BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 10, Number 1, January 1984

0273-0979/84 $1.00 + $.25 per page

The geometry of Coxeter groups, by H. Hiller, Research Notes in Mathematics, Vol. 54, Pitman Advanced Publishing Program, Boston, 1982, 213 pp., $21.95. ISBN 0-2730-8517-4

Quotients of Lie groups appear throughout mathematics and physics. To an algebraist, the nicest Lie groups are the simple Lie groups over the complex numbers. The most interesting quotients formed from these groups are the *flag manifolds* G/P obtained by dividing out by a "parabolic" subgroup. The algebraic geometric study of flag manifolds reveals several interesting interactions between algebraic geometry, representation theory, and combinatorics. The main topic of the book under review is the description of the cohomology rings of complex flag manifolds with an accent on the combinatorial aspects thereof.

By far the most famous of the flag manifolds are the Grassmannians. A Grassmannian $G_d(V)$ consists of the set of all d-dimensional subspaces of an n-dimensional complex vector space V together with a suitable topology. To express $G_d(V)$ in the form G/P, first note that the group of linear transformations with determinant 1, the special linear group $\mathrm{SL}(V)$, acts transitively on d-dimensional subspaces. Pick any d-dimensional subspace W, and let P be the subgroup of $G = \mathrm{SL}(V)$ which stabilizes W. Then G/P with the quotient topology is the desired Grassmannian. More generally, let $W_1 \subset W_2 \subset \cdots \subset W_k$ be a strictly increasing sequence of subspaces of V (a *flag*), and let P be the subgroup of G which stabilizes these subspaces. Most often a maximal flag ($k = n$) is taken, and then P is a "Borel" subgroup B. The resulting manifold G/B is sometimes referred to as *the* flag manifold.

Grassmannians can be generalized in a second direction. Let G be the special orthogonal group $\mathrm{SO}(V)$ or symplectic group $\mathrm{Sp}(V)$. These are the subgroups of $\mathrm{SL}(V)$ which preserve symmetric or antisymmetric bilinear forms, respectively. Then the parabolic subgroups are again stabilizers of (suitably defined) flags of subspaces. All the simple Lie groups are known. Up to simple connectedness, there are only five other complex simple Lie groups: E_6, E_7, E_8, F_4, and G_2. Flag manifolds can be formed from these groups using the general definition of parabolic subgroup given below.

What is a Coxeter group? There is a general definition, but Hiller's book is mainly concerned with finite Coxeter groups. Ignoring the dihedral groups, there are only two finite Coxeter groups which are not *Weyl* groups. Let E be an n-dimensional Euclidean space. A Weyl group is a finite subgroup of the orthogonal group $\mathrm{O}(E)$ which is generated by n reflections and which leaves an n-dimensional lattice of points in E invariant (hence the chemists' terminology: "point crystallographic groups"). Irreducible Weyl groups have been classified. There are 3 infinite families: the symmetries of the regular n-simplex (the symmetric group S_n), the symmetries of the n-cube, and a certain index 2 subgroup of the cube group. And again there are five exceptional cases.

Now for the connection between Coxeter groups and Lie groups. First we will switch to algebraic geometric language and talk of algebraic groups rather than Lie groups. The structure of simple algebraic groups is very beautiful and well known **[Hum]**. The usual approach is to fix a subgroup T (for torus) which is isomorphic to the algebraic group of $n \times n$ diagonal matrices, with n as large as possible, and then to fix a maximal connected solvable subgroup B (a *Borel* subgroup) containing T. A parabolic subgroup is by definition any closed subgroup containing a Borel subgroup. These subgroups are the only subgroups P having the property that G/P is a projective variety. All parabolic subgroups are conjugate to some parabolic subgroup containing the fixed Borel subgroup B, and there are only 2^n of these. They are indexed by subsets of nodes of the Dynkin diagram. The *Weyl group of* G is defined to be the normalizer of the torus modulo the torus, $W = N_G(T)/T$. This definition is admittedly uninspiring to any but the most hardcore algebraists, but space does not permit us to describe how this W actually acts upon a Euclidean space associated to G. Every Weyl group arises from one or two algebraic groups in this manner. In the present context the main role of the Weyl group is to provide a set of coset representatives for the double coset decomposition of G with respect to B: $G = \bigcup_{w \in W} BwB$, the Bruhat decomposition. If $G = \mathrm{SL}(n+1)$, then T is taken to be the diagonal matrices and B is taken to be the upper triangular matrices. Then the Weyl group can safely be thought of as the permutation matrices. The Bruhat decomposition projects down onto flag manifolds: $G/P = \bigcup_{w \in W^J} BwP/P$, where W^J is a set of coset representatives for W/W_J, and W_J is the "parabolic" subgroup of W corresponding to $P \supseteq B$. The closures $\overline{BwP/P}$ of the cells are called *Schubert varieties*.

