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BOOK REVIEWS

Cartanian geometry, nonlinear waves, and control theory, by Robert Hermann, Interdisciplinary Mathematics, volumes 20 and 21, Math. Sci. Press, Brookline, Mass., Part A, 1979, xv + 501 pp., $50.00; Part B, 1980, xii + 585 pp., $60.00.

Cartanian geometry, A and B are the twentieth and twenty-first volumes in the series *Interdisciplinary mathematics* (IM) which is authored by Robert Hermann. (A companion series is *Lie groups*: *history, frontiers and applications*.)

The volumes under review represent both a refinement and an extension of earlier work in the IM series. Most particularly, for the purpose of this review, we need to refer to Vol. 3 (*Algebraic topics in system theory*), Vol. 8 (*Linear system theory and Introductory algebraic geometry*), Vol. 9 (*Geometric structure of systems—Control theory and physics*, *Part* A), Vol. 11 (*Geometric structure of systems—Control Theory*, *Part* B) and Vol. 13 (*Algebro—Geometric and Lie theoretic techniques in systems theory*, *Part* A—coauthored with Clyde Martin). As is the case with most of the series, *Cartanian Geometry* A,B cover an enormous number of topics. Broadly grouped, they fall into the categories (1) systems and control theory; (2) nonlinear waves; (3) quantum mechanics and (4) a translation (by Michael Ackerman) of Sophus Lie's papers *General investigations of differential equations which admit a finite continuous group* and *Foundations of the theory of infinite continuous transformation groups*. I. This last item is of value to historians of modern mathematics as well as those whose interest is in Lie groups or differential geometry.

Because of the wealth of ideas dealt with in these volumes and the need to keep this review to a manageable length, attention will be restricted to the material on the matrix Riccati equation and on the relationship between control/system theory and vector bundle theory. (Those interested in the topic of nonlinear waves may wish to see Hermann's review [**33**] of *Elements of soliton theory* by G. L. Lamb, Jr. The presentation in Lamb's book is from a different perspective from that of Hermann's work in this area, but Hermann has taken the opportunity of the review to give an overview of the geometric theory of nonlinear waves.)

Before we discuss in depth any of the mathematical content, a few words are in order about the author's philosophy. Perhaps the first two paragraphs of the preface to Part A best sum up his attitude.

> "I began in 1970 to write this series of books in order to develop a *unified* mathematical science and technology. After all, if subjects like category theory, logic, differential topology are accepted and integrated into the mathematical world, why not system theory, mathematical elementary particle theory,

> relativity, etc.? I had no master plan, but intended to write down what I could, as best I could, and see where it led.
>
> Twenty volumes are now completed and I can say more definitively that the unifying theme is the role that *geometry* plays in physics and engineering. 'Applied mathematics' is usually thought of as involving the more concrete parts of analysis and certain areas like numerical analysis and combinatorics, which interface computer science; but my vision is quite different. To a large extent I am inspired by the historical example of the 19th century, where the basis of much of the fruitful interchange between mathematics and physics was precisely in the area we call 'geometry' or 'the geometric theory of differential equations.'"

Surely none of us could quarrel with the notion that there is a symbiotic relationship between mathematics and physics (as well as other branches of science and engineering). In fact, an argument can be made that mathematical ability is what differentiates the scientist from the nonscientist. C. P. Snow in the preface of *The search* **[62]** says:

> "There is just one basic difficulty. All children have a dash of scientist in them. Watch any bright child if you tell him about the stars or atoms or dinosaurs. He will want to find out some more. The urge to investigate, which is the scientific urge, isn't anything very special or academic. It is one of the most human things about us. In that sense, as I said, all children are scientists. But all children are not mathematicians, and that is the core of the difficulty. I don't know how many people are mathematically blind to the extent that some of us are tone deaf, but I suspect a larger proportion than the educational psychologists usually allow. Thinking of twenty acquaintances, who have all done pretty well in various sorts of intellectual life, I should say that at least five were, if not mathematically blind, at least grossly deficient in mathematical sense. That means that though, sensibly educated, they could have got a good working idea of how physical science goes about its business, they would never have reached the fundamental concepts. I suggest we have got to accept the fact that, for a lot of people of high intelligence and imagination, this is as near as they are going to come to the real stuff. It is much better than nothing, but there are limits, and it is just as well to be clear-sighted about them in advance."

