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DYER-LASHOF OPERATIONS IN K-THEORY

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Dyer-Lashof operations were first introduced by Araki and Kudo in [**1**] in order to calculate $H_*(\Omega^n S^{n+k}; Z_2)$. These operations were later used by Dyer and Lashof to determine $H_*(QY; Z_p)$ as a functor of $H_*(Y; Z_p)$ [**5**], where $QY = \bigcup_n \Omega^n \Sigma^n Y$. This has had many important applications. Hodgkin and Snaith independently defined a single secondary operation in K-homology (for p odd and $p = 2$ respectively) which was analogous to the sequence of Dyer-Lashof operations in ordinary homology [**7**, **13**], and this operation has been used to calculate $K_*(QY; Z_p)$ when Y is a sphere or when $p = 2$ and Y is a real projective space [**11**, **12**]. In this note we describe new primary Dyer-Lashof operations in K-theory which completely determine $K_*(QY; Z_p)$ in general.

We shall remove the indeterminacy of the operation by lifting it to higher torsion groups. First we establish notation. X will always denote an E_∞-space [**9**] and Y will denote an arbitrary space, considered as a subspace of QY via the natural inclusion. We write $K_*(Y; r)$ for $K_0(Y; Z_{p^r}) \oplus K_1(Y; Z_{p^r})$; in particular K-theory is Z_2-graded and we write $|x|$ for the mod 2 degree of x. There are evident natural maps

$$p^s_*\colon K_\alpha(Y; r) \to K_\alpha(Y; r+s) \quad \text{if } s \geq 1,$$
$$\pi\colon K_\alpha(Y; r) \to K_\alpha(Y; t) \quad \text{if } 1 \leq t \leq r,$$

and

$$\beta_r\colon K_\alpha(Y; r) \to K_{\alpha-1}(Y; r).$$

THEOREM 1. *For each $r \geq 2$ and $\alpha \in Z_2$ there is an operation*

$$Q\colon K_\alpha(X; r) \to K_\alpha(X; r-1)$$

with the following properties, where x, $y \in K_*(X; r)$.

(i) *Q is natural with respect to E_∞-maps.*

$$\text{(ii)} \qquad Q(x+y) = \begin{cases} Qx + Qy - \pi\left[\displaystyle\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}\right] & \text{if } |x| = |y| = 0, \\ Qx + Qy \quad \text{if } |x| = |y| = 1. \end{cases}$$

(iii) *$Q\phi = 0$, where $\phi \in K_0(X; r)$ is the identity element.*

$$\text{(iv)} \qquad Q(xy) = \begin{cases} Qx \cdot \pi(y^p) + \pi(x^p) \cdot Qy + p(Qx)(Qy) & \text{if } |x| = |y| = 0, \\ Qx \cdot \pi(y^p) + p(Qx)(Qy) & \text{if } |x| = 1, |y| = 0, \\ (Qx)(Qy) & \text{if } |x| = |y| = 1. \end{cases}$$

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(v)
$$\sigma Qx = \begin{cases} Q\sigma x & \text{if } |x| = 0, \\ \pi(\sigma x)^p + pQ\sigma x & \text{if } |x| = 1, \end{cases}$$

where $\sigma : \tilde{K}_\alpha(\Omega X; r) \to K_{\alpha+1}(X; r)$ *is the homology suspension.*

(vi) *If* k *is prime to* p, *then* $Q\psi^k = \psi^k Q$, *where* ψ^k *is the* k*th Adams operation.*

(vii)
$$\beta_{r-1}Qx = \begin{cases} Q\beta_r x - p\pi(x^{p-1}\beta_r x) & \text{if } |x| = 0, \\ \pi(\beta_r x)^p + pQ\beta_r x & \text{if } |x| = 1. \end{cases}$$

$Q\pi x = \pi Qx$ *if* $r \geq 3$ *and*

(viii)
$$Qp_*x = \begin{cases} x^p & \text{if } |x| = 0, r = 1, \\ p_*Qx - (p^{p-1} - 1)x^p & \text{if } |x| = 0, r \geq 2, \\ 0 & \text{if } |x| = 1, r = 1, \\ p_*Qx & \text{if } |x| = 1, r \geq 2. \end{cases}$$

(ix) *Let* $p = 2$. *If* $x \in K_1(X; 1)$ *then* $Q\beta_2 2_* x = x^2$. *If* $x \in K_1(X; 2)$ *then* $(\pi x)^2 = (\pi\beta_2 x)^2$; *in particular* $(\pi x)^2 \in K_0(X; 1)$ *is zero if* $x \in K_1(X; r)$ *with* $r \geq 3$.