One of the main constructions in Hiller's book is that of *Bruhat order*. This order is defined on the elements of a Weyl group by ordering the Schubert varieties of a flag manifold G/B; namely, $w \leqslant w'$ iff $\overline{BwB/B} \subseteq \overline{Bw'B/B}$. Alternatively, an equivalent definition can be easily formulated in abstract Coxeter group terms and applied to arbitrary Coxeter groups. Either definition can be extended to the coset spaces W^J corresponding to arbitrary G/P. The Bruhat orders comprise an important, combinatorial aspect of Lie theory: Not only are they finite ordered discrete structures (and thus combinatorial in the modern sense), but their elements and ordering can usually be described with permutations of multisets and tableaux (and are thus combinatorial in the traditional sense) **[Pr1]**. These nice descriptions arise because the three infinite families of Weyl groups are either symmetric groups or closely related to symmetric groups. Hiller summarizes many of the known combinatorial results concerning Bruhat posets in a section of the last chapter of his book. Three of the most interesting of these results are **[Ver, St1**, and **B-W]**.

Bruhat orders are used most often in the book under review to index nice bases for the cohomology rings $H^*(G/P)$. Since the spaces G/P are manifolds, subvarieties define cohomology classes, and intersection of subvarieties corresponds to cup product. The Schubert varieties provide a cellular decomposition (or a finite CW-decomposition) for G/P, implying that the classes defined by the Schubert varieties form an additive basis for the cohomology. The need to study the cohomology of Grassmannians reachs back to the work of Schubert.

He considered such enumerative questions as: How many lines in complex projective 3-space intersect four given lines? (Answer: 2.) Schubert's methods were not rigorous, but he was able to answer this and many other much harder questions correctly. Widespread concern over the validity of Schubert's methods led Hilbert to pose the rigorous justification of Schubert's results as his 15th problem in 1900. Cohomology theory today provides the correct setting for these computations. Two good expository papers on this subject have been written by Kleiman and Laksov [**K-L, Kle**]. Each Schubert variety in $G_d(n)$ consists of a set of d-subspaces satisfying certain linear conditions. Roughly speaking, requiring a d-subspace to satisfy various conditions can correspond to intersecting these subvarieties or taking the cup product. Thus a certain description of $H^*(G_d(n))$ which provides for effective computations has come to be known as the *Schubert calculus*. Describing a Schubert calculus for an arbitrary flag manifold is the central topic of the book under review. It should be noted that the standard Schubert calculus for the Grassmannian is not sufficient to justify all of Schubert's computations. Schubert's claim that 666,841,088 space quadrics are tangent to 9 quadrics in general position has only recently been confirmed by DeConcini and Procesi [**D-P**]. The standard Schubert calculus is most effective with problems concerning linear varieties in projective space.

As the author observes in a supplementary section to Chapter III, there are many methods and/or viewpoints by which the cohomology of complex Grassmannians can be derived and/or described, including invariant differential forms [**Sto**] and the theory of symmetric functions [**St2, Las**]. One viewpoint which generalizes to arbitrary flag manifolds is that of Lie algebra cohomology [**Kos**]. Hiller considers two approaches to cohomology, initially in both cases just for Grassmannians. The first approach is the Schubert calculus viewpoint: "Special" Schubert classes (which turn out to be the normal Chern classes) are selected, and the cup products of these classes with arbitrary Schubert classes are described (Pieri's formula). Then an arbitrary Schubert class is expressed as a cup product polynomial in the special Schubert classes (Giambelli's formula). Griffiths and Harris also cover this material [**G-H**]. Hiller and Boe have recently found explicit, combinatorial descriptions of Pieri's formula in the cases $\mathrm{SO}(2n+1)/\mathrm{SL}(n)$ and $\mathrm{Sp}(2n)/\mathrm{SL}(n)$ [**H-B**]. The second approach to the cohomology of the Grassmannians (which is only outlined) is to describe $H^*(G_\alpha(n))$ in terms of Chern class generators and relations.

