_n and, for every $u \in \mathcal{U}$, $C_G(u)$ is connected and hence $A(u) = \{e\}$. Thus in this case we have a bijection of $\bigcup_{u\in C} S(u) \times S(u)$ onto W. Let $u \in \mathcal{U}$. We may assume u is in Jordan canonical form. If $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$ are the sizes of the Jordan blocks, then $\lambda(u) = (\lambda_1, \ldots, \lambda_r)$ is a partition of n and the conjugacy class of u is completely determined by the partition $\lambda(u)$. Thus we have a canonical bijective map from $\mathrm{Cl}(\mathcal{U})$ to the set $\mathcal{P}_n$ of partitions of n. On the other hand the set of irreducible representations of $W = S_n$ is canonically parametrized by the set $\mathcal{P}_n$. If $\lambda \in \mathcal{P}_n$, then the dimension of M_λ, the corresponding irreducible S_n-module, is equal to the number of "standard tableaux of shape λ". But there is a geometric connection. By a direct and elementary geometric argument, the author defines a natural bijection from the set $S(u)$ to the set of standard tableaux of shape $\lambda(u)$. In particular, $\dim M_{\lambda(u)} = |S(u)|$. Using the correspondence between $S(u)$ and standard tableaux, the bijection $\bigcup_{u\in C} S(u) \times S(u) \to S_n$ can be interpreted as a bijection from the set of all pairs of standard tableaux of the same shape to the symmetric group S_n. But such a bijection is well known to combinatorialists; it is the Robinson-Shenstead correspondence. A previously unpublished proof (due independently to Steinberg and the author) that the above bijection agrees with the Robinson-Shenstead correspondence is given in this book.

We only have space to discuss a few of the results of Chapter II. However the flavour of the results is clear. There are remarkable and unexpected connections between the varieties $\mathfrak{B}_u$ and the Weyl group and its representations. And for the case of $GL_n(K)$ one gets surprising new geometric interpretations of classical combinatorial concepts related to the symmetric group.

Chapter III discusses a possible duality theory for unipotent conjugacy classes. If $u, v \in \mathfrak{U}$, then we say that $Cl(u) \leqslant Cl(v)$ if $Cl(u)$ is contained in the closure of $Cl(v)$. For $G = GL_n(K)$ there is a natural order-reversing duality map, d: $Cl(\mathfrak{U}) \to Cl(\mathfrak{U})$, defined as follows: $d(Cl(u)) = Cl(v)$ if $\lambda(v)$ is the conjugate partition of $\lambda(u)$. In this case d is a bijection and d^2 is the identity. To define what the author calls a duality map in the general case we need more notation. If P is a parabolic subgroup of G, then the Richardson class $C_P \in Cl(\mathfrak{U})$ is the class which contains a dense open subset of the unipotent radical of P. We let $C_P^* \in Cl(\mathfrak{U})$ be the class containing the regular unipotent elements of a Levi subgroup of P. A *duality map* is a map d: $Cl(\mathfrak{U}) \to Cl(\mathfrak{U})$ satisfying the following conditions: (i) if $C_1 \leqslant C_2$, then $d(C_1) \geqslant d(C_2)$; (ii) if P is a parabolic subgroup of G, then $d(C_P^*) = C_P$; and (iii) $C \leqslant d(d(C))$ for every C. The author shows that for every reductive G there exists a duality map d, and that if G is a classical group, then d is unique. The approach is purely empirical and depends on complete information on the classification of unipotent classes and the order relation on these classes.

Chapter IV consists of tables which summarize most of the known information on classification of unipotent classes, closures of classes, induced conjugacy classes, centralizers of unipotent elements, etc. This information will be extremely useful to experts.

As mentioned earlier, the author considers throughout most of this book unipotent elements in (not necessarily connected) algebraic groups G such that G^0 is reductive. This greater generality is a considerable complication. The author manages to handle the resulting technical problems in an impressive way. However, the greater generality means that most of the proofs are more complicated than for the case of connected groups. This has the unfortunate effect of making the book much more difficult to read than it would have been had the author restricted attention to the case of connected reductive groups.

In conclusion, this is a careful and elegant treatment of a difficult and important topic in the theory of algebraic groups. It is certainly a very valuable book for the expert. However, it is not a book for beginners.

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