REMARKS. (i) There are no Adem relations.

(ii) If $x \in K_*(X; 1)$ has $\beta x = 0$ then x lifts to $y \in K_*(X; 2)$. Thus one can define a secondary operation $\overline{Q}$ on $\ker \beta$ by $\overline{Q}x = Qy$. The element y is well defined modulo the image of p_*, and thus Theorem 1 (viii) shows that $\overline{Q}x$ is well defined modulo pth powers if $|x| = 0$ and has no indeterminacy if $|x| = 1$. This is essentially the operation defined by Hodgkin and Snaith (although their construction is incorrect when p is odd, as shown in [**10**]).

The next result shows that, in contrast to ordinary homology, $K_*(QY; 1)$ will in general have nilpotent elements.

THEOREM 2. $\pi(\beta_r x)^{p^r} = 0$ *in* $K_0(X; 1)$ *if* $x \in K_1(X; r)$.

If $x \in K_*(Y; r)$, we write $Q^s x \in K_*(QY; r - s)$ for the sth iterate of Q when $s < r$. These elements give a family of indecomposable generators in $K_*(QY; 1)$, but in general there can be other generators as well. For example, if $x \in K_1(Y; 1)$ with $\beta x \neq 0$ then $x(\beta x)^{p-1}$ has zero Bockstein by Theorem 2, hence it lifts to an element $z \in K_1(QY; 2)$, and it turns out that Qz is indecomposable (note that we cannot apply the Cartan formula to Qz). The next theorem allows us to deal systematically with elements like z; in particular it gives the higher Bocksteins of such elements.

THEOREM 3. *For each* $r \geq 1$ *there is an operation*

$$R : K_1(X; r) \to K_1(X; r + 1)$$

with the following properties, where x, $y \in K_1(X; r)$.

(i) R *is natural for* E_∞*-maps.*

(ii) $p_*Rx = Rp_*x$, $\pi Rx = Qp_*x - x(\beta_r x)^{p-1}$, *and if* $r \geq 2$, $R\pi x = Qp_*x - p^{p-1}x(\beta_r x)^{p-1}$.

(iii) $\beta_{r+1}Rx = Q\beta_{r+2}p_*^2 x$.

(iv) *If* $r \geq 2$, *then* $QRx = RQx$.

(v) *If k is prime to p, then $R\psi^k = \psi^k R$.*

(vi)
$$\sigma Rx = \begin{cases} p_*[(\sigma x)^p] & \text{if } r = 1, \\ p_*[(\sigma x)^p] + p_*^2 Q\sigma x & \text{if } r \geq 2. \end{cases}$$

(vii)
$$R(x+y) = Rx + Ry - \sum_{i=1}^{p-1} \left[\frac{1}{p}\binom{p}{i}(p_*x)(\beta_{r+1}p_*x)^{i-1}(\beta_{r+1}p_*y)^{p-i} + \binom{p-1}{i}\beta_{r+1}p_*(xy)(\beta_{r+1}p_*x)^{i-1}(\beta_{r+1}p_*y)^{p-i-1} \right].$$

Theorems 1 and 3 imply that $\pi Q^s R^t x$ is decomposable if $x \in K_1(Y;r)$ and $s < r+t-1$. If $s = r+t-1$ and $\pi\beta_r x \neq 0$ then this element turns out to be indecomposable.

In order to give a Cartan formula for R and to provide generators for the higher terms of the Bockstein spectral sequence, we next give a K-theory analogue for the Pontryagin pth power introduced in ordinary homology by Madsen [8] and May [4]. Note, however, that by part (viii) of the following theorem this operation does not give rise to new families of indecomposables in $K_*(QY;1)$.

THEOREM 4. *For each $r \geq 1$ there is an operation*

$$\mathcal{Q}\colon K_0(X;r) \to K_0(X;r+1)$$

with the following properties.