The main subject of this book is a description of the cohomology of arbitrary flag manifolds which combines these two approaches, following the work of Bernstein, Gelfand, and Gelfand [**BGG**] and Demazure [**De2**] of the early 70s. Let the Weyl group W of G be realized in the usual fashion $W \subset \mathrm{O}(E)$. Then W also acts on the polynomial functions on the complexification V of E, which form a graded ring $S(V)$. Let I_W be the homogeneous ideal of $S(V)$ which is generated by the W-invariant polynomials of positive degree. Then the quotient $S_W = S(V)/I_W$ is known as the *coinvariant ring of* W. Borel showed in 1953 that S_W and $H^*(G/B, \mathbf{C})$ are isomorphic as graded rings [**Bo1**]. But this fact doesn't contain as much geometric information as one would like. The Weyl groups of $SO(2n + 1)$ and $Sp(2n)$ coincide, implying that their coinvariant algebras do, but the manifolds $SO(2n + 1)/B$ and $Sp(2n)/B$ are different. What is needed is knowledge of the cup product multiplication *with respect to some geometrically defined basis for* $H^*(G/B)$, i.e. a generalization of the Schubert calculus. Chevalley found an analog to Pieri's formula for general G/P in 1958 [**Che, De1**]. Demazure and BGG independently combined the results of Borel and Chevalley in 1973: They constructed a Schubert calculus for G/B in the abstract algebraic setting of S_W. Hiller presents this construction of the Schubert calculus in some detail. Unfortunately, he only has time to outline BGG's proof that the ring so constructed coincides with $H^*(G/B)$. Giving full treatment to the proof would require spending some time on the subject of line bundles on flag manifolds.

An interesting present day descendent of the papers of Demazure and BGG is the work of Lascoux and Schützenberger concerning "Schubert polynomials" and "flag Schur functions" [**L-S**]. One consequence of this work is a third proof (in addition to [**St3**] and [**E-G**]) of the following difficult conjecture of Stanley: The number of ways of passing from the permutation $1, 2, \ldots, n - 1, n$ to $n, n - 1, \ldots, 2, 1$ using adjacent transpositions is equal to the number of standard Young tableaux on the perfect staircase shape with $n - 1$ squares in the first row.

The smallest flag manifolds G/P are the most tractable; these occur when P is a maximal (by containment) parabolic subgroup. Grassmannians are of this form. But Grassmannians have an additional nice property: Whenever a Schubert class $[X]$ is multiplied by the unique codimension 1 Schubert class $[U]$ in $H^*(G_d(n))$, the result is the sum (with all coefficients 1) of the Schubert classes lying just above $[X]$ in the Bruhat order. A few other flag manifolds formed with maximal P have this property: one other nontrivial infinite family $SO(k)/P_{\text{spin}}$, two trivial infinite families, and two exceptionals E_6/D_5 and E_7/E_6. Flag manifolds with this property are called *minuscule*. It seems that anything true for Grassmannians is true for all minuscule cases. The minuscule flag manifolds occupy a very pretty area of overlap between algebraic geometry, representation theory, and combinatorics. Topics related to the minuscule flag manifolds include the 27 lines on a cubic curve in projective 3-space, the spin representation of the orthogonal group, and the ballot problem from combinatorics.

One interesting connection between algebraic geometry and combinatorics arises during the computation of the self-intersection multiplicity of the unique codimension 1 Schubert variety U of a minuscule flag manifold G/P. This computation is performed in the last chapter of Hiller's book, and also appears as a paper [**Hi1**]. If the complex dimension of G/P is d, then the intersection of d generic varieties cohomologous to U is a union of N points. In cohomology this is expressed as $[U]^d = N[Z]$, where Z is the unique 0-dimensional Schubert variety in G/P and N is the self-intersection multiplicity of U. Using the Bruhat poset description of $H^*(G/P)$ together with the fact that all coefficients are 1 in the sum for $[U] \cup [X]$ in the minuscule case, it is easy to see that N must also be the number of strictly increasing maximal chains in the

