BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 4, Number 1, 1981

# STRONG REDUCIBILITIES

BY PIERGIORGIO ODIFREDDI[1]

*Dedicated to my teacher Flavio Previale*

## Table of Contents



**Acknowledgements.** During the writing of the first draft I have been helped in many ways by Carl Jockusch, Jr., who patiently explained to me an incredible number of results on these and other topics. R. Shore and R. Soare have extensively commented on it and given many suggestions for the Introduction. Sincere thanks to them.

I'm also indebted to Marat Arslanov, Richard Epstein, Yuri Ershov, Georg Kreisel, Donald Martin, Anil Nerode and Steve Simpson for help, advice and corrections.

**1. Introduction.** The notion of algorithm permeates the experience of every mathematician, and goes back to the very origins of mathematics. To answer positively a problem asking for an algorithmic solution, the intuitive notion of algorithm is sufficient since we can simply exhibit a solution and convince ourselves that the procedure is effective. This has been done for centuries. If we have instead reasons to believe that the answer is negative, we need a formal characterization of algorithm to prove that each one is ruled out as a solution. Such a characterization was only obtained in the late 1930's with the work of Herbrand, Gödel, Church, Turing, Post and Kleene. (See **[Da1]**.) They defined various notions of recursive function (on the set of natural numbers) and the related concept of recursively enumerable (r.e.) set, using for example abstract machines, routine rules of calculation or generative grammars. All of these prima facie different approaches defined the same class of functions and of sets. These are the precise analogues of the intuitive notions of computable function and effectively generated set.

With the formal version of computability at hand, we can talk of an

Parts of this paper were presented in 1979 in talks at the University of Chicago, Iowa State University and the University of Connecticut; received by the editors June 1, 1980.

1980 *Mathematics Subject Classification*. Primary 03D30; Secondary 03D25.

[1]The author was partially supported by a grant of CNR, Italy.


0002-9904/81/0000-0002/$13.50

unsolvable or undecidable problem, that is one with no recursive solution. Unsolvable problems naturally arise in many fields of mathematics: the halting problem (to decide if a Turing machine halts or not on a given input), Hilbert's tenth problem (to decide if a polynomial Diophantine equation with integral coefficients has solutions in the integers), the word problem for finitely generated groups (to decide if two words, that is two strings of generators, are equal in the group: proposed by Dehn in 1911), the ergodicity problem for homogeneous random media (to see if there is exactly one time-independent probability measure on the set of states), the homeomorphism problem for $n$-manifolds for $n \geqslant 4$ (to decide, given two $n$-manifolds, if they are homeomorphic; for $n = 2$ it is a classic result of Riemann that the problem is solvable). (See **[Da2]** and **[Ku]**.)

Furthermore, undecidability results can be used to show that, although a group is completely defined by a presentation, it is not possible to obtain effectively from the presentation very much information about the group itself (e.g., we cannot even tell if the group is trivial or not). Indeed for any fixed algebraic property of finitely presented groups which is not trivial (i.e., some finitely presented group has it and some does not) and hereditary with respect to finitely presented subgroups, the problem of deciding–given a presentation–whether the group defined by it has the property or not is undecidable (Adjan-Rabin, see **[Da2]**).

A surprising result in this direction is a recursion-theoretic characterization of a purely algebraic property. A finitely generated group is isomorphic to a subgroup of a finitely presented group if and only if it is recursively presentable, i.e., the set of words equal to 1 is recursively enumerable (Higman, see **[Da2]**).

Rather than simply classifying problems into decidable and undecidable (or sets as recursive and nonrecursive) we want a precise measure of the relative difficulty of such problems. The proofs of unsolvability suggest a natural classification system. The idea of those proofs is to first construct an unsolvable problem $A$, roughly by Cantor's method of diagonalization over the recursive sets. Then to prove $B$ undecidable, one gives an algorithm for $A$ that uses $B$, i.e., an algorithm that would answer every question of the form "$x \in A$?" if we knew $B$. Of course, if $B$ were decidable then so would $A$ be as well. In general if such an algorithm exists we say $A$ is Turing-reducible to $B$ or $A$ is recursive in $B$, written $A \leqslant_T B$. This yields a natural equivalence relation $A \equiv_T B$ (when $A \leqslant_T B$ and $B \leqslant_T A$) whose equivalence classes we call Turing degrees ($T$-degrees). If a $T$-degree contains an r.e. set, we call it an r.e. $T$-degree.

$T$-degrees and r.e. $T$-degrees arise naturally in various branches of mathematics. For example, for any $T$-degree there is a finitely generated group whose word problem has that degree, and if the $T$-degree is r.e. the group may be chosen as finitely presented. (See **[Be]**.) Similarly, for any r.e. $T$-degree there is a recursive class of $n$-manifolds ($n \geqslant 4$) with homeomorphism problem of that degree (see **[Da2]**). Thus the classification of (r.e.) degrees is equivalent to the classification of such algebraic problems. For example, the existence of infinitely many r.e. degrees implies that there are infinitely many genuinely different word problems for finitely presented groups.

The first attempt at such a classification began with Post **[Po]**. In this attempt he introduced several reducibilities stronger than $T$-reducibility. The reductions used in actual undecidability proofs are almost always of these sorts. They usually produce an algorithm $f$ such that questions of the form "$x \in A$?" are reduced to questions of the form "$f(x) \in B$?". In this case we say $A$ is many-one reducible to $B$ ($A \leqslant_m B$). Moreover, the algorithm usually gives a one-one function $f$, in which case we write $A \leqslant_1 B$. A more general kind of reduction (but again stronger than $\leqslant_T$) allows not only one question of the form "$f(x) \in B$?" but a finite Boolean combination of them. It is called truth-table reduction ($A \leqslant_{tt} B$). All these concepts have corresponding notions of degrees, and these are the degrees we will be dealing with in the paper. Word problems for finitely generated (resp. presented) groups again correspond to the (r.e.) $tt$-degrees, but not to the $m$-degrees. The r.e. $m$-degrees correspond instead to the word problems for finitely presented semigroups, whose word problem was posed by Thue in 1914 (see **[Be]**, **[Da2]**).

Two sets of numbers are said to be *recursively isomorphic* if one is the image of the other under a recursive permutation of the natural numbers. The naturalness and importance of 1-degrees comes from the fact that they are exactly the recursive isomorphism types (a result which is the recursive analogue of the Schroeder-Bernstein theorem on the set-theoretical isomorphism of two sets such that each one can be one-one mapped into the other). The analogy between classical set theory and recursive set theory can be pursued along the lines above (giving for example an analogue of Cantor's theorem on the nonexistence of an onto function between $\omega$ and ${}^{\omega}\omega$) with some interesting new features (infinite sets $A$ not recursively isomorphic to $A - \{x\}$ for $x \in A$ exist here but not classically, unless the axiom of choice fails). The most surprising analogy, which is actually more than that since it subsumes the classical theory and produces new results, is the one between functions on $\omega$ recursive in a parameter and functions on ${}^{\omega}\omega$ (or, homeomorphically, the irrationals) continuous with respect to the topology which is the product of the discrete topology on $\omega$. In this analogy the r.e. sets correspond to the open sets and there are effective (and nontrivial!) analogues of Borel and projective hierarchies. The most delicate point in the analogy is probably the introduction of the analogue of $\aleph_1$, which turns out to be the first (countable) ordinal not representing the order type of a recursive well-ordering of $\omega$.

All theorems of the classical descriptive set theory (like Souslin's theorem saying that the Borel sets are exactly the analytic-coanalytic sets) with the sole exception of the decomposition of every coanalytic set into $\aleph_1$ disjoint Borel sets have effective versions of which they are consequences (but not conversely!) (see **[Mt2]** or **[Hi]**). Also, the effective theory has produced new classical results of which no classical proof is known, like the following extension of the old fact that every uncountable analytic set has cardinality $2^{\aleph_0}$: any coanalytic equivalence relation (on ${}^{\omega}\omega$) has either countably many or $2^{\aleph_0}$ equivalence classes (Silver, see **[KM]**).

Recursion theory has provided new tools for the study of this subject. Sophisticated methods of proof like priority arguments, which were invented to solve Post's problem of classifying r.e. degrees, have been applied here, for example to prove the determinacy of Borel games (Martin, [**Mt1**]) and to obtain closed relations with no uniformizing functions of given countable Borel level (Harrington). Fine methods of classification of sets of reals have been introduced by analogy with $m$-reducibility (Wadge degrees, obtained from: $A \leqslant_w B$ iff for some continuous function $f$, $x \in A \Leftrightarrow f(x) \in B$) or $T$-reducibility (Kleene degrees). Although the Wadge degrees were (at least in some sense) studied classically (when considering properties of sets preserved under inverse images of continuous functions), the concept of Kleene degree really requires the machinery of recursion theory even for its definition.

The general problem of effective analogues of mathematical structures has been in the mind of recursion theorists since the beginning of the subject. The original paper in which Turing introduces his concept of mechanical computability (see [**Da1**]) is really devoted to giving a precise notion of computable real number. It is interesting to note that the usual equivalent definitions of real numbers (infinite decimals, Dedekind cuts, Cauchy sequences, nested interval sequences, complete ordered field) remain equivalent when effectivized. They define a countable subfield of $\mathbf{R}$ with computable field operations (but not computable equality) and containing all the real numbers of practical interest, like the algebraic numbers, $\pi$, $e$, the real zeros of the Bessel functions, etc. Since classical methods of isolating and computing real roots are effective, the recursive real numbers are real closed and the recursive complex numbers are algebraically closed. Interesting new phenomena occur in recursive analysis, for example that every recursively continuous function on all reals is uniformly continuous, making the subject nontrivial (and very much related to intuitionistic analysis) (see [**Go**]).

