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SUBDIFFERENTIABILITY SPACES AND NONSMOOTH ANALYSIS

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We use the words "subdifferentiability space" as a collective name for classes of Banach spaces characterized by subdifferentiability properties of l.s.c. nonconvex functions defined on them. The classes were introduced by analogy with Asplund (differentiability) spaces [**1**], but the initial impulse came from nonsmooth analysis where one specific class had appeared quite naturally [**2**]. The purpose of the note is to introduce four classes of subdifferentiability spaces, describe their properties and the role of one of the classes in nonsmooth analysis.

1. All spaces are assumed Banach and the functions extended real-valued. For a function f on X we set

$$\operatorname{dom} f = \{x \mid |f(x)| < \infty\},$$
$$U(f,z,\delta) = \{x \in \operatorname{dom} f \mid \|x-z\| < \delta,\ |f(x)-f(z)| < \delta\},$$
$$d^- f(z;h) = \liminf_{\substack{u\to h\\ t\searrow 0}} t^{-1}(f(z+tu)-f(z))$$

($z \in \operatorname{dom} f$); B, B^* denote unit balls in X, X^*.

DEFINITION 1. Let f be a function on X, $\epsilon > 0$, and $z \in \operatorname{dom} f$. We denote by $\varphi_\epsilon^-(z)$ the set of all $x^* \in X^*$ such that

$$\liminf_{\|h\|\to 0} \|h\|^{-1}(f(z+h)-f(z)-\langle x^*,h\rangle) \geq -\epsilon,$$

and by $\partial_\epsilon^- f(z)$ the set of all $x^* \in X^*$ such that

$$\langle x^*,h\rangle \leq d^- f(z;h) + \epsilon\|h\|;$$

$\varphi_\epsilon^- f(z)$ and $\partial_\epsilon^- f(z)$ will be respectively called the *Fréchet* and the *Dini ϵ-subdifferential* of f at z.

If $z \notin \operatorname{dom} f$, we set $\varphi_\epsilon^- f(z) = \partial_\epsilon^- f(z) = \varnothing$.

DEFINITION 2. X is a *subdifferentiability* (*weak subdifferentiability*) space or S-space (WS-space) if for any $\epsilon > 0$ every l.s.c. function f on X is Fréchet (Dini) ϵ-subdifferentiable on a dense subset of $\operatorname{dom} f$.

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DEFINITION 3. X is a *trustworthy space* or T-space if for any two l.s.c. functions f_1, f_2 on X, any $z \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, any $\epsilon > 0$, $\delta > 0$ and any weak* neighborhood $V \subset X^*$ of zero,

$$\varphi_\epsilon^-(f_1 + f_2) \subset \bigcup_{x_i \in U(f_i, z, \delta)} (\varphi_\epsilon^- f_1(x_1) + \varphi_\epsilon^- f_2(x_2) + V).$$

Replacing φ^- by ∂^- in the inclusion, we obtain the definition of *weak trustworthy spaces* (WT-spaces) [**2**].

(Other definitions of "trustworthiness" postulating one or another embryonic form of calculus are also possible.)

2. The principal result will be stated below. In the statement we denote by P the property "X is a P-space"; A means "Asplund", WA—"weak Asplund", and F-spaces (G-spaces) are characterized by the property: "there exists a Fréchet (locally Lipschitz and Gâteaux) differentiable bump function" (i.e. a function $g(x)$ such that $g(0) > 0$ and $g(x) \leq 0$ if $||x|| \leq 1$).

THEOREM 1. *The following implications are valid:*

$$\begin{array}{ccccccc} \mathrm{F} & \Rightarrow & \mathrm{T} & \Rightarrow & \mathrm{S} & \Rightarrow & \mathrm{A} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \mathrm{G} & \Rightarrow & \mathrm{WT} & \Rightarrow & \mathrm{WS} & & \mathrm{WA}. \end{array}$$

Implications F$\Rightarrow$G, S$\Rightarrow$WS, A$\Rightarrow$WA are trivial. Implication S$\Rightarrow$A was actually proved by Ekeland and Lebourg [**3**]; a much simpler proof can be obtained from Kenderov's characterization of Asplund spaces [**4**] which can be equivalently formulated in terms of Fréchet ϵ-subdifferentials: *X is an Asplund space iff for any $\epsilon > 0$ every concave continuous function on X is Fréchet ϵ-subdifferentiable on a dense set.*

The inclusions Theorem 1 actually announces are F$\Rightarrow$T$\Rightarrow$S, G$\Rightarrow$WT$\Rightarrow$WS and T$\Rightarrow$WT, of which F$\Rightarrow$T and G$\Rightarrow$WT are the main. A detailed paper with proofs of Theorem 1 and Theorem 2 will appear in the Annals of the New York Academy of Sciences [**5**].