Bruhat poset. The posets for the two nontrivial infinite families of minuscule flag manifolds have arisen independently in combinatorics [**Li1**, **Li2**]. Counting strict maximal chains in the poset corresponding to the Grassmannian is equivalent to a special case of the ballot problem: Given k candidates in an election which eventually receive $a_1 \geqslant a_2 \geqslant \cdots a_k \geqslant 0$ votes, respectively, what is the probability that if the ballots are counted one at a time that candidate #1 will never trail candidate #2, candidate #2 will never trail candidate #3, ... in the intermediate vote tallies? This problem was solved by Bertrand in 1887 for $k = 2$ and by MacMahon in 1915 for general k. But Schubert and Frobenius obtained expressions in the 1890s for equivalent quantities in their contexts, the Schubert calculus and representations of the symmetric group, respectively. To get strict maximal chains in the Bruhat poset associated to $G_d(n)$, take $k = d$ and $a_i = n - d$ for $1 \leqslant i \leqslant d$. The general ballot problem is equivalent to finding the number of standard Young tableaux of a given shape, which in the theory of symmetric groups is the dimension of the irreducible representation indexed by that shape. It was in this context that Frame, Robinson, and Thrall derived their beautiful but mysterious "hook" formula for this quantity. And (as Hiller notes) a similar formula developed by Schur during his study of projective representations of the symmetric group, the hook formula for *shifted* standard Young tableaux, counts the number of strict maximal chains in the Bruhat posets for the minuscule flag manifolds $\mathrm{SO}(k)/P_{\mathrm{spin}}$.

Besides ordinary cohomology, another major (but interrelated) topic concerning flag manifolds is the study of line bundles L on these spaces and the associated sheaf cohomology $H^*(G/P, L)$. For a start in this subject, see the nice introductory article [**Bo2**]. *Very ample* line bundles on an abstract projective variety correspond to embeddings of the variety in projective space. Line bundles on G/P's have the additional property that they carry representations of the group G. In fact, isomorphism classes of very ample line bundles on G/B correspond naturally with equivalence classes of finite dimensional representations of G.

In addition to providing the connection between the Schubert structure erected on the coinvariant ring S_W and $H^*(G/B)$, line bundles on flag manifolds also appear in Hiller's book when he gives a brief summary of a recent result of Seshadri [**Ses**]. Any flag manifold formed with a maximal parabolic subgroup has a very ample line bundle L which gives an embedding of the manifold analogous to the Plücker embedding of the Grassmannian. Once a Borel subgroup (and therefore a root system) has been fixed, there is a natural way to index both maximal parabolic subgroups P_j of G and fundamental highest weights λ_j of G with positive simple roots α_j, $1 \leqslant j \leqslant n$. If L is the Plücker-like line bundle for P_j, then by the Borel-Weil theorem the vector space of line bundle global sections $H^0(G/P_j, \otimes^m L)$ is an irreducible G-module with highest weight $m\lambda_{i(j)}$, where i is the Weyl automorphism on the positive simple roots. But it is also known that $H^0(G/P_j, \otimes^m L) \cong R_m$, the mth graded piece of the homogeneous coordinate ring for G/P in the given embedding. Now we can state Seshadri's result: If P_j is a minuscule maximal parabolic subgroup, then there is a vector space basis for R_m indexed by *weakly increasing* chains with m elements in the Bruhat poset associated to $G/P_{i(j)}$. But the Bruhat posets associated to G/P_j and $G/P_{i(j)}$ are isomorphic. Thus the number of weak m-chains in these posets is the dimension of the mth graded piece of the coordinate ring of G/P_j in this embedding, whereas the number of strict maximal chains is the self-intersection multiplicity of the Schubert variety $U \subset G/P$. It is natural to ask, as does Hiller in **[Hi1]**, if there is any theoretical connection here.