The reason for introducing recursive analysis was that among the basic structures of mathematics, $\mathbf{R}$ does not have a natural effective character, while $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{Q}$ do. Consider for example $\mathbf{Q}$. When we abstract some of its properties to obtain the concept of field, much gets lost. The natural topology induced by the order is not considered, and to restore part of the original richness topological fields are introduced. Another interesting feature not considered is the fact that $\mathbf{Q}$ has recursive domain and recursive operations. The notion of recursive field is hence defined to recapture this effective character. Thus a theory of recursive algebra can be developed, in which some classical facts remain true, sometimes requiring new effective proofs (e.g., every recursive field has a recursive algebraic closure) but others fail (e.g., a recursive field has a unique–up to recursive isomorphism–recursive algebraic closure iff it has a splitting algorithm, i.e., an algorithm to determine whether a polynomial on it is irreducible or not). This type of result which depends upon the priority method from the theory of r.e. degrees shows that some constructions cannot be done effectively starting with effective structures, such as finding a transcendence basis given the field operations (see [**MN**]).

As in the case of descriptive set theory, algebra receives from recursion theory new concepts and tools. For a given (recursively presented) algebraic structure, the lattice of the recursively enumerable substructures of it can be considered, and a theory analogous to (but not consequence of) the one for recursively enumerable sets can be developed. The concept of degree can be used to give a measure of the complexity of a structure, although it is usually not enough to define the degree of a structure as its degree as a set (e.g., an independent subset of a recursive vector space may not always be effectively extended to a basis, so it seems more natural to consider the degree of the dependence relation itself). Another interesting part of the theory studies which structures may be thought of as effective (e.g., a finitely presented group is a recursive group iff it has a solvable word problem) and, among the effective structures, which are invariantly effective in the sense that any structure of the same kind isomorphic to it is already recursively isomorphic and thus the recursive structure is unique (e.g., a recursive Boolean algebra is invariantly recursive iff it has a finite number of atoms).

We have talked until now about classifications of undecidable problems, but the decidable ones can also be classified by combining the strong reducibilities with notions and methods of complexity theory. Reducibilities that nontrivially split the class of the recursive sets are very strong, and will not be dealt with in this paper (but we will treat them in [**O**]). The idea needed to define a hierarchy among the decidable problems is to order them for example by the time needed to perform an algorithm for the problem (as a function of the input). Upper and lower bounds are known for many decidable problems (see [**FeR**]), and the degrees of recursive sets obtained from various notions of complexity have a structure similar in many respects to the one of r.e. $T$-degrees (for example they are a dense upper semilattice which is not a lattice, see [**Lad**]). We only quote here (a formulation of) a challenging open problem in the area, known as $P = NP$. Given a formula in the propositional calculus, there is an algorithm to decide whether it is satisfiable which runs in exponential time (the usual method of truth-tables). Is there any procedure running in polynomial time?

This last example brings in the relationships between recursion theory and logic. Again decidability and undecidability arise naturally when we look at theories formulated in the first order predicate calculus. For example, the theories of groups, fields, lattices, the algebra of closed sets of the plane, the sentences in the language of plus and times for $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{Q}$ are undecidable. On the other hand the theories of abelian groups, finite fields, Boolean algebras, the algebra of closed sets of Cantor's discontinuum, the sentences in the language of plus and times for $\mathbf{R}$ and $\mathbf{C}$ are decidable (see [**ELTT**]). Indeed, any $tt$-degree is the degree of a theory and any r.e. $tt$-degree is the degree of a finitely axiomatizable theory. This correspondence does not hold however for (r.e.) $m$-degrees (see [**Be**]).

Recursion theory provides once again useful concepts to analyze theories. A surprising connection is for example: a consistent axiomatizable theory is not independently axiomatizable iff there is an enumeration $\{x_0, x_1, \ldots\}$ of it such that the set $\{n\colon x_{n+1}$ is deducible in the predicate calculus from

$x_0 \ldots x_n\}$ is hypersimple (a concept defined later). Another is that a theory is effectively extensible (i.e., there is an effective procedure to get, given any r.e. set of axioms consistent with the theory, a sentence undecidable in the theory obtained by extending the original one by the new axioms) iff the sets of theorems and of refutable formulas are effectively inseparable. Also, any two effectively inseparable theories are mutually one-one reducible in a way preserving propositional connectives and relative derivability **[PE]**, **[MPE]**.

It may be worthwhile to sketch the methods usually used to prove the undecidability of a theory $T$. We say that a set $A$ is representable in $T$ if there is a formula $\varphi$ in the language of $T$ such that, for every $x$, $x \in A$ iff $\varphi(x)$ is provable in $T$. The first obvious way to have undecidability for $T$ is to prove that some nonrecursive set $A$ is representable in $T$: any recursive decision procedure for $T$ would give a procedure for $A$. Note that if $T$ is axiomatized in the predicate calculus, then every set representable in it must be recursively enumerable, because to generate it we simply generate the theorems of the form $\varphi(x)$. So in practice the method has always been applied using as $A$ the most obvious recursively enumerable nonrecursive set: the one obtained by diagonalization of the recursively enumerable set (and called $K$ later, as usual).

There is also a less obvious method to prove the undecidability of $T$: instead of representing in it something nonrecursive, we may represent in $T$ all the recursive sets. There is no reason, prima facie, to believe that the possibility of representing in $T$ all the effective objects forces $T$ to be undecidable, but it is so. In the case of (say) Peano Arithmetic both possibilities arise, and generally if we represent $K$ in $T$ then we represent all the recursively enumerable sets in it (since $K$ codes all of them) and therefore in particular all the recursive ones. A nice application of the infinity injury method **[Sh]** gives a theory for which this second method is applicable but not the first one. It is worth noting that this was actually the first application of this powerful method, which has since proven so useful in the study of r.e. sets and degrees.

Indeed the motivation for Post's original study of the r.e. degrees was to determine whether the first method to prove undecidability (applied with $A = K$, the only known example of recursively enumerable nonrecursive set at that time) was actually "the only method", in the sense of being at least in theory applicable to prove the undecidability of every undecidable axiomatizable theory. In this direction Post showed that there are nonrecursive r.e. sets to which $K$ is not $m$-reducible, while the method applies only when $K \leqslant_m T$. He also asked whether in general a $T$-reduction of $K$ was enough, i.e. if for every nonrecursive r.e. set $A$, $A \equiv_T K$. This is the famous Post's problem, and we are going to begin our study from it in §2. In §3 we study the mutual relationships of various reducibilities, whose structure theories are considered in the remaining paragraphs.

We consider this paper as a companion to **[So1]**, in which Robert Soare has given a very elegant and readable survey of the present state of the theory of r.e. sets and r.e. $T$-degrees. We review the same topic from the point of view of stronger reducibilities, although we will not always restrict our attention to r.e. degrees. We will follow Soare's approach, giving only sketches of proof and avoiding as much as possible technical details. The complete treatment will appear in our forthcoming book **[O]**. Our source of notations is Rogers' book **[Ro]**. Letters $A, B, \ldots$ are used for sets of natural numbers and $a, e, i, \ldots$ for natural numbers. In particular, $W_e$ and $\varphi_e$ are respectively the r.e. set and the partial recursive function with index $e$. $W_{e,s}$ is the finite approximation of $W_e$ at stage $s$. Also $\varphi_e^B$ is the partial function computed from oracle $B$ by the $e$th procedure. We will occasionally identify a set and its characteristic function. We suppose that the reader has a working knowledge of the basic concepts of recursion theory. When the method of proof is priority, we will simply give the strategy for a single requirement. The reducibilities we are concerned with in §§2–7 are

(a) *one-one reducibility*: $A \leqslant_1 B$ if for some recursive one-one function $f$ and for all $x$, $x \in A \Leftrightarrow f(x) \in B$.
(b) *many-one reducibility*: $A \leqslant_m B$ if for some recursive function $f$ and for all $x$, $x \in A \Leftrightarrow f(x) \in B$.
(c) *truth-table reducibility*: let $\{\sigma_n\}_{n\in\omega}$ be an effective enumeration of all the propositional formulas built from the atomic ones "$n \in X$" for $n \in \omega$ (*tt*-conditions). Then $A \leqslant_{tt} B$ if for some recursive function $f$ and for all $x$, $x \in A \Leftrightarrow B$ satisfies $\sigma_{f(x)}$ (abbreviated, $B \vDash \sigma_{f(x)}$). A useful characterization of *tt*-reducibility (Nerode, see **[Ro**, p. 143]) is $A \leqslant_{tt} B$ iff for some $e$ such that for all $X$ $\varphi_e^X$ is total, $A = \varphi_e^B$.
(d) *Turing reducibility*: $A \leqslant_T B$ if for some $e$, $A = \varphi_e^B$.

Two more reducibilities will be considered in §8.

All these reducibilities are reflexive and transitive, so if $\leqslant_r$ is any of them then $A \leqslant_r B \wedge B \leqslant_r A$ is an equivalence relation, whose equivalence classes are called $r$-degrees. A degree is said to be r.e. if it contains an r.e. set (only for 1- and $m$-degrees does an r.e. degree contain only r.e. sets; for *tt*- and $T$-degrees this is true only for $\mathbf{0}$, to be defined later). Since $A \leqslant_1 B \Rightarrow A \leqslant_m B \Rightarrow A \leqslant_{tt} B \Rightarrow A \leqslant_T B$, every degree of a weaker reducibility is the union of degrees of stronger reducibilities. There are three things to study here.

(1) The structure of degrees of a stronger reducibility inside the degrees of a weaker reducibility.

(2) The structure of degrees (under the partial ordering induced on the equivalence classes by the given reducibility; if no confusion arises, we will simply name this order $\leqslant$).

(3) The structure of degrees of r.e. sets.

