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SELF-DUAL CONNECTIONS AND THE TOPOLOGY OF SMOOTH 4-MANIFOLDS

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1. Introduction, statement of result. To any compact oriented 4-manifold X there is associated a quadratic form Q, defined on the cohomology group $H^2(X;\mathbf{Z})$ by $Q(\alpha) = (\alpha \cup \alpha)[X]$. Poincaré duality requires that it be a "unimodular" form—given by a symmetric matrix of determinant ± 1 with respect to any base for the torsion free part of H^2. It is known from arithmetic that there are many such forms that are positive definite and not equivalent (over the integers) to the standard form [**4**, Chapter 5]. The problem of finding which forms are realised by simply-connected 4-manifolds was raised, for example, in [**3**]; a partial answer for smooth 4-manifolds is announced here in the form of

THEOREM. *If X is a smooth, compact, simply-connected oriented 4-manifold with the property that the associated form Q is positive definite, then Q is equivalent, over the integers, to the standard diagonal form.*

As a particular application, the theorem shows that it is impossible to remove smoothly, by surgery, all three hyperbolic factors in a K3 surface (which has quadratic form $E_8 + E_8 + \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) + \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) + \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$) since this would give a simply-connected smooth 4-manifold with definite form $E_8 + E_8$.

2. Method of proof. I give, in this note, an outline of the proof; a detailed account will appear soon. The idea of the proof is to exploit topological information that emerges from a study of "self-dual connections" or "instantons"; I take [**1**] as a general reference for background in this area, and for notation. Suppose throughout that X is a 4-manifold satisfying the hypotheses of the theorem, and that we are given some Riemannian metric.

There is, up to isomorphism, a unique principal $SU(2)$ bundle P over X with characteristic class $c_2(P)[X] = -1$. One forms the space of all equivalence classes of connections on P as the quotient of the affine space $\mathcal{A}$ of connections by the action of the "gauge group" $\mathcal{G}$ of automorphisms of P. A Hausdorff topology descends to $\mathcal{A}/\mathcal{G}$ and the dense open subset representing *irreducible* connections can be made into a Banach manifold. On the other hand a *reducible* connection corresponds to a reduction of P to an S^1 bundle and in the neighbourhood of such a point the space $\mathcal{A}/\mathcal{G}$ has the structure of

$$(\text{Real Banach Space}) \times (\text{Complex Banach Space}/S^1).$$

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The S^1 action arises as the action of the stabiliser in $\mathcal{G}$ of a reducible connection.

A simple calculation with characteristic classes shows that the number of topologically distinct reductions of P, and so the number of components of the singular subset of $\mathcal{A}/\mathcal{G}$, is given in terms of the form Q by

$$n(Q) = \tfrac{1}{2}\#\{\alpha \in H^2(X;\mathbf{Z})\colon Q(\alpha) = 1\}.$$

3. Self-dual connections. A connection of P is called "self-dual" if its curvature Ω satisfies the equation $^*\Omega = \Omega$. This property is invariant under $\mathcal{G}$ so we may define the *moduli space* $\mathcal{M} \subset \mathcal{A}/\mathcal{G}$ of equivalence classes of self-dual connections on P.

Atiyah, Hitchin, and Singer [**1**, §6] showed that, provided a certain cohomology group vanishes, the *irreducible* elements of $\mathcal{M}$ form a smooth finite-dimensional submanifold of $\mathcal{A}/\mathcal{G}$. Their formula for the dimension gives in the special case at hand

$$\dim \mathcal{M} = 8 - \left(\tfrac{3}{2}\right)(\chi(X) - \tau(X)) = 5.$$

In general we do not know that the vanishing condition will be satisfied, but this is not important for the topological application since one may show

LEMMA 1. *For suitable generic perturbations of the self-duality equations the perturbed moduli space $\mathcal{M}^* \subset \mathcal{A}/\mathcal{G}$ is a smooth 5-manifold at all points representing irreducible connections.*

This is a straightforward extension of a standard general position argument to an infinite-dimensional setting; using the methods of Kuranishi [**2**], or of Smale [**5**].

It is not obvious that there are any self-dual connections on P at all. Their existence follows from a recent theorem of C. H. Taubes [**6**, Theorem 1.2], once one observes that an equivalent form of the hypothesis that the quadratic form be positive is the statement that all harmonic 2-forms on X are self-dual.

Then, using the techniques of [**6**] together with results of K. Uhlenbeck [**7**, **8**], and guided by what is known for the case when $X = S^4$ [**1**], one has a good understanding of the "boundary" of $\mathcal{M}$.

LEMMA 2. *There is an open subset $\mathcal{U} \subset \mathcal{M}$ which is a smooth 5-manifold diffeomorphic to $X \times (0,1)$, and $\mathcal{M}\backslash\mathcal{U}$ is compact.*

Notes. (i) In proving this lemma I use the fact that X is simply connected.

(ii) The perturbed moduli space $\mathcal{M}^*$ of Lemma 1 has the same boundary properties as $\mathcal{M}$.

At each reducible connection in $\mathcal{M}^*$ the 5-manifold inherits a quotient singularity from the ambient space; around such a point, $\mathcal{M}^*$ has the structure of $\mathbf{C}^3/S^1$: a cone on $\mathbf{CP}^2$. $\mathcal{M}$, and so $\mathcal{M}^*$, meets each component of the reducible connections in $\mathcal{A}/\mathcal{G}$ exactly once, so there are $n(Q)$ singular points. Thus *there is a smooth compact 5-manifold with boundary the disjoint union of X and $n(Q)$ copies of $\mathbf{CP}^2$.*

Finally, an argument based on properties of the index for families of operators shows that this 5-manifold is orientable. (This was first proved by M. F. Atiyah.)

4. Proof of theorem. Note first that for any unimodular positive form Q, $n(Q) \leq \text{rank}(Q)$ with equality if and only if Q is equivalent to the standard form.

Then apply the fact that signature is an invariant of oriented cobordism to the 5-manifold above. The signature of X is rank(Q) so there must be at least rank(Q) copies of $\mathbf{CP}^2$. Hence $n(Q) = \text{rank}(Q)$ and Q is equivalent to the standard diagonal form.

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