Before answering this, we will make some additional remarks on Seshadri's work. The language of **[Ses]** and succeeding papers is heavily algebraic geometric, and the introductions emphasize applications to sheaf cohomology vanishing theorems. But it should be noted that the central results, describing bases for finite dimensional representations in terms of posets or poset-like objects, can be expressed entirely in terms of the language of representation theory. Since this result is of great interest in the context of representation theory alone, it would be desirable to have a proof which does not use algebraic geometry. When the Bruhat poset for the Grassmannian is described combinatorially, Seshadri's chains become special cases of the semistandard Young tableaux often used in physics texts to describe bases for representations of $\mathrm{SL}(n)$. In later papers Seshadri and coworkers Lakshmibai and Musili have extended this "standard monomial" theory not only to all representations of $\mathrm{SL}(n)$ (thereby producing all semistandard Young tableaux), but also to any representation of $\mathrm{SO}(k)$ or $\mathrm{Sp}(2n)$ **[LMS]**. These results have many applications. In addition to sheaf cohomology computations, they also can be used to prove that the homogeneous coordinate rings of certain Schubert varieties are arithmetically Cohen-Macaulay **[D-L, H-L]**. (These two papers use in addition the combinatorial-topological notion of the shellability of certain simplicial complexes associated to the Bruhat posets **[B-W]**.)

Now for the answer: Yes, there is a theoretical connection. The *degree* of an embedded projective variety of dimension d is the number of points M left after intersecting the variety with d generic hyperplanes in the ambient projective space. It is easily computed if one knows the Hilbert polynomial $h(m) = \dim_{\mathbf{C}} R_m$ $(m \geqslant 0)$ for the homogeneous coordinate ring: $M = d!h_d$, where h_d is the leading coefficient of the Hilbert polynomial. Now it is an easy fact that if a finite partially ordered set has $d+1$ elements in its longest strict maximal chains, then the number of weakly increasing chains with m elements is a polynomial $z(m)$ of degree d. Furthermore, the number of strict maximal chains is $N = d!z_d$, where z_d is the leading coefficient of $z(m)$. But by Seshadri's result, $h(m) = z(m)$. Thus $M = N$, and the degree of G/P_j is equal to the number of strict maximal chains in the associated Bruhat poset. (Stanley used this line of reasoning to show that the degree of a skew Schubert variety in a Grassmannian could be computed by finding the number of skew standard Young tableaux of a certain shape **[St2]**.) But N is also the self-intersection multiplicity of the Schubert variety U. Therefore the degree of the embedded manifold is equal to the self-intersection multiplicity of the unique codimension 1 Schubert variety. But this must be true, since it is known that under the given Plücker-like embedding this Schubert variety *is* a hyperplane section for the overlying space! Because of the two roles (intersection theory and representation theory) played by isomorphic Bruhat posets for G/P_j and $G/P_{i(j)}$, the algebraic geometry picture matches up perfectly with the combinatorial picture, right down to a two variable generating function identity which arises in each context independently [**Pr2**]. In spite of this pretty picture, the quickest way of computing the degree of *any* Plücker-embedded G/P_j with P_j maximal is to use no combinatorics at all. Since R_m is the irreducible G-module with highest weight $m\lambda_{i(j)}$, the Weyl dimension formula can be used to express $h(m)$ as a polynomial in m. This method dates back to Hirzebruch [**Hir**], who found $M = 13{,}110$ for the exceptional flag manifold E_7/E_6, as opposed to Hiller's miscounted 13,188. Robert Steinberg notes that $13{,}110 = 2 \cdot 3 \cdot 5 \cdot (18 + 1) \cdot 23$ and $78 = 2 \cdot 3 \cdot (12 + 1)$ for E_6/D_5, where 18 and 12 are magic numbers (the "Coxeter numbers") for E_7 and E_6 (personal communication).

Chapter I of *The geometry of Coxeter groups* consists of a handy collection (with proofs) of some basic facts about Coxeter groups. Included are Tits' theorem that the canonical reflection representation of any Coxeter group generated by a finite number of reflections is faithful and discrete, the classification of finite Coxeter groups, and basic facts about the Bruhat order. The standard reference for Coxeter groups is [**Bou**]. Chapter II presents standard material on the invariant theory of finite Coxeter groups in the context of complex reflection groups. There is no shortage of accessible treatments of this subject; e.g. see [**Fla, St4**], or Chapters 9 and 10 of [**Car**]. Chapters III and IV deal with the cohomology of Grassmannians and general flag manifolds, respectively, as described above. Chapter IV is probably the first appearance of [**BGG**] and [**De2**] in book form. This chapter also appears as [**Hi2**]. The last chapter, Chapter V, consists of miscellaneous topics involving the Bruhat order, including the self-intersection multiplicity computations and the description of Seshadri's result mentioned above.