For every reducibility, we will call the greatest r.e. degree $\mathbf{0}'$. It always exists since if $K = \{e: e \in W_e\}$ then, for every r.e. set $A$, $A \leqslant_1 K$. We would like to call the least degree $\mathbf{0}$, but this must be done with some care. No problem arises for *tt*- and $T$-reducibilities, since if $A$ is recursive then $A \leqslant_{tt} B$ for every $B$: given $x$, every *tt*-condition which is 1 if $x \in A$ and 0 otherwise will reduce $A$ to $B$. In particular, the recursive sets are exactly the elements of $\mathbf{0}$ for *tt*- and $T$-reducibilities. For $m$-reducibility the situation is almost as good: there are only three $m$-degrees containing recursive sets, but two of them are the singletons $\{\varnothing\}$ and $\{\omega\}$ and the last contains every other recursive set. We call this one $\mathbf{0}$ and forget the other two, since their relation is

$$\begin{tikzcd}[column sep=small] \{\varnothing\} \arrow[dr, dash] & \\ & \mathbf{0} \\ \{\omega\} \arrow[ur, dash] & \end{tikzcd}$$

(to be connected from left to right means to be less than, to be one above the other means to be incomparable). Moreover, if $A \in \mathbf{0}$ and $B \neq \varnothing, \omega$ then $A \leqslant_m B$ via $f$ such that, for fixed $a \in B$ and $b \in \bar{B}$

$$f(x) = \begin{cases} a & \text{if } x \in A, \\ b & \text{if } x \in \bar{A}. \end{cases}$$

For 1-reducibility there are some additional problems, since if $A \leqslant_1 B$ then $x \in A \Leftrightarrow f(x) \in B$ for some one-one $f$; so it follows $|A| \leqslant |B|$ and $|\bar{A}| \leqslant |\bar{B}|$ (with $|A|$ we denote the cardinality of $A$). Hence the situation for recursive sets is the following

$$\begin{tikzcd}[column sep=small] \mathbf{a}_0 \arrow[r, dash] & \mathbf{a}_1 \arrow[r, dash] & \mathbf{a}_2 \arrow[r, dash] & \cdots \arrow[dr, dash] & \\ & & & & \mathbf{0} \\ \mathbf{b}_0 \arrow[r, dash] & \mathbf{b}_1 \arrow[r, dash] & \mathbf{b}_2 \arrow[r, dash] & \cdots \arrow[ur, dash] & \end{tikzcd}$$

where $\mathbf{a}_n$ contains the sets with $n$ elements, $\mathbf{b}_n$ the sets whose complement is in $\mathbf{a}_n$ and $\mathbf{0}$ the infinite, coinfinite recursive sets. Moreover, it's not true that if $A \in \mathbf{0}$ and $B$ is infinite and coinfinite, then $A \leqslant_1 B$ (neither $B$ nor $\bar{B}$ can be immune, see later). So it is better for our purposes to consider only the 1-degrees above $\mathbf{0}$ and suppress the others. Having the notion of $\mathbf{0}$ we can say that a nonzero degree is *minimal* if there aren't degrees between it and $\mathbf{0}$. If $\mathbf{a}$ is minimal, $\{\mathbf{0}, \mathbf{a}\}$ is the simplest example of an *initial segment*, i.e. of a set of degrees closed downward with respect to the order.

We will call $\mathbf{a} \cup \mathbf{b}$ and $\mathbf{a} \cap \mathbf{b}$, respectively, the least upper bound and the greatest lower bound of the degrees $\mathbf{a}$ and $\mathbf{b}$, when they exist. By $A \oplus B$ we mean the set $\{2x\colon x \in A\} \cup \{2x + 1\colon x \in B\}$, which is like the disjoint union of $A$ and $B$ (note that it is r.e. when $A$ and $B$ are r.e.). An r.e. set is said to be *r-complete* if it is in $\mathbf{0}'$ (relative to $\leqslant_r$).

**2. Where it all started: Post's problem and $T$-completeness.** The strong reducibilities we are concerned with were introduced by Post in his classic paper **[Po]**. We briefly sketch now the path he followed, and see some more recent progress along the same line.

The great problem underlying all of Post's work was: find a structural property of r.e. sets which implies nonrecursiveness and non-$T$-completeness. In terms of $T$-degrees: a property such that every r.e. set with this property has degree strictly between $\mathbf{0}$ and $\mathbf{0}'$. We know that the class of r.e. nonrecursive and non-$T$-complete sets is nonempty because in 1956 Friedberg and Muchnik invented the priority method to exhibit such a set, but a positive solution of Post's problem in the style of Post's work was found only in 1976. Soare (see **[So1**, Theorem 3.2**]**) proved that no property invariant under automorphisms of the lattice of r.e. sets under inclusion can insure incompleteness. He also found (see [**Sol**, Theorem 5.1]) a different answer to Post's problem.

Since we will be dealing with the concept of $T$-completeness, it may be useful to begin with a criterion which is a very nice generalization of both the recursion theorem in fixed-point form [**Ro**, Theorem 11.1] and the Martin-Lachlan criterion [**Sol**, Theorem 5.2]. The proof is from Soare [**So2**].

THEOREM 2.1 (ARSLANOV). *An r.e. set $A$ is complete iff there is a function $f \leqslant_T A$ with no fixed-point, i.e. for all $x$ $W_{f(x)} \neq W_x$.*

PROOF. Let $A$ be $T$-complete; $\{x\colon 0 \in W_x\}$ is r.e., hence recursive in $A$. So there is $f \leqslant_T A$ such that

$$W_{f(x)} = \begin{cases} \{0\} & \text{if } 0 \notin W_x, \\ \{\bar{0}\} & \text{if } 0 \in W_x, \end{cases}$$

and for every $x$ $W_{f(x)} \neq W_x$.

Let $f \leqslant_T A$ be such that for every $x$ $W_{f(x)} \neq W_x$, and let $e$ be an index of $f$ relative to $A$, i.e. $f = \varphi_e^A$. Choose enumerations $\{A_s\}_{s\in\omega}$ and $\{K_s\}_{s\in\omega}$ of $A$ and $K$, and let $\psi(x) =$ least $s$ such that $x \in K_s$ if $s$ exists, and $\psi(x)$ undefined otherwise. To have $K \leqslant_T A$ (and hence $A$ $T$-complete) it is enough to majorize $\psi$ recursively in $A$, since then for $s \geqslant \psi(x)$, $x \in K \Leftrightarrow x \in K_s$. Define the recursive function $\hat{f}(s, x) = \varphi_{e,t}^{A_t}(x)$ where $t \geqslant s$ is minimal such that $\varphi_{e,t}^{A_t}(x)$ is defined. By the recursion theorem with parameters define the recursive function $g$ by

$$W_{g(x)} = \begin{cases} W_{\hat{f}(\psi(x),\, g(x))} & \text{if } x \in K, \\ \varnothing & \text{otherwise.} \end{cases}$$

The idea is that we use finite approximations to force a change in $A$, because we know that $W_{f(x)} \neq W_x$, so the part $A_{\psi(x)}$ used in the computation is wrong. Find, recursively in $A$, the least $s$ such that $A$ and $A_s$ agree up to $fg(x)$: then $s \geqslant \psi(x)$. □

After the introduction of the set $K$, Post noted the following simple fact: $\bar{K}$ contains an infinite r.e. set. In fact, by definition, if $W_x \subseteq \bar{K}$ then $x \in \bar{K} - W_x$ (if $x \in W_x$ then $x \in \bar{K}$ because $W_x \subseteq \bar{K}$, but also by definition of $K$ $x \in K$, a contradiction; hence $x \notin W_x$ and $x \in \bar{K}$). To generate an infinite r.e. subset of $\bar{K}$, start with $a_0$ s.t. $W_{a_0} = \varnothing$: then $W_{a_0} \subseteq \bar{K}$ and $a_0 \in \bar{K}$; then take $a_1$ s.t. $W_{a_1} = \{a_0\}$: $W_{a_1} \subseteq \bar{K}$ and $a_1 \in \bar{K} - \{a_0\}$, etc. So it was natural to define the notion of simplicity: an r.e. set $A$ is *simple* if $\bar{A}$ is immune, i.e. infinite and with no infinite r.e. subsets. We call $S$ a Post simple set if it is defined in the following way: $x \in S$ iff for some $e$, $x$ is the first element generated by $W_e$ such that $x \geqslant 2e$. This makes $\bar{S}$ infinite and, for all $e$, if $W_e$ is infinite then $W_e \cap S \neq \varnothing$, so $W_e \not\subseteq \bar{S}$. Post proved that *a simple set can't be m-complete*. Let us prove something a little more general. First a definition: $A$ and $B$ are *recursively inseparable* if they are disjoint and there doesn't exist a recursive set $R$ such that $A \subseteq R$ and $B \subseteq \bar{R}$. If for some set $B$, $A$ and $B$ are recursively inseparable, we say that $A$ is part of an r.i. pair. In most cases such as Theorems 2.2 and 2.4 $A$ and $B$ will be r.e.

THEOREM 2.2. (a) *$B \leqslant_m A$ and $B$ part of an r.i. pair of r.e. sets $\Rightarrow A$ nonsimple.*

(b) *$K$ is part of an r.i. pair of r.e. sets.*

PROOF. (a) Let $x \in B \Leftrightarrow f(x) \in A$ with $f$ recursive. There is $C$ r.e. such that $B$ and $C$ form an r.i. pair. Let $D = f(C)$: if $D$ were finite, then $R = f^{-1}(D)$ would be a recursive set separating $B$ and $C$. So $D$ is infinite and contained in $\bar{A}$, and $A$ is not simple.