There are three questions that remain open: does WS imply WA, does S (WS) imply T (WT) and does A imply S? All obviously related to long standing problems: whether any Banach space with a Gâteaux differentiable norm is a WA-space and whether any A-space has a Fréchet differentiable norm.

THEOREM 2. *If X is an* S*-space* (WS*-space*)*, then so are* (a) $X \times R^n$; (b) *any closed subspace of X*; (c) *any quotient space X/L (with closed L). In particular, any Banach space which is a continuous image of an* S*-space* (WS*-space*) *is an* S*-space* (WS*-space*).

It is also possible to show that *a closed subspace of a* T*-space* (WT*-space*) *is a* T*-space* (WT*-space*)*, and that the property of being a* T*- or* WT*-space is an isomorphic property.*

3. The specific class that naturally arises in nonsmooth analysis is the class of WT-spaces. To explain how it happens we have to consider an extended version of approximate subdifferentials first introduced in [6]. The results of this section are proved in [2].

We shall call a collection $\mathcal{L}$ of closed subspaces of X *admissible* if it is a directed set w.r.t. inclusion and every $x \in X$ belongs to some $L \in \mathcal{L}$. By $f_Q(x)$ we denote the restriction of f to $Q \subset X$, i.e. the function coinciding with f on Q and equal to ∞ outside of Q.

DEFINITION 4. Let $\mathcal{L}$ be an admissible collection of subspaces of X. The set

$$\partial_A^{\mathcal{L}} f(z) = \bigcap_{\substack{\epsilon>0 \\ L\in\mathcal{L}}} \overline{\bigcup_{x\in U(f,z,\delta)} \partial_\epsilon^- f_{x+L}(x)}$$

(the bar denotes the weak* closure) is called the (*broad*) *analytic approximate* $\mathcal{L}$-*subdifferential* of f at z.

(There are also "narrow" subdifferentials in [2] which we do not consider here.)

THEOREM 3. *If f is l.s.c., then $\partial_A^{\mathcal{L}} f(z)$ is the same for all admissible families $\mathcal{L}$ formed by* WT-*subspaces of* X.

Thus the notation $\partial_A f(z)$ may be used and (since every finite-dimensional space is a WT-space as follows from Theorem 1) $\partial_A f(z)$ coincides with the "grande sous-différentielles approchées" introduced in [6], hence having the whole set of analytic properties listed there.

The last theorem we are going to state describes the place of approximate subdifferentials in nonsmooth analysis. Let us say that we are given a *subdifferential* on X if for any l.s.c. function f and any $x \in X$ a weak* closed (possibly empty) set $\partial f(x)$ is defined in such a way that:

(a) $0 \in \partial f(x)$ if f attains a local minimum at x;

(b) if f is convex, then $\partial f(x)$ is the subdifferential in the sense of convex analysis;

(c) $\partial(f+g)(x) \subset \partial f(x) + \partial g(x)$ if one of the functions is Lipschitz near x.

We shall say that ∂ is a u.s.c. subdifferential if, in addition, the set-valued map $x \to \partial f(x)$ is u.s.c. (from the norm topology of X into the weak* topology of X^*) at every x near which f is Lipschitz. A well-known example of a u.s.c. subdifferential is the generalized gradient of Clarke; ∂_A is another example.

THEOREM 4. *Let ∂ be a u.s.c. subdifferential on X. Then $\partial_A f(x) \subset \partial f(x)$ whenever f is Lipschitz near x.*

Thus ∂_A is the minimal subdifferential for Lipschitz functions. We note in this connection that properties (a)–(c) are very natural for any "good" subdifferential and the upper semicontinuity property is absolutely unavoidable in many situations.

We refer to [2] for further "geometric" approximate subdifferentials coinciding with ∂_A on Lipschitz functions and smaller than ∂_A on all other functions which have stronger minimality properties.

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