(i) *$\mathcal{Q}$ is natural for E_∞-maps.*

(ii) *$\pi\mathcal{Q}x = x^p$ and $\mathcal{Q}p_*x = p^{p-1}p_*\mathcal{Q}x$. If $r \geq 2$ then $\mathcal{Q}\pi x = x^p$.*

(iii) *$\pi\beta_{r+1}\mathcal{Q}x = x^{p-1}\beta_r x$.*

(iv) *Let p be odd. Then*

$$R(xy) = \begin{cases} (Rx)(\mathcal{Q}y) & \text{if } |x| = 1, |y| = 0 \text{ and } r = 1, \\ (Rx)(\mathcal{Q}y) + p_*^2[(Qx)(Qy)] & \text{if } |x| = 1, |y| = 0 \text{ and } r \geq 2. \end{cases}$$

$$\mathcal{Q}(xy) = (\mathcal{Q}x)(\mathcal{Q}y) \quad \text{if } |x| = |y| = 0.$$

(v) *If k is prime to p, $\psi^k\mathcal{Q} = \mathcal{Q}\psi^k$.*

(vi)
$$\mathcal{Q}(x+y) = \mathcal{Q}x + \mathcal{Q}y + \sum_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}p_*(x^i y^{p-i}).$$

(vii)
$$\sigma\mathcal{Q}x = \begin{cases} 0 & \text{if } p \text{ is odd}, \\ 2^{r-1}2_*[(\sigma x)(\beta_r\sigma x)] & \text{if } p = 2. \end{cases}$$

(viii)
$$Q\mathcal{Q}x = \begin{cases} 0 & \text{if } r = 1, \\ \sum_{i=1}^{p}\binom{p}{i}p^{i-2}x^{p^2-ip}p_*[(Qx)^i] & \text{if } r \geq 2. \end{cases}$$

REMARK. The formulas in part (iv) have analogues when $p = 2$, but some of the coefficients in this case have not yet been determined.

Using the operations Q and R we can completely describe $K_*(QY;1)$. We shall assume that Y is a finite complex, although this condition can be avoided. First recall the construction CY from [**9**]. By [**4**, Theorem I.5.10] we have $K_*(QY;1) \cong (\pi_0 Y)^{-1} K_*(CY;1)$, and so it suffices to give $K_*(CY;1)$.

Next recall the reduced K-theory Bockstein spectral sequence $E_*^r Y$ from [**2**]. If Y is a finite complex we have $E_*^n Y = E_*^\infty Y$ for some n, and we can choose a subset $A_\infty \subset \tilde{K}_*(Y;Z)$ projecting to a basis for $E_*^\infty Y$. Proceeding inductively, we can choose subsets $A_r \subset \tilde{K}_*(Y;r)$ such that

$$A_\infty \cup A_{n-1} \cup \beta_{n-1}(A_{n-1}) \cup \cdots \cup A_r \cup \beta_r(A_r)$$

projects to a basis of $E_*^r Y$ for $1 \leq r \leq n-1$. We write A_{r0} and A_{r1} for the zero- and one-dimensional subsets of A_r. Let BY be the quotient of the free strictly commutative algebra generated by the four sets

$$\{\pi Q^s x | x \in A_r, 0 \leq s < r\}, \quad \{\pi\beta_{r-s} Q^s x | x \in A_{r0}, 0 \leq s < r < \infty\},$$

$$\{Q^{r+s} R^{s+1} x | x \in A_{r1}, r < \infty, 0 \leq s\}, \quad \text{and} \quad \{\pi\beta_{r+s} R^s x | x \in A_{r1}, r < \infty, 0 \leq s\}$$

by the ideal generated by the set

$$\{(\pi\beta_{r+s} R^s x)^{p^{r+s}} | x \in A_{r1}, r < \infty, 0 \leq s\}.$$

The Dyer-Lashof operations Q and R give an additive homomorphism $\lambda\colon BY \to K_*(CY;1)$, which is a ring homomorphism if p is odd but not if $p = 2$. Our main theorem is

THEOREM 5. *λ is an isomorphism.*

REMARKS. (i) Theorems 1, 3, and 5 also give the ring structure of $K_*(CY;1)$ when $p = 2$. First recall that mod2 K-theory is noncommutative [**2**], in fact the commutator of x and y is $(\beta x)(\beta y)$. Now

$$\beta(Q^{r+s} R^{s+1} x) = (\beta_{r+s+1} R^{s+1} x)^{2^{r+s}}$$

if $x \in A_{r1}$ with $r < \infty$ and $s \geq -1$, and all other generators (except $Q^{r-1}x$ for $x \in A_{r0}$, $r < \infty$, whose Bockstein is the generator $\beta Q^{r-1}x$) have zero Bockstein and hence lie in the center. Further, all odd-dimensional generators have square zero except in the following cases:

$$(\pi Q^{r-2} x)^2 = (\beta_r x)^{2^{r-1}} \quad \text{if } x \in A_{r1}, 2 \leq r < \infty;$$

$$(Q^{r+s} R^{s+1} x)^2 = (\pi\beta_{r+s+2} R^{s+2} x)^{2^{r+s}} \quad \text{if } x \in A_{r1}, r < \infty, s \geq -1.$$

These facts, together with Theorem 5, determine the ring structure.