(b) If $e \in A \Leftrightarrow \varphi_e(e) = 0$ and $e \in B \Leftrightarrow \varphi_e(e) = 1$, then $A$ and $B$ are an r.i. pair (otherwise if $A \subseteq R$ and $B \subseteq \bar{R}$ with $R$ recursive, define

$$f(x) = \begin{cases} 1 & \text{if } x \in R, \\ 0 & \text{if } x \in \bar{R}. \end{cases}$$

If $f = \varphi_e$ then for $x = e$ we have a contradiction). Since $K$ is complete let $A \leqslant_1 K$ via $g$; then $K$ and $g(B)$ are recursively inseparable. □

The next step was to ask: can a simple set be $T$-complete? Since a simple set can't be recursive (otherwise its complement is r.e. and infinite), a negative answer would provide a solution of Post's problem. But the set $S$ is already $T$-complete. In fact $S$ is not only simple, but *effectively simple*, in the sense that there is a recursive function $g$ such that $W_e \subseteq \bar{S} \Rightarrow |W_e| \leqslant g(e)$ (namely $g(e) = 2e$), and every effectively simple set is $T$-complete (let $f \leqslant_T S$ such that $W_{f(x)} =$ the first $g(e) + 1$ elements of $\bar{S}$: then for all $x$ $W_{f(x)} \neq W_x$, so $S$ is $T$-complete by Theorem 2.1). By changing the set $S$ a little, Post constructed a simple set which is in fact $tt$-complete. The next result proves that the set $S$ can already be $tt$-complete itself, if we choose the enumeration $\{W_x\}_{x \in \omega}$ of the r.e. sets in a suitable way. A much more difficult argument proves that $S$ can be $tt$-incomplete **[La7]**. So we have a property ($S$ is $tt$-complete) which depends on the chosen enumeration of the r.e. sets. Jockusch and Soare **[JSo]** were the first to note that such a situation is possible for another set constructed by Post.

THEOREM 2.3 (LACHLAN **[La7]**). *Post's simple set can be tt-complete.*

PROOF. Given $\{W_x\}_{x \in \omega}$ we define $\{W'_x\}_{x \in \omega}$ so that $x \in K \Leftrightarrow B_x \subseteq S'$, where $B_x$ is a finite set (obtained effectively from $x$) and $S'$ is the Post's set constructed from $\{W'_x\}$. Note that if there exists $e$ such that $2e < z \wedge W'_e = \{z\}$, then $z$ is put into $S'$. Also we want, among the $W'_x$, all the $W_x$. So we can define

> $W'_0 = \varnothing$, $W'_{2^x} = W_x$, if $x \in K$ then $W'_{2^x+1} = \{2^{x+1} + 3\}$, so that $(2^{x+1} + 3) \in S'$, since $2(2^x + 1) < 2^{x+1} + 3$. In general, for $0 < n < 2^x$, $W'_{2^x+n} = \{2^{x+1} + 2n + 1\}$ if $x \in \bar{K}$ then, for $0 < n < 2^x$, $W'_{2^x+n} = \varnothing$.

Now let $B_x = \{2^{x+1} + 2n + 1 \colon 0 < n < 2^x\}$. If $x \in \bar{K}$ then no element of $B_x$ is put into $S'$ by $W'_{2^x+n}$ for $0 < n < 2^x$, but perhaps by some other $W'_y$ for $y \leqslant x$. Moreover at most $x$ such elements are put into $S'$ so $\overline{S'} \cap B_x \neq \varnothing$. □

Post then tried to extend the notion of simple set and the fact that simple sets are not $m$-complete, introducing hypersimple sets and proving that they are not $tt$-complete. An r.e. set $A$ is *hypersimple* if its complement is hyperimmune, i.e. infinite and there doesn't exist a recursive sequence of (codes of) disjoint finite sets, each one intersecting $\bar{A}$. If we only consider singletons instead of finite sets, we have the notion of simple set. Note that the most straightforward generalization of the notion of simple sets, i.e. there doesn't exist a recursive sequence of disjoint nonempty finite sets each one contained in $\bar{A}$, is equivalent to simplicity (from such a sequence we can obtain an r.e. subset of $\bar{A}$ by picking an element in each set). The point here is that if $A$ is simple then there can exist such a sequence, because for every one of the finite sets we know that one element is in $\bar{A}$, but we could not know which, so that it could be impossible to obtain an r.e. subset of $\bar{A}$. In fact e.g. $S$ is simple but not hypersimple, since the sequence $B_n = \{2^n, 2^n + 1, \ldots, 2^{n+1} - 1\}$ is a counterexample. The fact that *the hypersimple sets are not tt-complete* can be extended in the style of Theorem 2.2a.

THEOREM 2.4 (DENISOV **[Den2]**). *$B \leqslant_{tt} A$ and $B$ part of an r.i. pair of r.e. sets $\Rightarrow A$ nonhypersimple.*

PROOF. Let $B$ and $C$ be r.i. and r.e. Given $n$, we want to effectively find an $m > n$ such that $\{n + 1, n + 2, \ldots, m\} \cap \bar{A} \neq \emptyset$. There is a recursive function $f$ such that $x \in B \Leftrightarrow A \vDash \sigma_{f(x)}$. Since $B \cap C = \emptyset$,

$$x \in B \wedge y \in C \Rightarrow A \vDash \sigma_{f(x)} \wedge A \nvDash \sigma_{f(y)}.$$

Let $z \in A^* \Leftrightarrow (z \leqslant n \wedge z \in A) \vee z > n$ (so $A$ and $A^*$ are the same up to $n$). If

$$x \in B \wedge y \in C \wedge (A^* \vDash \sigma_{f(x)} \Leftrightarrow A^* \vDash \sigma_{f(y)})$$

then $A$ and $A^*$ must be different, so for some $z > n$ used in the two computations, $z \notin A$ (since $z \in A^*$). The problem is that we don't know $A^*$ effectively since $A$ is not recursive, in general. But there are only $2^{n+1}$ possibilities for membership in $\{0, \ldots, n\}$; let $A_i^*$ ($i < 2^{n+1}$) correspond to these possibilities, and note that there exist $x$ and $y$ such that

$$x \in B \wedge y \in C \wedge (\forall i < 2^{n+1})\left[A_i^* \vDash \sigma_{f(x)} \Leftrightarrow A_i^* \vDash \sigma_{f(y)}\right]$$

since otherwise we could recursively separate $B$ and $C$ because

$$x\sigma y \Leftrightarrow (\forall i < 2^{n+1})\left[A_i^* \vDash \sigma_{f(x)} \Leftrightarrow A_i^* \vDash \sigma_{f(y)}\right]$$

is a recursive equivalence relation with only finitely many equivalence classes. (If $x, y$ did not exist we could separate $B$ and $C$ by taking the union of the equivalence classes containing elements of, say, $B$.) So given $n$ we can find $x$, $y$ as above and define $m$ to be the maximum element used in $\sigma_{f(x)}, \sigma_{f(y)}$. □

We don't yet know if hypersimple sets exist and if they can be $T$-complete. Post constructed a hypersimple set, and this is how far he went: he did not know the answer to the second question. Dekker found a method to answer both questions together, but this was only in 1954 **[Dek]**. As Jockusch and Soare **[JSo]** proved, Post's hypersimple set can be $T$-complete or not, depending on the enumeration of the r.e. sets.

THEOREM 2.5 (DEKKER). *Every nonrecursive r.e. T-degree contains a hypersimple set.*

PROOF. Let $A$ be r.e. nonrecursive, and $f$ be a recursive one-one function with range $A$. Then the set $B$ of deficiency stages in the enumeration of $A$ given by $f$, i.e.

$$x \in B \Leftrightarrow (\exists y)(y > x \wedge f(y) < f(x))$$

is hypersimple and in the same $T$-degree of $A$. In fact

(a) $x \in B$ iff $(\exists y < f(x))(y \in A - \{f(0), \ldots, f(x)\})$, so $B \leqslant_{tt} A$.
(b) $\bar{B}$ is certainly infinite, so given $x$ find $y \in \bar{B}$ such that $f(y) > x$ (recursively in $B$). Then $x \in A$ iff $x \in \{f(0), \ldots, f(y-1)\}$, so $A \leqslant_T B$.
(c) If $\bar{B}$ is not hyperimmune, there is a recursive sequence $\{F_n\}_{n\in\omega}$ of disjoint finite sets, each one intersecting $\bar{B}$. Let $g(n) = 1 + \max \bigcup_{i<n} F_i$: $g$ is recursive and $g(n)$ is greater than the $n$th element of $\bar{B}$. So $x \in A$ iff $x \in \{f(0), \ldots, f(g(x))\}$ and $A$ is recursive, contradiction. □

Even without knowing whether or not there was a $T$-complete hypersimple set, Post introduced the following extension of the concept: an r.e. set $A$ is *hyperhypersimple* if $\bar{A}$ is hyperhyperimmune, i.e. infinite and there does not exist a recursive sequence of (finite) disjoint r.e. sets (given via r.e. indices), each one intersecting $\bar{A}$. The word finite is in parentheses because it turns out **[Y1]** that the same class of r.e. sets is obtained, with or without the restriction. Lachlan proved that the hyperhypersimple sets are exactly those coinfinite r.e. sets whose r.e. supersets are (under inclusion, modulo finite sets) a Boolean algebra (see **[So1**, Theorem 7.1]). Let us turn for a moment to Turing reducibility. To say $A \leqslant_T B$ means that for some $e$, $A = \varphi_e^B$: to see for example if $x \in \bar{A}$, we generate the computation relative to $\varphi_e^B(x)$, and this is an r.e. procedure plus some finite questions using the $B$-oracle. If $\{D_n\}_{n\in\omega}$ is an effective enumeration of the finite sets, we can say that there is an r.e. relation $R$ such that

$$x \in \bar{A} \Leftrightarrow (\exists u)(\exists v)\left[R(x, u, v) \wedge D_v \subseteq B \wedge D_u \subseteq \bar{B}\right].$$

If $A$ is r.e. the existence of such an $R$ is actually equivalent to $A \leqslant_T B$: to see if $x$ is in $A$ or not, we simultaneously generate $A$ and $\{(u, v): R(x, u, v)\}$ and see if $x \in A$ or, for some $u$ and $v$, $R(x, u, v) \wedge D_v \subseteq B \wedge D_u \subseteq \bar{B}$. When $B$ too is r.e., we can absorb into $R$ the part $D_v \subseteq B$. So, for $A$ and $B$ r.e., $A \leqslant_T B$ iff for some r.e. $R$

$$x \in \bar{A} \Leftrightarrow (\exists u)\left[R(x, u) \wedge D_u \subseteq \bar{B}\right].$$