Nor, we feel, can one deny that differential geometry and topology, in particular, provide a very powerful and succinct language for describing physical phenomena. A few examples show how significant these languages have become in the description of diverse areas of physics and engineering and how manifold are their application.

(1) The association of a symplectic manifold with the phase space of a mechanical system (with finitely many degrees of freedom) introduces great clarity into classical mechanics. From here it is a short jump conceptually (though not necessarily technically) to studying continuum problems on symplectic manifolds modelled on Banach spaces [**1**, **2**, **13**, **25**, **46**, **47**, **65**, **69**, **70**, **76**].

(2) Fiber bundles give a structure to the gauge theories of quantum field theory whereby a gauge potential is a connection one-form on a principal fiber bundle over space-time and the curvature form of that connection defines the field strength of the gauge field. The many tools of differential geometry (e.g. Chern classes, Atiyah-Singer Theorem, etc.) have been brought to bear on physical problems with good results [**3**, **4**, **16**, **19**, **30**].

(3) In geometric quantization, one seeks to formulate the relationship between classical and quantum mechanics in a geometric language. Not only does this approach lead to an intuitive mathematical interpretation of quantum mechanics, but it has led, in particular, to new insights into the connection between the concepts of symmetry in the two theories [**3**, **4**, **41**, **60**, **61**, **63**, **72**].

(4) General relativity was, of course, a geometric theory from its inception. Little more need be added than to say that it is now *de rigueur* to use the invariant calculus of modern differential geometry [**22, 26, 52, 58, 71**]. Complex manifold theory has come to play an increasing and exciting role here and in mathematical physics in general [**20, 21, 44, 56, 74, 75**].

(5) In the materials sciences, geometric and topological methods have been valuable in describing dislocations in simple bodies and defects in ordered media [**5, 51, 53, 54, 57, 73**].

(6) A vast array of mathematical tools, including not only the traditional analytic and linear algebraic ones but more recently those of algebraic and differential geometry and ring theory, have been brought to bear in the study of control systems. Lie theoretic techniques are especially useful in the study of nonlinear systems [**7–9**, **15**, **27**, **29**, **31**, **34–39**, **42**, **43**, **45**, **50**, **66–68**].

This list could go on to include other disciplines (e.g. Walrasian economics [**17**]), but we conclude this brief description with a quote from [**52**, p. 302]: "Nature likes theories which are simple when stated in coordinate free geometric language."

A word about the title is in order. *Cartanian geometry*, of course, means geometry in the sense of Elie Cartan. That in turn can mean many things, not the least of which is a systematic use of the ideas of exterior differential systems. Hermann is to be praised for making this material available in a modern setting, as there is a striking absence at the textbook level (the works of Dieudonné [**18**] and Sternberg [**64**] being exceptions) of in-depth information on exterior systems. In his own work he has made significant application of these methods in his collaboration with Estabrook and Wahlquist on nonlinear waves. The interest in exterior differential systems is quite high these days. There are some articles in the proceedings of the conference held at Park City, Utah [**11**]. In addition, between the time this review is written and it appears, there will be two conferences which will delve into this subject. S. S. Chern and R. Bryant will have lectured on it in January, 1982 at the University of New Mexico and R. Gardner will discuss some of its aspects at Michigan Technological University in July, 1982 at a conference on differential geometric control theory.

There can be little question then, at least from the mathematician's point of view, that differential geometry has had significant impact on the development of physical sciences and of mathematical engineering. Hermann among others has proselytized long and hard for its use, not only for the clarity and beauty thereby introduced but also, perhaps more to the point, as a way of circumventing difficulties (e.g. in quantum field theory) that have arisen from too rigid an adherence to analytical techniques. Of course, for each specific application a value judgment needs to be made. Thus, there are alternative views of the value of Hermann's work. Both sides of this question are presented in the author's essay *Twenty-five years as a mathematician* and the other reflections which cover the first 82 pages of Part B. Included in those pages are not only referees' reports which question whether there will be scientific rewards from the approach espoused in the book but also a series of rebuttal letters written by Hermann defending his approach. Prospective readers of these volumes may wish to examine these reflections before making the decision whether or not to go through the material themselves.

