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ON REPRESENTATIONS IN COHOMOLOGY OVER PSEUDO-HERMITIAN SYMMETRIC SPACES

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A significant part of the problem of classifying representations of groups is discovering which representations are *unitarizable,* i.e. in which cases do the vector spaces of the representation admit an invariant Hilbert space structure. In large measure, the difficulty here is finding an appropriate realization of the abstract space of the representation as a concrete space of functions, forms, cohomology classes, etc. One class of representations where the unitary structure (if it exists) is not well understood is given by considering the natural action of groups on holomorphic, homogeneous line bundles over pseudo-hermitian symmetric spaces, and then the induced action on cohomology with coefficients in the sheaf of holomorphic sections of these line bundles.

It has been known for some time [1], [2] that the representations of $SU(2, 2)$ on certain cohomology spaces over $SU(2, 2)/S(U(1) \times U(1, 2))$ are equivalent to subrepresentations of the well-known "metaplectic" representation of $SU(2, 2)$, and are therefore unitarizable.

The purpose of this note is to present the analogous theorem for representations of $SU(p, q)$ in cohomology over $SU(p, q)/S(U(k) \times U(p - k, q))$.

Abstractly, we have the following situation: there is a pseudo-Kahler bundle, E, over the pseudo-hermitian symmetric space G/H, a Kahler bundle E' over the hermitian symmetric space G'/H' and an inclusion $i: E \hookrightarrow E'$ which preserves both the real affine connection and the real symplectic structure. This inclusion allows us to pull back holomorphic functions on E' to partially holomorphic functions on E, which we can then change to $\bar{\partial}$-closed forms and hence cohomology classes.

While G acts by Kahler isometries on E, and the action can be extended to all of E', G acts only by symplectic transformations on E' and not Kahler isometries. Hence, the geometric action of G on functions on E' will not take holomorphic functions to holomorphic functions. This can be remedied by using the metaplectic action of $G \subset$ (symplectic automorphisms of E'). An important fact is that the geometric action on cohomology, and the metaplectic transformations on functions then correspond.

Other cases in which such a generalized Harish-Chandra embedding exists

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include the spaces $SP(2m, \mathbf{R})/U(p, q)$, and we therefore believe that analogous theorems will be true in these cases also.

In each of the examples considered below, the cohomology behaves as if the symmetric space were a product of a compact, positive definite space, and an open, negative definite space. The cohomology in question will live in the dimension of the maximal compact "factor". For the case $SU(p, q)/S(U(k) \times U(p-k, q))$, the compact "factor" is $SU(p)/S(U(k) \times U(p-k))$ and hence the dimension will be $k(p-k)$.

Let $\mathbf{C}^{p,q}$ denote the complex vector space $\mathbf{C}^p \oplus \mathbf{C}^q$ with the indefinite hermitian inner product $\Sigma_{i=1}^p u_i\bar{u}_i' - \Sigma_{i=1}^q w_i\bar{w}_i'$. Then

$$M_{p(k-p),q} = SU(p, q)/S(U(k) \times U(p-k, q))$$

may be identified with {k-dim subspaces, $W \subset \mathbf{C}^{p,q}$, so that the restriction of the inner product to W is positive definite}. Let E^* be the line bundle over $M_{k(p-k),q}$ whose fiber over a k-plane, W, is the one-dimensional space $\underbrace{W \wedge \cdots \wedge W}_{k}$. Sections of the lth power of the dual, E^l, of E^* may then be identified with functions, f, on E^* so that f is a holomorphic function, homogeneous of degree l, on $W \wedge \cdots \wedge W/0$, for each W.

On the other hand, forms $\tilde{\alpha}$, on $M_{k(p-k),q}$ can be pulled back to forms, α on $E^*/0$, where, if X is any vectorfield along the fibers of E^*, $X \lrcorner \alpha = X \lrcorner d\alpha = 0$. Since the fiber is connected, the space of forms on $M_{k(p-k),q}$ may be identified with forms on $E^*/0$ with this property. Therefore, forms on the base with values in E^l may be identified with forms, α, on $E^*/0$ of the form, $\alpha = \Sigma f_i\alpha_i$, where f_i are holomorphic, homogeneous of degree l, along the fibers and $X \lrcorner \alpha_i = X \lrcorner d\alpha_i = 0$. If Γ_0 is the $(1, 0)$ vectorfield generating the action of $\mathbf{C}^*$ on E^*, we can state the above conditions as $\Gamma_0 \lrcorner \alpha = \bar{\Gamma}_0 \lrcorner d\alpha = 0$ and $\Gamma_0 \lrcorner d\alpha = l\alpha$.

The space $V_k = \underbrace{\mathbf{C}^{p,q} \times \cdots \times \mathbf{C}^{p,q}}_{k}$, thought of as $(p+q) \times k$ matrices, $\binom{w}{u}$, has an action of $SU(p, q)$ on the left and $Gl(k, \mathbf{C})$ on the right and the two actions commute. Let $V_k^+ \subset V_k$ be those k-tuples of vectors in $\mathbf{C}^{p,q}$ which span a k-dim positive definite subspace. Then $V_k^+/(\text{action of } Sl(k, \mathbf{C}))$ may be identified with $E^*/0$.