It's natural to see what happens when we only use single questions in the oracle, instead of finite questions; that is when, for some r.e. $R$,

$$x \in \bar{A} \Leftrightarrow (\exists u)\left[R(x, u) \wedge u \in \bar{B}\right].$$

By uniformity there is a recursive function $f$ such that $R(x, u) \Leftrightarrow u \in W_{f(x)}$, so $x \in \bar{A} \Leftrightarrow W_{f(x)} \cap \bar{B} \neq \varnothing$ or $x \in A \Leftrightarrow W_{f(x)} \subseteq B$. We say that $A$ is $Q$-reducible to $B$ ($A \leqslant_Q B$) when there is a recursive function $f$ such that, for every $x$, $x \in A \Leftrightarrow W_{f(x)} \subseteq B$. When $A$ and $B$ are r.e., $A \leqslant_Q B \Rightarrow A \leqslant_T B$ but there is no implication between $Q$ and $tt$-reducibility. It's useful to note that we can always assume $W_{f(x)}$ finite, since we could replace it with a new r.e. finite set $W_{g(x)} \subseteq W_{f(x)}$ this way: generate $W_{f(x)}$, $A$ and $B$ simultaneously, and put elements of $W_{f(x)}$ into $W_{g(x)}$ until either $x \in A$ or a $z$ has been generated in $W_{g(x)}$ but not yet in $B$ (if at a later stage we find $z \in B$, we go on). If $x \in A$ then $W_{g(x)}$ is finite and $W_{f(x)} \subseteq B$; if $x \in \bar{A}$ then $W_{f(x)} \cap \bar{B} \neq \varnothing$ and we stop generating $W_{g(x)}$ when we run into the first $z \in W_{f(x)} \cap \bar{B}$. The interest of $Q$-reducibility lies in the next result.

THEOREM 2.6 (SOLOVE'V [Sl]). *Hyperhypersimple sets are not $Q$-complete.*

PROOF. Let $K \leqslant_Q A$, i.e. $x \in K \Leftrightarrow W_{g(x)} \subseteq A$ for some recursive function $g$. To prove that $A$ is not hyperhypersimple, suppose we already have $W_{f(0)} \ldots W_{f(n)}$ (finite): we want $W_{f(n+1)}$ such that

$$W_{f(n+1)} \cap \bar{A} \neq \varnothing \quad \text{and} \quad W_{f(n+1)} \cap \left(\bigcup_{i=0}^{n} W_{f(i)}\right) = \varnothing.$$

We try to have $f(n+1) = g(a)$ for some $a \in \bar{K}$, so that $W_{g(a)} \cap \bar{A} \neq \varnothing$. The idea is to define an r.e. set $B \subseteq \bar{K}$ such that

$$x \in \bar{B} \Rightarrow W_{g(x)} \cap \left(\bigcup_{i=0}^{n} W_{f(i)}\right) = \varnothing$$

and then take $a$ such that $B = W_a$: $W_a \subseteq \bar{K} \Rightarrow a \in \bar{K} - B$ and we are done. To have $B \subseteq \bar{K}$ we ask

$$x \in B \Rightarrow W_{g(x)} \cap \bar{A} \neq \varnothing.$$

Putting together the two conditions we have

$$x \in B \Leftrightarrow W_{g(x)} \cap \bar{A} \cap \left(\bigcup_{i=0}^{n} W_{f(i)}\right) \neq \varnothing.$$

This doesn't give us exactly what we want; we won't obtain a sequence of r.e. disjoint sets–they will only be disjoint on $\bar{A}$. But it's easy to see that this doesn't cause difficulties (extract a new sequence of really disjoint sets, by simultaneously generating the $W_{f(x)}$'s and avoiding repetitions; since what matters is that on $\bar{A}$ there is no overlapping, everything works). Since $\bar{A}$ is not r.e., the true definition of $B$ will be

$$x \in B \Leftrightarrow (\exists s \geqslant x)\left[ W_{g(x),s} \cap \bar{A}_s \cap \bigcup_{i=0}^{n} W_{f(i),s} \neq \varnothing \right].$$

The reason to have $s \geqslant x$ is that, since $\bigcup_{i=0}^{n} W_{f(i)}$ is finite, for $x$ (and hence $s$) big enough it will be true that $x \in B \Rightarrow x \in \bar{K}$. So we don't have $B \subseteq \bar{K}$, but at least $B \subseteq^* \bar{K}$ (i.e. $B \cap K$ is finite). Let $a_0$ be such that $B = W_{a_0}$; if $a_0 \in \bar{B}$ then $a_0 \in \bar{K}$ (by definition of $K$), but if we find $a_0 \in B$ then $a_0 \in K$ and $W_{g(a_0)} \subseteq A$, so this doesn't work. In this case, let $W_{a_1} = B - \{a_0\}$ and see if $a_1 \in W_{a_1}$ etc. We define a sequence $a_0, a_1, \ldots$ but we get stuck after a finite number of steps; otherwise (from $B \subseteq^* \bar{K}$) we would have an $a_i$ such that $a_i \in W_{a_i}$ but $a_i \in \bar{K}$, contradiction. Then we let $W_{f(n+1)} = \bigcup_i W_{g(a_i)}$ (note that if $a_i \in K$ then $W_{g(a_i)} \subseteq A$, so there is no problem on $\bar{A}$). $\square$

As usual, we still have to see if hyperhypersimple sets exist and if they can be $T$-complete. To answer the first question, we introduce a concept easier to work with: an r.e. set $A$ is *maximal* if $\bar{A}$ is cohesive, i.e. infinite and for every r.e. set $B$, $B \cap \bar{A}$ is finite or $\bar{B} \cap \bar{A}$ is finite. Historically, the concept was defined by Myhill as the ultimate step toward the solution of Post's problem along the line proposed by Post himself: maximal sets have the thinnest possible complement, because an equivalent condition for $A$ being maximal is that for every r.e. superset $B$ of $A$, either $B$ is cofinite or $B - A$ is finite. Of course, every r.e. coinfinite set has at least that kind of r.e. superset, so the requirement is really minimal: if even maximal sets do not solve Post's problem, the situation gets desperate. Friedberg **[Fr]** proved the existence of maximal sets, and Yates **[Y1]** destroyed the hopes that an r.e. set with sufficiently thin complement is incomplete.

THEOREM 2.7 (FRIEDBERG, YATES). *There is a T-complete maximal set.*

PROOF. We want to have, for every $e$,

$$W_e \text{ infinite} \Rightarrow W_e \cap \bar{A} \text{ finite or } \bar{W}_e \cap \bar{A} \text{ finite.}$$

The idea of the construction is to have

$$W_e \cap \bar{A} \text{ infinite} \Rightarrow \text{from a certain point on, every element of } \bar{A} \text{ is in } W_e.$$

To obtain this, we try to construct $\bar{A}$ in such a way that its $n$th element is in $W_0 \cap \cdots \cap W_n$ or, if this is not possible, in $W_{i_0} \cap \cdots \cap W_{i_m}$ where $(i_0, \ldots, i_m)$ is the best possibility with respect to the priority ordering $W_0, W_1, W_2, \ldots$. Technically, at stage $s + 1$ we have $A_s, W_{0,s}, \ldots, W_{n,s}$ and $a_0, \ldots, a_{n-1}$ and we take as $a_n$ the least element in $\bar{A}_s \cap W_{i_0,s} \cap \cdots \cap W_{i_m,s}$ greater than $a_{n-1}$, where

> $i_0$ is the least $i \leqslant n$ such that $W_{i,s} \cap \bar{A}_s \neq \varnothing$ (above $a_{n-1}$), $i_1$ is the least $i \leqslant n$ such that $W_{i,s} \cap W_{i_0,s} \cap \bar{A}_s \neq \varnothing$ (above $a_{n-1}$), $i > i_0$, etc.

This gives a maximal set. To have a $T$-complete maximal set it is enough to modify the construction, so as to have $A$ effectively simple. This can be obtained by allowing the $n$th element of $\bar{A}$ in $W_e$ only if $n < e$, so that in the end $W_e \subseteq \bar{A} \Rightarrow |W_e| \leqslant e + 1$. □

Lachlan has noted that the maximal set originally constructed by Friedberg is already $T$-complete, so the last modification of the proof is really not necessary. He calls $A$ *effectively maximal* if there is a recursive function $g$ such that for every $e$ the sequence of 0's and 1's consisting of the values of the characteristic function of $W_e$ on the elements of $\bar{A}$ (in increasing order) has at most $g(e)$ alternations (note: $W_e \cap \bar{A}$ or $\bar{W}_e \cap \bar{A}$ is finite). Take $W_{f(e)}$ = the first elements of $\bar{A}$ up to the $g(e) + 1$ alternations of the characteristic function of $A$ (both $A$ and $\bar{A}$ are infinite). Then $f$ is recursive in $A$ and has no fixed point, so by Arslanov's criterion $A$ is $T$-complete. The set $A$ constructed in Theorem 2.7 is automatically effectively maximal (take $g(e) = e + 1$) and hence $T$-complete.

