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# THE GÖDEL CLASS WITH IDENTITY IS UNSOLVABLE

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The Gödel Class with Identity (GCI) is the class of closed, prenex formulas of pure quantification theory extended by inclusion of the identity sign "=" whose prefixes have the form $\forall x \forall y \exists z_0 \cdots \exists z_n$. At the end of [2], Gödel claims that the GCI can be shown to contain no infinity axioms—and hence to be decidable (for satisfiability)—"by the same method" as he employed to show this for the analogous class without identity. (An infinity axiom is a satisfiable formula that has no finite models.) Gödel's claim has been questioned for almost twenty years; since no obvious extension of Gödel's method seemed to apply to the GCI, the decision problem for this class has been deemed open. Gödel's claim is, in fact, erroneous; below we explicitly construct an infinity axiom $F$ in the GCI. Moreover, by exploiting further properties of $F$, we can encode an undecidable problem into the GCI. Hence the GCI is undecidable.

The formula $F$ contains the monadic predicate letter $Z$ and the dyadic letters $S$, $P_1$, $P_2$, $Q$, $N$, $R_1$, $R_2$. $F$ is designed so that, in every model $\mathcal{M}$ of $F$, there will be a unique element $\overline{0}$ such that $\mathcal{M} \models Z\overline{0}$, a unique element $\overline{1}$ such that $\mathcal{M} \models S\overline{1}\,\overline{0}$, a unique element $\overline{2}$ such that $\mathcal{M} \models S\overline{2}\,\overline{1}$, and so on *ad infinitum*. Thus $Z$ acts as the predicate "is zero", and $S$ as the successor relation. The other letters are used to insure the existence of such $\overline{0}, \overline{1}, \overline{2}, \ldots$, and are meant to act as follows. Elements of $\mathcal{M}$ can be taken to encode pairs of integers. Suppose $b$ encodes $\langle p, q\rangle$; then $P_1$ holds between $b$ and the element $\overline{p}$, $P_2$ between $b$ and $\overline{q}$, $Q$ between $b$ and $\overline{q+1}$, $N$ between $b$ and an element that encodes $\langle p+1, q\rangle$, $R_1$ between $b$ and any element that encodes $\langle q+1, r\rangle$ for some $r$, and $R_2$ between $b$ and any element that encodes $\langle r, q+1\rangle$ for some $r$.

Let $F$ be a prenex form of $\forall x \forall y \exists z_0 H$, where $H$ is the conjunction of the following ten clauses:

$$(1) \qquad Zx \wedge Zy \to x = y;$$

$$(2) \qquad Zz_0 \wedge \neg Sz_0x \wedge \bigwedge_{\delta=1,2} (P_\delta x z_0 \wedge P_\delta xy \to y = z_0);$$

$$(3) \qquad (\exists z) Szx;$$

$$(4) \qquad \neg Zx \wedge x \neq y \to (\exists w)(Sxw \wedge -Syw);$$

$$(5) \qquad Sxy \to (\exists z)(Qzx \wedge P_2zy \wedge P_1zz_0);$$

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(6) $(\exists z)[Nxz \wedge (Qxy \rightarrow Qzy) \wedge (R_1xy \rightarrow R_1zy) \wedge (R_2xy \rightarrow R_2zy)]$;

(7) $Nxy \rightarrow (\exists z)(P_2xz \wedge P_2yz) \wedge (\exists w)(\exists u)(P_1xw \wedge Suw \wedge P_1yu)$;

(8) $Qxy \rightarrow (\exists z)(P_1xz \wedge (Syz \rightarrow P_2xz))$;

(9) $\bigwedge_{\delta=1,2} [P_\delta xy \wedge \neg Zy \rightarrow (\exists z)(\exists w)(R_\delta zx \wedge P_2zw \wedge P_1zz_0 \wedge Syw)]$;

(10) $\bigwedge_{\delta=1,2} [R_\delta xy \rightarrow (\exists z)(\exists w)(P_1xz \wedge Swz \wedge (P_\delta yw \rightarrow P_2xz))]$.

$F$ is satisfiable. Indeed, let $\pi\colon \mathbf{N}^2 \rightarrow \mathbf{N}$ be a bijective pairing function. Interpret the predicate letters over $\mathbf{N}$ as indicated two paragraphs back, where $\overline{0}, \overline{1}, \overline{2}, \ldots$ are identified with $0, 1, 2, \ldots$ and an integer $k$ is taken to encode $\langle p, q\rangle$ iff $k = \pi(p, q)$. These interpretations yield a model for $F$ with universe $\mathbf{N}$.

Now let $\mathcal{M}$ be any model for $F$. We find distinct elements $\overline{0}, \overline{1}, \overline{2}, \ldots$ of $\mathcal{M}$ such that, for each integer $p$,

(A) for all $c$ in $\mathcal{M}$, $\neg S\overline{0}c$, and $Zc$ iff $c = \overline{0}$;

(B) for all $c$ in $\mathcal{M}$, $S\overline{p}c$ iff $p > 0$ and $c = \overline{p-1}$;

(C) for all $c$ in $\mathcal{M}$, if $p > 0$ and $Sc\overline{p-1}$, then $c = \overline{p}$;

(D) for $\delta = 1, 2$ and all $c, b$ in $\mathcal{M}$, if $P_\delta c\overline{p} \wedge P_\delta cb$, then $b = \overline{p}$.

(An expression like "$P_\delta cb$" is short for "$\mathcal{M} \models P_\delta cb$".)