(ii) The effect of $(Qf)_*\colon K_*(QY;1) \to K_*(QZ;1)$ for any $f\colon Y \to Z$ can be ascertained from Theorems 1, 3, and 5 if $f_*\colon K_*(Y;r) \to K_*(Z;r)$ is known for all $r \geq 1$ (although the formulas can become complicated unless f_* takes the chosen sets A_r for Y into the corresponding sets for Z). In particular if $f\colon S^2 \to S^2$ is the degree p map then Theorem 1 (ii) implies that $(Qf)_*$ is nonzero on $K_*(QS^2;1)$. Thus $K_*(QY;1)$ is not a functor of $K_*(Y;1)$, a fact first noticed by Hodgkin [**7**].

(iii) Theorem 5 specializes to give an independent proof of the computations of Hodgkin [**6**] and Miller and Snaith [**11, 12**]. The operation R did not arise in those computations since in the cases considered A_{r1} was empty for all $r<\infty$.

Finally, we describe the Bockstein spectral sequence for CY.

THEOREM 6. *For $1\le m<\infty$, $E_*^m(CY)^+$ is additively isomorphic to the quotient of the free strictly commutative algebra generated by the six sets*

$$\{\pi Q^s x | x\in A_r, m\le r-s, s\ge 0\},$$
$$\{\pi\beta_{r-s}Q^s x | x\in A_{r0}, m\le r-s<\infty, s\ge 0\},$$
$$\{\pi \mathcal{Q}^{m-r+s}Q^s x | x\in A_{r0}, 1\le r-s<m, s\ge 0\},$$
$$\{\pi\beta_m \mathcal{Q}^{m-r+s}Q^s x | x\in A_{r0}, 1\le r-s<m, s\ge 0\},$$
$$\{\pi Q^{t-m}R^{t-r}x | x\in A_{r1}, t\ge \max(m,r+1), r<\infty\},$$

and

$$\{\pi\beta_t R^{t-r}x | x\in A_{r1}, t\ge\max(m,r), r<\infty\}$$

by the ideal generated by the set

$$\{(\pi\beta_t R^{t-r}x)^{p^{t+1-m}} | x\in A_{r1}, t\ge\max(m,r), r<\infty\}.$$

If p is odd or $m\ge 3$ the isomorphism is multiplicative. The differential in $E_*^m(CY)^+$ is determined by the formula

$$\pi\beta_m Q^{t-m}R^{t-r}x = (\pi\beta_t R^{t-r}x)^{p^{t-m}}$$

for $x\in A_{r1}$, $t\ge\max(m,r)$, $r<\infty$.

The construction of the operations is as follows. Let M_r be the Moore spectrum $S^{-1}\cup_{p^r} e^0$ and let K be the integral K-theory spectrum. By definition, any $x\in K_\alpha(X;r)$ is represented by a stable map

$$x\colon S^\alpha\to K\wedge \Sigma M_r\wedge X.$$

Since the dual of ΣM_r is M_r, such a map induces

$$x'\colon \Sigma^\alpha M_r\to K\wedge X.$$

Applying the stable extended power functor D_p and using the fact that $K\wedge X$ is an H_∞ ring spectrum [**3**] one obtains a composite

$$x''\colon D_p\Sigma^\alpha M_r\to D_p(K\wedge X)\to K\wedge X.$$

Finally, if $e\in K_\alpha(D_p\Sigma^\alpha M_r;s)$ for some s one has the composite

$$\Sigma^\alpha M_s\xrightarrow{e'} K\wedge D_p\Sigma^\alpha M_r\xrightarrow{1\wedge x''} K\wedge K\wedge X\xrightarrow{\mu\wedge 1} K\wedge X,$$

where μ is the K-theory product. This composite represents an element of $K_\alpha(X;s)$ depending only on e and x. The operations Qx, $\mathcal{Q}x$ and Rx are obtained in this way for various choices of e, and the proofs of Theorems 1, 3, and 4 reduce in each case to the analysis of e. The construction has the further advantage that the proof of Theorem 5 is reduced, after some diagram chasing, to the universal case $Y=\Sigma^\alpha M_r$. Details will appear in [**3**].

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