Since, obviously, a maximal set is hyperhypersimple, we have the same result for hyperhypersimple sets; in particular, they are not, in general, $T$-incomplete. But since they are non-$Q$-complete and $Q$-reducibility is so near to $T$-reducibility, we may hope to find some property that reduces $T$-completeness to $Q$-completeness, so that this together with hyperhypersimplicity would reach our goal. The only trouble (for r.e. sets) seems to be the reduction of an expression of the kind $D_u \subseteq \bar{A}$ to one of the kind $u \in \bar{A}$. Let's call a set $A$ *semirecursive* if there is a recursive function $f$ of two variables, such that for all $x$ and $y$

$$f(x, y) = x \quad \text{or} \quad f(x, y) = y,$$
$$x \in A \vee y \in A \Rightarrow f(x, y) \in A.$$

THEOREM 2.8 (MARCHENKOV [**Ma4**]). *If $A$ is r.e. and semirecursive and $B \leqslant_T A$ then $B \leqslant_Q A$; so if $A$ is $T$-complete then $A$ is also $Q$-complete.*

PROOF. Let $A$ be semirecursive with respect to $f$, and $D$ be a finite set with elements $x_0, \ldots, x_n$. Since $x \in A \vee y \in A \Leftrightarrow f(x, y) \in A$ it's enough to define $z_0 = x_0$ and $z_{i+1} = f(z_i, x_{i+1})$ to have $D \subseteq \bar{A}$ iff $z_n \in \bar{A}$. □

Now we really have the solution at hand: a hyperhypersimple semirecursive set is not $T$-complete! It only remains to construct such a set. The simplest condition that insures semirecursiveness is the following: $A$ is r.e. and there is an enumeration $\{a_0, a_1, \ldots\}$ of $\bar{A}$ (not necessarily recursive) and a partial recursive function $\varphi$ such that $\varphi(a_0) = a_0$ and $\varphi(a_{n+1}) = a_n$ (a set with the property that $\bar{A}$ has above is called *regressive*; if $\{a_0, a_1, \ldots\}$ is the enumeration in order of magnitude, the set is called *retraceable*). To define $f$ as desired, given $x$ and $y$ we simultaneously generate $A$, $\{\varphi^n(x)\}_{n\in\omega}$ and $\{\varphi^n(y)\}_{n\in\omega}$ ($\varphi^n$ is the $n$th iteration of $\varphi$) and see what happens first: if $x \in A$ then we set $f(x, y) = f(y, x) = x$; if $x \in \{\varphi^n(y)\}_{n\in\omega}$ then we set $f(x, y) = f(y, x) = y$, and we are finished.

THEOREM 2.9 (JOCKUSCH [**J1**]). *Every nonrecursive r.e. $T$-degree contains a hypersimple semirecursive set.*

PROOF. Take the set $B$ as in Theorem 2.5: it's enough to see that it is coretraceable. We want $\varphi$ partial recursive such that if $x \in \bar{B}$ then $\varphi(x)$ is the greatest element of $\bar{B}$ less than $x$. But since if $x \in \bar{B}$ then $y > x \Rightarrow f(x) < f(y)$, to see if something less than $x$ is in $\bar{B}$ it's enough to check the values of $f$ below $x$. So we can define $\varphi(x)$ as the greatest $z < x$ such that $z < y \leqslant x \Rightarrow f(z) < f(y)$ (if any). □

Jockusch [**J1**] has extended the result as follows: every nonrecursive (r.e.) $tt$-degree contains a semirecursive (r.e.) set. As remarked before, we would like to prove now the existence of hyperhypersimple semirecursive sets, but once again the solution to Post's problem escapes us.

THEOREM 2.10 (MARTIN). *No hyperhypersimple set is semirecursive.*

PROOF. Since if a set is semirecursive, so is its complement, it's enough to prove that $A$ immune semirecursive $\Rightarrow A$ nonhyperhyperimmune. First note that $A$ is the lower cut of a recursive linear ordering $\leqslant_0$, i.e. $x \leqslant_0 y \wedge y \in A \Rightarrow x \in A$. Let $f$ be as in the definition of semirecursiveness. We define $\leqslant_0$ inductively on the integers up to $n$. Let $0 \leqslant_0 0$ and let $x_0, \ldots, x_n$ be the integers in $\{0, \ldots, n\}$ such that $x_0 \leqslant_0 x_1 \leqslant_0 \cdots \leqslant_0 x_n$. To insert $n + 1$ somewhere, we do this

–if $f(n + 1, x_0) = n + 1$ then $x_0 \in A \Rightarrow n + 1 \in A$, so $n + 1 \leqslant_0 x$.
–if $f(n + 1, x_n) = x_n$ similarly we let $x_n \leqslant_0 n + 1$.
–otherwise, let $i$ be the greatest such that $f(n + 1, x_i) = x_i$ and $f(n + 1, x_{i+1}) = n + 1$ (note that $f(n + 1, x_0) = x_0$ and $f(n + 1, x_n) = n + 1$). Then $x_{i+1} \in a \Rightarrow n + 1 \in A \Rightarrow x_i \in A$, so $x_i \leqslant_0 n + 1 \leqslant_0 x_{i+1}$.

Now let $x \in B \Leftrightarrow (\forall y)(x \leqslant y \Rightarrow x \leqslant_0 y)$; $B$ is an infinite subset of $A$ (if $x \in B$, let $y \in A \wedge x \leqslant y$: then $x \leqslant_0 y$ and $x \in A$. For the infinity see **[J1]**), and it is retraceable by a total recursive function, because if $x \in B$ and $z < x$ then $z \in B \Leftrightarrow (\forall y)(z \leqslant y \leqslant x \Rightarrow z \leqslant_0 y)$, so that we can let

$$g(x) = \begin{cases} b_0 & \text{if } x \leqslant b_0, \\ \max z \text{ s.t. } z < x \wedge (\forall y)(z \leqslant y \leqslant x \Rightarrow z \leqslant_0 y) & \text{otherwise,} \end{cases}$$

where $b_0$ is the least element of $B$. Since $g(x) \leqslant x$, for every $x$ there is $n$ such that $g^{n+1}(x) = g^n(x)$. Then

$$F_n = \{x : n \text{ is the least } i \text{ such that } g^{i+1}(x) = g^i(x)\}$$

is a sequence of r.e. disjoint sets, each one intersecting $A$ (if $b_0, b_1, \ldots$ is an enumeration of $B$ in order of magnitude, $g^{n+1}(b_n) = g^n(b_n)$ so $b_n \in F_n$). □

At this point, Theorems 2.7 and 2.10 suggest one good possibility (with, as Russell would say, all the advantages of theft over honest toil) to give up. An even better suggestion comes from Theorems 2.6 and 2.8: to relax the requirement of hyperhypersimplicity, in a way that doesn't spoil non-$Q$-completeness but allows coexistence with semirecursiveness. Let us try this honest path, and replace in the prior definitions equality with an r.e. equivalence relation $\eta$, in the style of [E3]. A set $A$ is *$\eta$-closed* if $(\forall x)(\forall y)(x \in A \wedge x\eta y \Rightarrow y \in A)$, i.e. $A$ consists of equivalence classes, and it is *$\eta$-finite* if it consists of finitely many equivalence classes (in particular an $\eta$-finite set is $\eta$-closed). By restricting our attention to $\eta$-closed sets and replacing the notion of finiteness with that of $\eta$-finiteness, we can relativize to $\eta$ the notions of this paragraph, and obtain the concepts of *$\eta$-simplicity*, *$\eta$-hypersimplicity*, *$\eta$-hyperhypersimplicity*, *$\eta$-maximality*. For example, an r.e. set $A$ is $\eta$-maximal if it is $\eta$-closed, co-$\eta$-infinite and the only possible r.e. $\eta$-closed supersets $B$ of $A$ are those such that either $B - A$ is $\eta$-finite, or $\bar{B}$ is $\eta$-finite. The implications among the unrelativized concepts remain true in this relativized context (e.g. an $\eta$-maximal set is $\eta$-hyperhypersimple), but note that now $\eta$-simplicity (in fact even $\eta$-maximality) doesn't necessarily imply nonrecursiveness. Only notational changes are required to obtain, for example,

THEOREM 2.11 (MARCHENKOV **[Ma4]**). *For every $\eta$, $\eta$-hyperhypersimple sets are not $Q$-complete.*

Also, Theorem 2.8 doesn't mention $\eta$ at all. Made suspicious by the previous disappointments, we almost do not dare to ask if, for some $\eta$, $\eta$-hyperhypersimple semirecursive (nonrecursive) sets may exist. But, at last, here it is: a phoenix is reborn from the ashes of Theorem 2.10, and gives a solution to Post's and our problems!

THEOREM 2.12 (DEGTEV **[Deg5]**). *There is an r.e. equivalence relation $\eta$, for which there exists an $\eta$-maximal semirecursive nonrecursive set.*

PROOF. In the construction we will define an infinite number of boxes, which should be thought of as the classes of $\eta$. One big box will constitute $A$ (so $A$ is $\eta$-closed), and in the end we want infinitely many boxes (so $\bar{A}$ is $\eta$-infinite). To make $A$ nonrecursive, we make it simple. To make $A$ $\eta$-maximal, we want (as in Theorem 2.7) that, for every $e$,

$$W_e \cap \bar{A} \text{ infinite} \Rightarrow \text{from a certain point on, every box of } \bar{A} \text{ intersects } W_e.$$

Since every box (except the one of $A$) will be finite, when $W_e$ is $\eta$-closed then $W_e \cap \bar{A}$ infinite will mean $W_e \cap \bar{A}$ $\eta$-infinite, and every box intersecting $W_e$ will actually be contained in $W_e$. Also, we'll make $A$ semirecursive.