Let us begin the mathematical review of this work with the material which does work quite well; namely, applying fiber bundle techniques to linear systems theory. This material, which was originally due to the author and Clyde Martin **[32, 48, 49]**, starts off with a linear system and interprets it in the frequency domain as a vector bundle over the Riemann sphere. The machinery of algebraic topology/differential geometry may then be brought to bear on the problem. We will outline this procedure and its consequences, after a short introduction to relevant concepts from linear systems theory.

Basic to the understanding of a linear time-invariant system Σ are the internal and external descriptions of such a system. When the system dynamics are known, then one has at hand three vector spaces X, U, Y (called the state, input and output spaces respectively) and a triple of constant matrices (A, B, C) of appropriate sizes so that the state x, the input u, and the output y are related by

$$\text{(1)} \qquad \Sigma: \quad \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned}$$

with $x(t) \in X$, $u(t) \in U$ and $y(t) \in Y$. When a control function u is chosen (frequently from the space of piecewise continuous U-valued functions), the differential equation can be integrated to produce, for any initial time t_0 and initial state x_0, an evolution of the state $x(t;\, t_0, x_0, u)$ which starts at x_0 at time t_0 and an associated output $y(t;\, t_0, x_0, u)$. Consequently, Σ can be viewed as a map, via the state trajectory $x(t;\, t_0, x_0, u)$, from input functions to output functions.

Two questions of concern about Σ are whether it is *controllable* and/or it is *observable*. The former means that it is possible in finite time and by suitable choice of u to connect any two points of the state space by a state trajectory.

Since the system is linear, this is equivalent to being able to drive any initial state to zero. For Σ to be observable means that one can reconstruct the initial state of the system from a reading of the input and output over a sufficient duration of time. Simple matrix criteria are known for determining whether Σ possesses these properties; to wit

$$\Sigma \text{ controllable} \Leftrightarrow n = \dim X = \operatorname{rank}([B, AB, \ldots, A^{n-1}B]) \qquad \text{(column partition)}$$

and

$$\Sigma \text{ observable} \Leftrightarrow n = \operatorname{rank}([C, CA, \ldots, CA^{n-1}]) \qquad \text{(row partition)}.$$

What has been given up to this point is an internal description in the time domain. Taking the Laplace transform $\hat{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$ (with zero initial condition), one obtains equations in the frequency domain

$$\text{(2)} \qquad \hat{\Sigma}: \quad \begin{aligned} s\hat{x}(s) &= A\hat{x}(s) + B\hat{u}(s), & \hat{x}(\infty) &= 0, \\ \hat{y}(s) &= C\hat{x}(s), & \hat{y}(\infty) &= 0, \end{aligned}$$

where the values at the ideal point, ∞, follow from setting $\tau = s^{-1}$ in $\hat{\Sigma}$ and letting τ go to zero. Consequently, the system is now defined over the Riemann sphere, S. Solving for $\hat{y}$ yields the input-output (i/o) map

$$\hat{y}(s) = C(sI - A)^{-1} B\hat{u}(s), \qquad s \in S\text{-spectrum}(A).$$

In the external ("black box") description, only the i/o map is known (i.e. $\hat{y}(s) = T(s)\hat{u}(s)$, where $T(s)$ is a proper rational matrix function called the *transfer function*). One then seeks a *realization* of the system, which is an internal description (= triple (A, B, C)) with the same i/o map. While many realizations are possible, a *minimal realization* is achieved when the dimension of the state space is minimal. It is a fundamental result of linear system theory that the realization is minimal when (A, B, C) is a controllable and observable triple (uncontrollable modes and unobservable modes do not contribute to the i/o map). The minimal state space dimension is called the *McMillan degree* of Σ. (For a matrix theoretic description of these ideas, see Brockett [**6**].)

The Hermann-Martin work, which is described next, has made interesting connections between vector bundle theory and the concepts of controllability, observability, and McMillan degree. Recall that a *vector bundle* over M is a (total) space P and a continuous map $\pi: P \to M$ such that the fiber $\pi^{-1}(m)$ is a vector space for each $m \in M$. The vector bundle is *nonsingular* if $\dim \pi^{-1}(m)$ is constant. A section of the bundle is a map $\gamma: M \to P$ with $\pi \circ \gamma = \mathrm{id}_M$. Working in the frequency domain, Hermann constructs the input-output bundle I/O for Σ by

$$I/O = \{(s, \hat{u}, \hat{y}) \colon \text{there is an } \hat{x} \text{ which solves (2)}\}.$$

This is a holomorphic vector bundle over the Riemann sphere S whose cross-sections are i/o maps for Σ. The object now is to attach geometric invariants to this bundle that have linear system theoretic significance.