This book is well written, especially for a set of lecture notes. It is perhaps closer to a "real" book in quality and care of preparation: not typeset but virtually no typographical errors, and a smallish but adequate index. There are plenty of references: 163 for a 212 page book, many to original sources. It should be readily accessible to second year graduate students. The chapter introductions and concluding sections are interesting and useful, containing both historical notes and remarks on contemporary related topics.

References

[**BGG**] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, *Schubert cells, and the cohomology of the spaces G/P*, Russian Math. Surveys **28** (1973), no. 3, 1–26.

[**B-W**] A. Björner and M. Wachs, *Bruhat order of Coxeter groups and shellability*, Adv. in Math. **43** (1982), 423–435.

[**Bo1**] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts*, Ann. of Math. (2) **57** (1953), 115–207.

[**Bo2**] ______, *Linear representations of semi-simple algebraic groups*, Algebraic Geometry, Arcata, 1974 (R. Hartshorne, ed.), Proc. Sympos. Pure Math., vol. 29, Amer. Math. Soc., Providence, R.I., 1975, pp. 421–441.

[**Bou**] N. Bourbaki, *Groupes et algèbres de Lie*, Chap. 4–6, Hermann, Paris, 1968.

[**Car**] R. W. Carter, *Simple groups of Lie type*, Wiley, New York, 1972.

[Che] C. Chevalley, *Sur les décompositions cellulaires des espaces G/B* (manuscrit non publié), circa 1958.

[D-L] C. DeConcini and V. Lakshmibai, *Arithmetical Cohen-Macaulayness and arithmetical normality for Schubert varieties*, Amer. J. Math. **103** (1981), 835–850.

[D-P] C. DeConcini and C. Procesi, *Complete symmetric varieties* (preprint).

[De1] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 53–88.

[De2] ______, *Invariants symétriques des groupes de Weyl et torsion*, Invent. Math. **21** (1973), 287–301.

[E-G] P. Edelman and C. Greene, *Combinatorial correspondences for Young tableaux, balanced tableaux, and maximal chains in the Bruhat order of S_n* (research announcement).

[Fla] L. Flatto, *Invariants of finite reflection groups*, Énseign. Math. (2) **24** (1978), 237–292.

[G-H] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

[Hi1] H. Hiller, *Combinatorics and intersections of Schubert varieties*, Comment. Math. Helv. **57** (1982), 41–59.

[Hi2] ______, *Schubert calculus of a Coxeter group*, Énseign. Math. (2) **27** (1982), 57–84.

[H-B] H. Hiller and B. Boe, *Pieri formula for SO_{2n+1}/U_n and Sp_n/U_n* (preprint).

[Hir] F. Hirzebruch, *Characteristic numbers of homogeneous domains*, Seminars on Analytic Functions, Vol. 2, Institute for Advanced Study, Princeton, N.J., 1957.

[Hum] J. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York, 1975.

[H-L] C. Huneke and V. Lakshmibai, *Arithmetic Cohen-Macaulayness and normality of the multicones over Schubert varieties in* $\mathrm{SL}(n)/B$ (preprint).

[K-L] S. L. Kleiman and D. Laksov, *Schubert calculus*, Amer. Math. Monthly **79** (1972), 1061–1082.

[Kle] S. L. Kleiman, Problem 15: *Rigorous foundation of Schubert's enumerative calculus*, Mathematical Developments Arising from Hilbert Problems (F. Browder, ed.), Proc. Sympos. Pure Math., vol. 28, Amer. Math. Soc., Providence, R.I., 1976, pp. 445–482.

[Kos] B. Kostant, *Lie algebra cohomology and generalized Schubert cells*, Ann. of Math. (2) **77** (1963), 72–144.

[Las] A. Lascoux, *Fonctions de Schur et grassmanniennes*, C. R. Acad. Sci. Paris Sér. A **281** (1975), 813–815; 851–854.

[L-S] A. Lascoux and M.-P. Schützenberger, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. A **295** (1982), 629–633.