We sketch the strategies separately. At the beginning the box for $A$ is empty, and the $n$th box for $\bar{A}$ is the singleton $\{n\}$. The idea to make $A$ simple is the usual one: if we find an element of $W_e$ in a box whose position is $\geqslant 2e$, we put all elements of the box into $A$ (so $W_e \cap A \neq \emptyset$ and $A$ will be $\eta$-closed). To make $A$ $\eta$-maximal, we try to have the $n$th box of $\bar{A}$ intersecting $W_0 \cap \ldots \cap W_n$; if for example at a certain stage we find $m$ and $n$ such that $m < n$ and $W_i$ $(i \leqslant m)$ intersects the $n$th box but not the $m$th, we put all the boxes from the $m$th through the $n$th together, and reindex the boxes. To make $A$ semirecursive, if $x \in A_{s+1} - A_s$ and $x \leqslant y \leqslant s$, then all elements of the box in which $y$ is go into $A_{s+1}$. This allows us to define directly the function $f$ that makes $A$ semirecursive.

$$f(x,y) = \begin{cases} \max\{x,y\} & \text{if } x,y \notin A_s, \\ x & \text{if } x \in A_s, \\ y & \text{if } x \notin A_s \wedge y \in A_s, \end{cases}$$

where $s = \max\{x,y\}$: if in the first case $x \leqslant y$ and $x$ goes into $A$ at a later stage, also $y$ goes into $A$ since that stage is bigger than $\max\{x,y\}$. $\square$

This proof is the first example of a priority argument in the form of the $e$-state method that produces directly an r.e. incomplete set. Incomplete maximal sets were obtained before, but by combining the $e$-state method with the usual kind of priority. The $e$-state method gave an incomplete set in Sacks' construction of a minimal $T$-degree below $\mathbf{0}'$, but its degree was not r.e.

Of course, the list of problems never has an end. The next problem is also the beginning of our list.

*Problem* 1. Characterize the $T$-degrees containing $\eta$-maximal ($\eta$-hyperhypersimple) semirecursive sets.

Some interesting results have been obtained by Miller **[Mi]**: $\eta$-maximal semirecursive sets are all low$_2$ (i.e. their double jump is $\mathbf{0}''$) and this of course greatly improves the mere incompleteness (some weaker results along this line were also obtained by Kobzev [**Ko4**]). In contrast, there are high $\eta$-hyper-hypersimple semirecursive sets (i.e. with jump $\mathbf{0}''$) and hence in some sense Theorem 2.11 is optimal. The classification of the $T$-degrees containing $\eta$-maximal semirecursive sets seems to be difficult since Miller has also proved that there are such degrees which are not low and that not every low degree contains $\eta$-maximal semirecursive sets. The existence of low degrees containing $\eta$-maximal semirecursive sets is easily proved by applying the permitting method to push the set constructed in Theorem 2.12 below any given nonzero degree (Marchenkov).

**3. The structure inside single degrees.** Suppose that $r_1$ and $r_2$ are two reducibilities such that $A \leqslant_{r_1} B \Rightarrow A \leqslant_{r_2} B$: then every $r_2$-degree consists of many $r_1$-degrees. The first thing to study is how many of these degrees there can be. The same question can be posed for r.e. degrees. Since if $A \leqslant_m B$ and $B$ is r.e. then $A$ is r.e., the degrees below $\mathbf{0}'$ are, for $m$- and 1-reducibility, all r.e. (and contain only r.e. sets). For $tt$- and $T$-reducibility this is of course not true anymore.

We begin with 1- and $m$-reducibility. A set $A$ is called a *cylinder* if for every $B$, $B \leqslant_m A \Rightarrow B \leqslant_1 A$. Here we have some equivalent characterizations. First let $A \cdot B$ be the set $\{\langle x,y\rangle: x \in A \wedge y \in B\}$ (where $\langle x,y\rangle$ is a recursive one-one onto function).

THEOREM 3.1 (ROGERS, YOUNG). *The following are equivalent.*
(a) *$A$ is a cylinder,*
(b) $A \equiv_1 A \cdot N$,
(c) *there is a recursive function $f$ such that, for all $x$, $W_{f(x)}$ is infinite and $x \in A \Rightarrow W_{f(x)} \subseteq A$ and $x \in \bar{A} \Rightarrow W_{f(x)} \subseteq \bar{A}$.*

PROOF. For (a) $\Rightarrow$ (b) use the fact that always $A \leqslant_1 A \cdot N$ (via $g(x) = \langle x, x\rangle$) and $A \cdot N \leqslant_m A$ (via $g(\langle x,y\rangle) = x$).

For (b) $\Rightarrow$ (c), if $A$ is cylinder then $A \cdot N \leqslant_1 A$ (because $A \cdot N \leqslant_m A$ always), say via $g$. Let $W_{f(x)} = \{g(z): z \in \{x\} \cdot N\}$, and this takes care of one direction. Finally, if $B \leqslant_m A$ let $f$ be as in (c) and use the facts on $W_{f(x)}$ to get a one-one reduction of $B$ to $A$ and thus (c) $\Rightarrow$ (a). $\square$

The following is simply an observation based on the fact that $A \equiv_m A \cdot N$ always holds.

PROPOSITION 3.2. *An $m$-degree consists of a single* 1*-degree iff it contains only cylinders.*

We still don't know if there are nonrecursive $m$-degrees consisting of only one 1-degree (there are two trivial recursive examples: $\{\varnothing\}$ and $\{\omega\}$). We'd like to have a form of Proposition 3.2 working when a property is true of one member of the $m$-degrees, not of all. We don't know of a necessary and sufficient condition, but below are some sufficient conditions. First let $K^A = \{e: e \in W_e^A\}$, $A'' = \{x: A \vDash \sigma_x\}$ and let's call $A$ *perfect* if $A$ is $\eta$-closed for some nontrivial r.e. equivalence relation $\eta$ for which the only $\eta$-closed recursive sets are $\varnothing$ and $\omega$ (also $\eta$ is called perfect).

THEOREM 3.3. *An m-degree consists of a single* 1*-degree if it contains a set of one of the following kinds.*

(a) $K^A$ *for any* $A$ *(Myhill),*

(b) $A^{tt}$ *for* $A$ *r.e. nonrecursive (Kobzev* **[Ko3]***),*

(c) $A$ *perfect (Ershov* **[E3]***).*

PROOF. (a) Consider $K$ (for $K^A$, relativize). Let $A \equiv_m K$: $A$ is r.e., so $A \leqslant_1 K$. Since $K \leqslant_m A$, let $x \in K \Leftrightarrow f(x) \in A$. As $W_x \subseteq \bar{K} \Rightarrow x \in \bar{K} - W_x$, if $W_{g(z)} = f^{-1}(W_x)$ and $h(x) = f(g(x))$ then $W_x \subseteq \bar{A} \Rightarrow h(x) \in \bar{A} - W_x$. We find $h^*$ one-one with the same property; this way: $h^*(0) = h(0)$. If $h(n+1) \notin \{h^*(0), \ldots, h^*(n)\}$ then $h^*(n+1) = h(n+1)$; otherwise let $W_{t(x)} = W_x \cup \{h(x)\}$ and $a = \max\{h^*(0), \ldots, h^*(n)\}$ and consider $h(n+1)$, $ht(n+1)$, $ht^2(n+1), \ldots$: either we find a repetition, or we find an element greater than $a$. In the first case let $h^*(n+1) = a+1$ (since here $W_{n+1} \nsubseteq \bar{A}$); in the second let $h^*(n+1)$ be the first element of the sequence greater than $a$. In the end, if $p$ is one-one and such that

$$W_{p(x)} = \begin{cases} \{h^*p(x)\} & \text{if } x \in K, \\ \varnothing & \text{otherwise,} \end{cases}$$

(by the recursion theorem), then $x \in K \Leftrightarrow h^*p(x) \in A$ and $K \leqslant_1 A$.

(b) Note the following properties of $A^{tt}$: for every $C$, if $C \leqslant_{tt} A^{tt}$ then $C \leqslant_1 A^{tt}$, hence $A \leqslant_1 A^{tt}$ and $A^{tt} \leqslant_1 \overline{A^{tt}}$. So, since $A$ is r.e. nonrecursive, $A$ and $A^{tt}$ are recursively inseparable. Now let $B \equiv_m A^{tt}$: $B \leqslant_m A^{tt}$, so $B \leqslant_1 A^{tt}$ by above. Let $A^{tt} \leqslant_m B$, say $A \vDash \sigma_x \Leftrightarrow f(x) \in B$; we'd like to have infinitely many choices to modify $f$ (to make it one-one). Let $h$ be such that $\sigma_{h(x,z)}$ is $(\sigma_z \wedge \sim \sigma_x) \vee (\sim \sigma_z \wedge \sigma_x)$;

$$\text{if } z \in A^{tt} \text{ then } (\forall x)\left(x \in A^{tt} \Leftrightarrow h(x,z) \in \overline{A^{tt}}\right),$$
$$\text{if } z \in \overline{A^{tt}} \text{ then } (\forall x)(x \in A^{tt} \Leftrightarrow h(x,z) \in A^{tt}).$$

Hence if $P_x = \{fh(x,z) : z \in A\}$ then by above (since $A$ and $A^{tt}$ are disjoint) $x \in A^{tt} \Rightarrow P_x \subseteq B$ and $x \in \overline{A^{tt}} \Rightarrow P_x \subseteq \bar{B}$, and each $P_x$ is infinite (if $P_a$ is finite, then $z \in R \Leftrightarrow fh(a,z) \in P_a$ defines a recursive set separating $A$ and $A^{tt}$).

(c) Proof similar to that of (a), using the fact that every equivalence class of a perfect equivalence relation is an infinite r.e. set (when passing from a given $h$ to a one-one $h^*$, in the case $h(n+1) \in \{h^*(0), \ldots, h^*(n)\}$). This works in both directions, and in the case $A \leqslant_m B \Rightarrow A \leqslant_1 B$ (when $A \leqslant_m B$ via $g$) the point is that if one cannot get a new value by generating $g(y)$ for $y$ in the $\eta$-closure of the element considered, then eventually $g^{-1}(F)$ would be $\eta$-closed for some finite $F$–a contradiction. □

A simple corollary is

THEOREM 3.4 (JOCKUSCH **[J2]**). *Every T-degree above* $\mathbf{0}'$ *and every r.e. tt-degree contain an m-degree consisting of a single* 1*-degree.*

PROOF. From parts (a) and (b) of Theorem 3.3, since every $T$-degree above $\mathbf{0}'$ contains a set of the form $K^A$, and since for every $A$ $A \equiv_{tt} A^{tt}$. □

*Problem* 2. Does every $T$-degree contain an $m$-degree consisting of only one 1-degree?