If $\Gamma_1, \ldots, \Gamma_{k^2-1}$ are $(1, 0)$-vectorfields generating the actions of $Sl(k, \mathbf{C})$ on V_k^+, and Γ is the generator of the action of $\mathbf{C}^*$, then a form, α, on V_k^+ will be the pull back of a form on $M_{k(p-k),q}$ with values in E^l exactly when

$$\begin{aligned}\Gamma_1 \lrcorner \alpha = \bar{\Gamma}_1 \lrcorner \alpha = \Gamma_1 \lrcorner d\alpha = \bar{\Gamma}_1 \lrcorner d\alpha = \cdots = \Gamma_{k^2-1} \lrcorner \alpha = \bar{\Gamma}_{k^2-1} \lrcorner \alpha \\ = \Gamma_{k^2-1} \lrcorner d\alpha = \bar{\Gamma}_{k^2-1} \lrcorner d\alpha = 0 \qquad \text{(I)}\end{aligned}$$

$$\Gamma \lrcorner \alpha = \Gamma \lrcorner \alpha = \bar{\Gamma} \lrcorner d\alpha = 0. \qquad \text{(II)}$$

$$\Gamma \lrcorner d\alpha = kl\alpha. \qquad \text{(III)}$$

The "k" arises from the fact that the push forward of Γ is $k\Gamma_0$ and not Γ_0.

To get cohomology classes in $H^s(M_{k(p-k),q}, \mathcal{O}(E^l))$, we need only to find $\bar{\partial}$-closed $(0, s)$-forms on V_k^+ satisfying the above conditions. Consider the Hilbert space $\bar{E} = \{f\colon V^k \longrightarrow \mathbf{C} \mid f$ is holomorphic in the coordinates $\bar{u}$ and w and $\|f\|_{\bar{E}}^2 = \int |f|^2 \exp(-\operatorname{tr}({}^t u\bar{u} + {}^t w\bar{w}))d\mu < \infty\}$ where $d\mu$ is Euclidean measure normalized so that $\|1\|_{\bar{E}}^2 = 1$. Then, both $u(1) \subset \mathbf{C}^*$ and $SU(k) \subset Sl(k, \mathbf{C})$ act unitarily on this space, and the actions commute. Let $\bar{E}_{0,kl}$ be the sub-Hilbert space of $\bar{E}$ defined by

$$\begin{cases} f(xg) = f(x) & \text{for all } x \in V_k,\ g \in SU(k),\ \lambda \in U(1), \\ f(\lambda x) = \bar{\lambda}^{kl} f(x). \end{cases}$$

A basis for this (infinite-dimensional) space may be found quite easily from classical invariant theory.

For $f \in \bar{E}_{0,kl}$, define the function

$$\tilde{f} = \int_{Gl(k,\mathbf{C})} \det(g)^l \det({}^t g\bar{g})^{p-k} f(\bar{u}\bar{g}, wg) \exp(-\operatorname{tr}({}^t g\,{}^t u\bar{u}\bar{g}))dg_{ij}d\bar{g}_{ij}.$$

Then it can be shown [3], using the reproducing kernel for $\bar{E}$, that

(1) $\tilde{f}$ is an analytic function defined on all of V_k^+,

(2) $\tilde{f}$ is $Sl(k, \mathbf{C})$ invariant,

(3) $\bar{\Gamma} \lrcorner d\tilde{f} = -kp\tilde{f}$ and $\Gamma \lrcorner d\tilde{f} = -k(p + l)\tilde{f}$.

Suppose that $\Gamma_1, \ldots, \Gamma_{k^2-1}$ are chosen to correspond to a basis of the Lie algebra of $Sl(k, \mathbf{C})$ so that $\Delta = \Gamma_1 \wedge \cdots \wedge \Gamma_{k^2-1}$ is $Sl(k, \mathbf{C})$-invariant. Define the $(k^2 - kp, 0)$-form $\eta = (\Gamma \wedge \Delta) \lrcorner \Omega_u$, where $\Omega_u = du_{11} \wedge \cdots \wedge du_{pk}$. For $f \in \bar{E}_{0,kl}$, the form $\alpha(f) = \tilde{f}\bar{\eta}$ is a $\bar{\partial}$-closed form on V_k^+ satisfying conditions (I) and (II) and $\Gamma \lrcorner d(\alpha(f)) = k(-p-l)\alpha(f)$. Thus $f \longrightarrow \alpha(f)$ maps $E_{0,kl}$ to $H^{k(p-k)}(M_{k(p-k),q}, \mathcal{O}(E^{-p-l}))$.

THEOREM [3]. *The map $\alpha\colon f \longrightarrow [\alpha(f)]$ is injective. Moreover, the image of α is an $SU(p, q)$-invariant subspace and the Hilbert space structure it inherits from $\bar{E}_{0,kl}$ is also $SU(p, q)$-invariant.*

The proof of injectivity follows from a formula which calculates the norm of f directly in terms of the form $\alpha(f)$ and noting that if $\alpha(f)$ were exact, the norm would vanish. However, this calculation relies heavily on the *form* of $\alpha(f)$, i.e. $\tilde{f}\bar{\eta}$, and it is not clear from the calculation that it is independent of the choice of representative for a cohomology class, or invariant under $SU(p, q)$.

The $SU(p, q)$-invariance of both the subspace and the inner product involves showing that for any $g \in SU(p, q)$, there is a unitary map Bg (essentially the metaplectic action) from $\bar{E}_{0,kl} \longrightarrow \bar{E}_{0,kl}$ so that $[\alpha(Bgf)] = g[\alpha(f)]$. This is completed in two stages. First it is shown that a Bg exists so that the difference

$[\alpha(Bgf)] - g[\alpha(f)]$ vanish on "enough" compact submanifolds of $M_{k(p-k),q}$, and then proceed by a spectral sequence argument [4] to show that the difference must vanish globally. Since Bg is a unitary map, this completes the proof.

We note that under the metaplectic representation of $SU(p, q)$, $\overline{E}_{0,kl}$ is irreducible, which follows from Howe's theory [5] of dual pairs (in this case $(U(k), U(p, q))$ in $\mathrm{Sp}(2k(p + q), \mathbf{R})$).

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