By clauses (1) and (2) of $F$, there is a unique $\overline{0}$ in $\mathcal{M}$ such that $Z\overline{0}$. Since the variable $z_0$ must always take $\overline{0}$ as its value, clause (2) of $F$ yields (A)–(D) for $p = 0$.

As induction hypothesis, suppose $\overline{0}, \ldots, \overline{k}$ are distinct elements of $\mathcal{M}$ obeying (A)–(D) for each $p \leq k$. An *$N$-chain* is a sequence $\langle c_0, \ldots, c_m\rangle$ of elements of $\mathcal{M}$ such that $Nc_ic_{i+1}$ for each $i < m$. An easy induction on $m$, using clause (7) of $F$ and (C) and (D), yields: for all $p, m \leq k$,

(E) suppose $\langle c_0, \ldots, c_m\rangle$ is an $N$-chain; if $P_2c_m\overline{p}$ then $P_2c_0\overline{p}$, and if $P_1c_0\overline{0}$ then $P_1c_m\overline{m}$.

LEMMA 1. *Let $a, b \in \mathcal{M}$ and suppose $Sa\overline{k}$ and $Sab$. Then $b = \overline{k}$.*

PROOF. By clause (5) there exists $c_0$ in $\mathcal{M}$ such that $Qc_0a \wedge P_2c_0b \wedge P_1c_0\overline{0}$. Iterated use of clause (6) yields an $N$-chain $\langle c_0, \ldots, c_k\rangle$ such that $Qc_ka$. By (E), $P_1c_k\overline{k}$. By clause (8), there exists $d$ in $\mathcal{M}$ with $P_1c_kd \wedge (Sad \rightarrow P_2c_kd)$. By (D), $d = \overline{k}$; since $Sa\overline{k}$, $P_2c_k\overline{k}$. By (E), $P_2c_0\overline{k}$. But $P_2c_0b$; hence, by (D), $b = \overline{k}$. □

LEMMA 2. *There is a unique $a$ in $\mathcal{M}$ such that $Sa\overline{k}$.*

PROOF. By clause (3) there is at least one $a$ in $\mathcal{M}$ with $Sa\overline{k}$. By (A), $\neg Za$. Let $b \in \mathcal{M}$, $b \neq a$. By clause (4) there exists $c$ in $\mathcal{M}$ with $Sac \wedge \neg Sbc$. By Lemma 1, $c = \overline{k}$. Thus $\neg Sb\overline{k}$. □

Now let $\overline{k+1}$ be the unique $a$ such that $Sa\overline{k}$. By (B), $\overline{k+1}$ is distinct from $\overline{0}, \overline{1}, \ldots, \overline{k}$.

LEMMA 3. *Let $\delta = 1$ or $2$, $c, b \in \mathcal{M}$; suppose $P_\delta c\overline{k+1}$ and $P_\delta cb$. Then $b = \overline{k+1}$.*

PROOF. By (A) and (D), $\neg Zb$. Hence by clause (9) there exist $c_0$, $d$ in $\mathcal{M}$ such that $R_\delta c_0 c \wedge P_2 c_0 d \wedge P_1 c_0 0 \wedge Sbd$. Iterated use of clause (6) yields an $N$-chain $\langle c_0, \ldots, c_k \rangle$ such that $R_\delta c_k c$. By (E), $P_1 c_k \overline{k}$. By clause (10) there exist $e, e'$ in $\mathcal{M}$ such that $P_1 c_k e \wedge Se'e \wedge (P_\delta ce' \rightarrow P_2 c_k e)$. By (D), $e = \overline{k}$. Thus $e' = \overline{k+1}$. Since $P_\delta c \overline{k+1}$, $P_2 c_k \overline{k}$. By (E), $P_2 c_o \overline{k}$. But $P_2 c_0 d$; hence, by (D), $d = \overline{k}$. Thus $Sb\overline{k}$, so $b = \overline{k+1}$. □

Lemmas 1–3 show that (A)–(D) hold for all $p \leq k+1$. Thus, by induction, there is an infinite sequence of distinct elements of $\mathcal{M}$.

We have shown that every model for $F$ contains an $\omega$-sequence of elements on which $S$ acts as the successor relation. Consequently, it is a simple matter to use $F$ to obtain undecidability. For example, let $G = \forall x \exists u \forall y K$ be any $\forall\exists\forall$-formula of pure quantification theory; we may suppose that the predicate letters of $G$ are distinct from those of $F$. A straightforward argument shows that $G$ is satisfiable if and only if $F \wedge \forall x \forall y \exists u (Sux \wedge K)$ is satisfiable; and the latter formula has a prenex equivalent in the GCI. Since the class of $\forall\exists\forall$-formulas is undecidable [3], we obtain the

THEOREM. *The Gödel Class with Identity is undecidable.*

The theorem may be sharpened. Using several additional predicate letters, we may construct an infinity axiom and encode $\forall\exists\forall$-formulas while using only one existential quantifier. Hence the Minimal GCI, i.e., the class of formulas with prefixes $\forall x \forall y \exists z$, is undecidable. This settles the decision problem for all prefix-classes of quantification theory with identity, for we now have the following division:

Decidable prefix-classes: $\exists\cdots\exists\forall\cdots\forall$ and $\exists\cdots\exists\forall\exists\cdots\exists$.

Undecidable prefix-classes: $\forall\exists\forall$ and $\forall\forall\exists$.

This dividing line differs from that in pure quantification theory, where the $\exists\cdots\exists\forall\forall\exists\cdots\exists$ class is decidable, so that the minimal undecidable prefix-classes are $\forall\exists\forall$ and $\forall\forall\forall\exists$ (see the Introduction to [1]).

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