To establish another link with Σ, he first considers the general class of kernel bundles associated with a pair of linear maps. Briefly described, the construction is as follows. Let $\alpha_0, \alpha_1: V \to W$ be linear maps between a pair of complex vector spaces V, W and define

$$A(s) = \ker(s_0\alpha_0 + s_1\alpha_1), \qquad s = (s_0, s_1) \in \mathbf{C}^2 - O.$$

Since $A(s)$ respects the equivalence relation which creates $P_1(\mathbf{C}) = S$ from $\mathbf{C}^2 - O$, there is obtained a (possibly singular) vector bundle over S with fibers $A(s)$. The Kronecker theory of pencils of linear maps [**23**, **28**] is now brought into play. Under a certain technical assumption about the nature of the Kronecker decomposition of the pencil, Hermann shows that the kernel bundle is nonsingular and then proceeds to give a concrete algebraic construction of the splitting of this bundle into complex line bundles (= vector bundles whose fibers are one dimensional complex vector spaces). This is the splitting ensured by the Grothendieck Theorem [**24**] on nonsingular holomorphic vector bundles over the Riemann sphere. In addition, Hermann's construction explicitly exhibits for each line bundle an integer n_i which is the Chern number of that bundle.

The machinery now being in place, the tie-in with linear system theory can be made. Form the kernel bundle K based on the spaces $V = XxU$, $W = X$ and the linear maps

$$\alpha_0(\hat{x}, \hat{u}) = \hat{x}, \qquad \alpha_1(\hat{x}, \hat{u}) = -(A\hat{x} + B\hat{u}).$$

Then, controllability of Σ makes K nonsingular (i.e. the aforementioned technical assumption of the Kronecker decomposition is then met). Controllability and observability of Σ imply that I/O and K are isomorphic vector bundles (i.e. there is a fiber preserving diffeomorphism between the total spaces which restricts to linear isomorphisms on fibers) and so the geometric invariants of K obtained by the Kronecker decomposition carry over to I/O.

To explore the nature of these invariants, let $\Sigma' = (A', B', C')$ be another controllable/observable system with associated pencil $P'(s) = (A' - sI, B')$ and input-output bundle I/O'. Continuing with the circle of ideas above, he establishes the equivalences: the kernel bundles K, K' of Σ, Σ' have the same Chern numbers for their decomposition into line bundles if and only if I/O and I/O' are isomorphic; if and only if the associated pencils are Kronecker equivalent, i.e. $\alpha P(s) = P'(s)\beta$ for invertible matrices α, β on appropriate spaces; if and only if (A, B) and (A', B') lie on the same orbit under the action of the general feedback group (which affects systems by state space and input space isomorphisms and by state feedback); if and only if Σ, Σ' have the same Brunovsky indices (= set of integers that label the above orbits). The Brunovsky indices (= Kronecker indices = controllability indices), which are of a linear system theoretic origin [**10**, **40**], are thus established as geometric invariants of the I/O bundle (also see Wonham [**77**] for a characterization of these indices as dimensions of (A, B)-controllability subspaces.) Moreover, the McMillan degree, as the sum of the Chern numbers of the I/O bundle, is also a geometric invariant.

Once Hermann has completed this discussion and made the analogy between the decomposition of kernel bundles into line bundles and the Grothendieck theory, he makes some speculative assertions about possible generalizations to higher dimensional projective spaces. That material is then mentioned on p. 436 where the Hermann-Martin papers are referred to.

The above illustrates both the strengths and weaknesses of the book. On the one hand the connection between control theory and holomorphic vector bundles is quite exciting. However, without having read the Hermann-Martin papers, it would be quite difficult to see where the circle of ideas discussed above was going. In fact, some aspects of the circle are mentioned only implicitly in the two volumes under review. One wonders why the author didn't incorporate more of the very interesting work from the Hermann-Martin papers in *Parts* A and B (of course, it may be that what the two reviewers see as the most important and interesting aspect of these ideas is different from what the author sees as most important). We would urge any readers of these two volumes to have the Hermann-Martin papers close at hand.