[Li1] B. Lindström, *Conjecture on a theorem similar to Sperner's*, Combinatorial Structures and their Applications (R. Guy, ed.), Gordon and Breach, New York, 1970.

[Li2] ______, *A partition of $L(3, n)$ into saturated symmetric chains*, European J. Combin. **1** (1980), 61–63.

[LMS] V. Lakshmibai, C. Musili and C. S. Seshadri, *Geometry of G/P*, Bull. Amer. Math. Soc. (N.S.) **1** (1979), 432–435.

[Pr1] R. Proctor, *Classical Bruhat orders and lexicographic shellability*, J. Algebra **77** (1982), 104–126.

[Pr2] ______, *Bruhat lattices, plane partition generating functions, and minuscule representations*, European J. Combin. (to appear).

[Ses] C. S. Seshadri, *Geometry of G/P*. I: *Theory of standard monomials for minuscule representations*, C. P. Ramanujan—A tribute, Tata Inst. Fund. Res. Studies in Math., 8, Springer, Berlin, 1978, pp. 207–239.

[St1] R. Stanley, *Weyl groups, the hard Lefschetz theorem and the Sperner property*, SIAM J. Algebraic Discrete Methods **1** (1980), 168–184.

[St2] ______, *Some combinatorial aspects of the Schubert calculus*, Combinatoire et Representation du Groupe Symétrique (D. Foata, ed.), Lecture Notes in Math., No. 579, Springer-Verlag, New York, 1977.

[St3] ______, *On the number of reduced decompositions of elements of Coxeter groups* (preprint).

[St4] ______, *Invariants of finite groups and their applications to combinatorics*, Bull. Amer. Soc. (N.S.) **1** (1979), 475–511.

[Sto] W. Stoll, *Invariant forms on Grassmann manifolds*, Ann. of Math. Studies, No. 89, Princeton Univ. Press, Princeton, N.J., 1977.

[Ver] D.-N. Verma, *Möbius inversion for the Bruhat ordering on a Weyl group*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 393–398.

ROBERT A. PROCTOR

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 10, Number 1, January 1984

0273-0979/84 $1.00 + $.25 per page

Angular momentum in quantum physics: Theory and application, by L. C. Biedenharn and J. D. Louck, Encyclopedia of Mathematics and its Applications, Vol. 8, Addison-Wesley Publishing Company, Reading, Mass., xxix + 716 pp., $54.50. ISBN 0-2011-3507-8

The Racah-Wigner algebra in quantum theory, by L. C. Biedenharn and J. D. Louck, Encyclopedia of Mathematics and its Applications, Vol. 9, Addison-Wesley Publishing Company, Reading, Mass., lxxxviii + 534 pp., $54.50 ISBN 0-2011-3508-6

Angular momentum is a physical quantity which appeared first in classical mechanics. Consider indeed the simplest case of a particle moving in $\mathbf{R}^3$. The observables of this physical system are (C^∞–) functions from $T^*\mathbf{R}^3 \simeq \mathbf{R}^6 \equiv \{(x, p) \mid x, p \in \mathbf{R}^3\}$. The Poisson bracket $\{\cdot,\cdot\}$ associates to every pair (f, g) of such functions the function $\{f, g\}$ defined by

$$\{f, g\} \equiv \sum_{k=1}^{3} \left(\partial_{x^k} f \cdot \partial_{p^k} g - \partial_{x^k} g \cdot \partial_{p^k} f\right). \tag{1}$$

Notice, in particular, that for all $k, l = 1, 2, 3$,

$$\{x^k, x^l\} = 0 = \{p^k, p^l\};\ \{x^k, p^l\} = \delta^{kl}1. \tag{2}$$

Then the angular momentum $L \equiv x \times p$ is the triple (L^1, L^2, L^3) of functions

$$L^j(x, p) \equiv x^k p^l - x^l p^k, \tag{3}$$

where (j, k, l) is any triple of indices obtained from $(1, 2, 3)$ by cyclic permutations. The Poisson brackets between the components of the angular momentum are

$$\{L^j, L^k\} = L^l, \tag{4}$$

where again (j, k, l) are cyclic permutations of $(1, 2, 3)$.

The angular momentum appeared in the quantum mechanical description of a particle, moving similarly in $\mathbf{R}^3$, as the triple $L = (L^1, L^2, L^3)$ of operators,