The problem of wondering what goes next occurs too often in the books. In a working draft of an article that is fine; however, in a book or seminar notes, it is a questionable approach. The books should be better organized so that similar results could be grouped together, rather than scattered through several sections of both volumes. For example, the Kronecker decomposition is discussed in *Part* A but not made use of until *Part* B. There are also incomplete ideas such as on p. 399 of *Part* A where, after showing that a certain procedure (involving algebraic geometry) yields the Gaussian elimination, Hermann says: "There are certain general features of these algorithms which have the same geometric and algebraic flavor; I plan to work further in this direction." On the next page, in the introduction to the *Surveys and development of new geometric methodology for application* section he says "I believe this topic, the QR algorithm, is an ideal one for this purpose and it *introduces tremendous new insight* into the applied problem. I plan much more work later on; this is the first tidbit" (emphasis is the author's). Excellent, then publish it when there is more than a tidbit. Moreover, while it is of interest to geometers to see other mathematics translated into their terms, it should be pointed out that there is already a geometric theory of the QR algorithm [**55**] that is in harmony with its natural linear space structure.

The beautiful subject of geometric methods in control theory has been undergoing an information explosion. The results of the Harvard conference of June, 1979, sponsored by NATO and NASA/Ames Research Center have been edited by C. Byrnes and C. Martin and published by D. Reidel [**12**]. These proceedings include many fine articles (especially the Byrnes, Hazewinkel, Martin and Rouchaleau article, the Byrnes article, the Brockett article, and the Martin article) which are well written and accessible even to the nonexpert. This is a better place to learn geometric control theory than are the Hermann books. (Of course, *Part* A was finished before the NATO conference and *Part* B must have been nearly complete at the time of the conference. Thus, it is likely that Hermann wrote most of his work without being able to make use of the preprints of the conference.)

We now turn to the matrix Riccati equation (MRE), which has a prominent place in the control theory literature. For example, its solutions determine the state feedback which gives optimal performance in the linear optimal control problem with quadratic performance functional. Naturally an equation of such importance has been studied from many points of view. Starting with a paper of C. R. Schneider **[59]**, the MRE has been viewed to be properly set geometrically as a differential equation on a Grassmann space. It is this interpretation which Hermann describes and builds upon in the book under review.

Consider R^{2n} as a linear symplectic manifold with respect to the alternating nondegenerate form ω given by

$$\omega(\xi_1, \xi_2) = x_1^T y_2 - y_1^T x_2, \qquad \xi_i = (x_i, y_i).$$

Define a subspace γ of R^{2n} to be *Lagrangian* if $\omega(v_1, v_2) = 0$ for all $v_1, v_2 \in \gamma$ and $\dim \gamma = n$. Let $\mathfrak{L}$ be the collection of such subspaces. $\mathfrak{L}$ is then a subset of the Grassmann space $\mathrm{GR}(R^{2n})$ of all subspaces of R^{2n} and is acted on transitively by the symplectic group $\mathrm{Sp}(\omega)$ (that is, the invertible $2n \times 2n$ matrices that leave ω invariant). As such, $\mathfrak{L}$ is an orbit in $\mathrm{GR}(R^{2n})$ and can be realized as the coset space $\mathrm{Sp}(\omega)/H$, where H is the isotropy subgroup at some element of $\mathfrak{L}$. This is an example of what Hermann terms a Grassmann coset space. What is the relevance of this idea for the study of the MRE?

The linear optimal control problem with quadratic performance index is given by a pair consisting of a linear differential equation

$$\text{(ODE)} \qquad \dot{x} = Ax + Bu, \qquad A, B \text{ constant}$$

and a performance index (over a finite time interval)

$$\text{(Index)} \qquad \int dt\left(x^T Q x + u^T R u\right), \qquad Q, R \text{ constant, positive definite.}$$

The goal is to choose the control function u so that it, together with the resulting state function as given by ODE, will minimize Index. It is a time-honored result that the optimal control is given by state feedback according to

$$u(t) = -R^{-1} B^T P(t) x(t)$$

where the symmetric matrix $P = P(t)$ satisfies

$$\text{(MRE)} \qquad -\dot{P} = A^T P + PA - PBR^{-1}B^T P + Q$$

Recast MRE as the equation

$$\text{(3)} \qquad \dot{\xi}_p = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \xi_p \quad \text{and} \quad \xi_p = [x, Px]^T.$$

If it is now noted that the set of $[x, Px]^T$, as x runs through R^n, is a Lagrangian subspace and that the symmetric matrices can be identified with $R^{n(n+1)/2}$, then we see that the map which sends P to the corresponding Lagrangian subspace is a chart on the compact $n(n+1)/2$-manifold $\mathfrak{L}$. In this setting, (3) tells us not only that MRE can be viewed as an equation on $\mathfrak{L}$ but also that the usual treatment is local, i.e. on a chart. (In this regard, Hermann develops a theory of global linear systems in *Part* B.)

A local description is in general inadequate for dealing with finite escape times or asymptotic phenomena. From the global viewpoint, MRE may continue to evolve on $\mathfrak{L}$. This offers the possibility of following the time evolution by employing other charts on $\mathfrak{L}$ (e.g. the one given by $P \to \{[Px, x]^T: x \in R^n\}$). Another pay-off comes in viewing solutions of the *algebraic* Riccati equation (set $\dot{P} = 0$ in MRE) as zeroes of a vector field on a manifold. These ideas are explored in Chapters 4 and 5 of Part A.

Thus far, this review has discussed both Hermann's philosophy and some of the mathematics in the volumes. We now turn to the pedagogy which is, unfortunately, a weak point of the volumes. In the preface to volume I of IM, the author says it is his intention that these are seminar notes and so we need to look at them in this light. The problem is that it is unclear for whom the notes are intended as the level of exposition is quite uneven. For example on p. 437 of *Part* B the author uses Chow's Theorem in a proof about controllability but never discusses it in detail, whereas five pages earlier he defines the pull-back map on differential forms and the Grassmann algebra! (In fairness, he does give reference to a paper by Hermann-Krener [**31**] in which a version of Chow's Theorem appears.) The three problems which we found most troublesome were the lack of an index, the enormous number of typographical errors, and the tendency alluded to earlier to start one subject, leave it and then return later. While a lack of index is understandable in a "seminar notes" type of format, large numbers of typos are not. Their presence makes the going very difficult, much more so than it should be. One example suffices: If $V = X \oplus U$ and $W = X$ then for any A, a linear map from X to X and B a linear map from U to X, $A(s) = (A - sI, B)$ for $s \in \mathbf{C}$, is called a *pencil.* Hermann says that $A(s)$ and $A'(s) = (A' - sI, B')$ are Kronecker equivalent if there is a pair $\gamma \in Gl(W)$ and $\beta \in Gl(V)$ such that $A(s) = A'(s)$. (Of course what is meant is $\gamma A(s) = A'(s)\beta$.) One wishes that the author had taken into account the problems which were cited in the Chernoff-Marsden review [**14**] of some of his earlier work.

Robert Hermann has made a significant contribution to mathematics both in differential geometry and in control/systems theory. These volumes, which describe some of his work, are from a philosophical viewpoint valuable. Their value could have been far greater if the author had (1) explored specific topics in more detail and/or omitted ideas whose depths had not yet been explored and (2) done a good job in proofreading. In his final remarks on exterior systems (*Part* A, p. 444) he states "I hope I have given some solid evidence for my belief that ideas of "modern" differential and algebraic geometry are very significant for systems-theoretic problems." For systems theory (to which we have restricted our review), this evidence has been given.

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CLARK R. GIVENS
RICHARD S. MILLMAN

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Bases in Banach spaces. II, by Ivan Singer, Springer-Verlag, Berlin and New York, 1981, viii + 880 pp., $78.00.

Introduction. In his book *Bases in Banach spaces*. II (BBS II), Ivan Singer takes all knowledge of bases and their generalizations to be his province. More precisely, he states in the preface that "this volume attempts to present the results known today on generalizations of bases in Banach spaces and some unsolved problems concerning them". *Bases in Banach spaces*. I (BBS I) was published in 1970 and BBS II in 1981. During the writing of these books, basis theory and its generalizations began to develop very rapidly. The task of the author became not that of describing a theory already essentially developed, but of presenting a theory in a very rapid state of development. Thus, in order to achieve his goal of a complete account of basis theory, its generalizations, and its applications, Ivan Singer is working on a third volume on applications, bases in concrete spaces, and perhaps some loose ends.

The book under review, BBS II, is encyclopaedic (at the Banach space level) with respect to its subject. It consists of twenty-one sections plus a section entitled *Notes and remarks*. The review will discuss the section on the solution