THEOREM 3.5 (JOCKUSCH [J2], SOARE). *Every r.e. $T$-degree contains an r.e. $m$-degree consisting of a single* 1-*degree.*

SKETCH OF PROOF. Yates [Y1] proved that every r.e. nonrecursive $T$-degree contains a simple nonhypersimple set $A$. Ershov [E3] proved that for such an $A$, $\{x: D_x \subseteq A\}$ is perfect (and, clearly, in the same $T$-degree of $A$). □

*Problem* 3. Does every r.e. $tt$-degree contain an r.e. $m$-degree consisting of only one 1-degree?

It's worth noting that not every $m$-degree consisting of a single 1-degree contains a perfect set (since it's possible to modify Theorem 2.4 and have $B \leqslant_{tt} A$ and $B$ perfect $\Rightarrow A$ nonhypersimple, see [Den2]).

*Problem* 4. Find a property $P$ such that the $m$-degree of $A$ consists of a single 1-degree iff $A$ has the property $P$.

What is the possible structure of 1-degrees inside (r.e.) $m$-degrees?

THEOREM 3.6 (YOUNG [Yo]). *An $m$-degree contains either only one or infinitely many* 1-*degrees.*

PROOF. Using part (c) of Theorem 3.2 we have

$$B \oplus B \leqslant_1 B \Rightarrow B \oplus B \text{ cylinder} \Rightarrow B \text{ cylinder}.$$

So if $B$ is not cylinder, its $m$-degree contains the chain $B <_1 B \oplus B <_1 \dots$. □

Young [Yo] proved that if there is more than one 1-degree in a given $m$-degree, then there is a dense linear ordering of 1-degrees in it.

*Problem* 5. If an $m$-degree has more than one 1-degree, does it contain an infinite antichain of 1-degrees (i.e. a set of mutually incomparable 1-degrees)?

Degtev [Deg5] proved that it is so if the $m$-degree is r.e.

THEOREM 3.7 (ROGERS, DEKKER). *In every $m$-degree there is always a greatest* 1-*degree, but not always* (*not even for r.e. ones*) *a least.*

PROOF. Let $A$ be in the $m$-degree. If $B \leqslant_m A$ then (from $A \leqslant_m A \cdot N$) $B \leqslant_m A \cdot N$, so $B \leqslant_1 A \cdot N$. Hence the 1-degree of $A \cdot N$ is independent of the choice of $A$ and it is the greatest.

If $A$ is simple, let $z \in \bar{A}$; then $A \cup \{z\} <_1 A$ and $A \cup \{z\} \equiv_m A$. Since if $A$ is simple and $B \equiv_m A$ and $B \leqslant_1 A$ then $B$ is simple, below $A$ there can't be a least 1-degree. □

Let's call a 1-degree in an $m$-degree *maximal* if there aren't 1-degrees between it and the greatest one, *minimal* if there aren't (in the $m$-degree) 1-degrees below it, except the least one (if this exists). In this terminology the greatest 1-degree is not maximal.

THEOREM 3.8 (DEGTEV [Deg6]). *No r.e. $m$-degree has a maximal* 1-*degree. If there are minimal* 1-*degrees, then there is the least one.*

PROOF. The first result is immediate, since we noted that if $B$ is not cylinder, then neither is $B \oplus B$, and $B <_1 B \oplus B$. Let $A \equiv_m B$ with $A$ and $B$

r.e. and of incomparable 1-degrees. If e.g. $x \in A \Leftrightarrow f(x) \in B$, let $x \in R \Leftrightarrow (\exists y < x)[f(x) = f(y)]$; $R$ is recursive. If $C = A \cup R$ then $C \leqslant_1 A$ and $C \leqslant_1 B$ and $C \equiv_m A$. But, since $A$ and $B$ are 1-incomparable, $C <_1 A$ and $C <_1 B$. So, if $A$ has minimal 1-degree, $C$ must have a least 1-degree (in the $m$-degree of $A$). $\square$

Degtev [**Deg6**] proved that there can be a least element and no minimal elements. Also, for every $n$ there can be exactly $n$ minimal elements.

*Problem* 6. Does there exist an r.e. $m$-degree with infinitely many minimal 1-degrees?

We turn now to the relationship between $tt$- and $m$-degrees. Here the case of r.e. degrees must be treated by itself. Let's study first the general case.

THEOREM 3.9 (JOCKUSCH [**J2**]). *Every nonrecursive tt-degree contains infinitely many m-degrees.*

SKETCH OF PROOF. Use the fact that every nonrecursive $tt$-degree contains an immune nonhyperimmune set which is retraceable by a total function (this is easy; given $B$, consider the set of sequence numbers coding initial segments of the characteristic function of $B$) and a property of immune retraceable sets (no disjoint sequence of finite sets of bounded cardinality can be the witness of the nonhyperimmunity of such a set) to prove that, if $A$ is such a set, then $\bar{A} <_m \bar{A} \cdot \bar{A} <_m \ldots$. (This is not immediate. See [**J2**].) $\square$

*Problem* 7. Does every nonrecursive $tt$-degree contain an infinite antichain of $m$-degrees?

The existence of two incomparable $m$-degrees is easy; as quoted above, every nonrecursive $tt$-degree contains a semirecursive set $A$, and in this case $A$ and $\bar{A}$ are $m$-incomparable.

THEOREM 3.10 (ROGERS, JOCKUSCH). *In every tt-degree there is always a greatest m-degree, but no nonrecursive tt-degree contains a least m-degree.*

PROOF. For every $A$, $A \equiv_{tt} A''$ and $B \leqslant_{tt} A \Rightarrow B \leqslant_1 A''$, so actually there is a greatest 1-degree. Also, from Theorem 4.6 we will see that every nonrecursive $tt$-degree contains a minimal pair of $m$-degrees, so no $m$-degree in it can be the least one. $\square$

*Problem* 8. What is the situation for maximal and minimal $m$-degrees inside a given $tt$-degree?

The knowledge of the relationship between r.e. $tt$-degrees and the r.e. $m$-degrees inside them is still not very great.

THEOREM 3.11 (DEGTEV [**Deg5**], FISCHER). *There are r.e. tt-degrees with only one r.e. m-degree, and others* (*e.g.* $\mathbf{0}'$) *with infinitely many.*

PROOF. To have an r.e. $tt$-degree with infinitely many r.e. $m$-degrees, the idea is to iterate the following reasoning. Let $A$ be the simple $tt$-complete set of Theorem 2.3, so for some $f$, $x \in K \Leftrightarrow B_x \subseteq A$. We have in general $A \leqslant_m A \cdot A$ and $A \equiv_{tt} A \cdot A$. Suppose $A \cdot A \leqslant_m A$; then for some recursive $g$, $x \in A \wedge y \in A \Leftrightarrow g(x, y) \in A$. Use this fact several times and the $tt$-reduction of $K$ to $A$, to have a recursive $h$ such that $x \in K \Leftrightarrow h(x) \in A$; contradiction, because a simple set is not $m$-complete. In the general case we don't have directly $K \leqslant_m A$, but only a reduction of $K$ to $A$ via "bounded truth-tables"; we derive a contradiction from this as in Theorem 2.4 (a simple set is not complete with respect to "bounded truth-tables", see §8). For the other part, see **[Deg5]**. □

*Problem* 9. Find a property such that the *tt*-degree of an r.e. set $A$ contains only one r.e. *m*-degree iff $A$ has $P$.

One sufficient condition is known (see **[Deg5]**): $A$ is an r.e. semirecursive set with *rigid* complement (i.e. any two subsets of $\bar{A}$ are *pm*-equivalent iff their symmetric differences is finite, where *pm*-reducibility is defined as *m*-reducibility but using partial functions in place of total functions).

*Problem* 10. Does every r.e. *tt*-degree contain either only one or infinitely many r.e. *m*-degrees?

*Problem* 11. Does every r.e. *tt*-degree have a greatest r.e. *m*-degree?

Note that $B''$ is r.e. only for $B$ recursive, since $B \leqslant_1 B''$ and $\bar{B} \leqslant_1 B''$. The situation for *T*-degrees is completely settled.

THEOREM 3.12 (JOCKUSCH **[J2]**, MARTIN). *For a T-degree* **a**, *the following are equivalent.*

(1) **a** *doesn't contain hyperimmune sets* (*i.e.* **a** *is hyperimmune-free*).

(2) **a** *consists of only one tt-degree.*

(3) **a** *consists of finitely many tt-degrees.*

(4) **a** *contains a greatest tt-degree.*

PROOF. First note that if **a** contains hyperimmune sets and $\mathbf{a} < \mathbf{b}$, then **b** too contains hyperimmune sets (using the fact that $A$ is hyperimmune iff $A$ is not majorized by a recursive function; so if $A \in \mathbf{a}$ and $B \in \mathbf{b}$ and $A = \{a_0, a_1, \dots\}$ in increasing order, let $C = \{2x\colon x \in A\} \cup \{2x + 1\colon (\exists i \in B)(a_i = x)\}$. From $A \leqslant_T B$ we have $C \leqslant_T B$. Also, $B \leqslant_T C$ since $i \in B \Leftrightarrow 2a_i + 1 \in C$. If $C = \{c_0, c_1, \dots\}$ in increasing order, then $c_{2n} \geqslant 2a_n \geqslant a_n$ so if $f$ recursive majorizes $C$, $g(x) = f(2x)$ majorizes $A$). It follows that **a** satisfies (1) iff every function of degree $< \mathbf{a}$ is majorized by a recursive function. (1) $\Rightarrow$ (2) since if $A \in \mathbf{a}$ and $B \leqslant_T A$ then let $B = \varphi_e^A$; if $f(x) =$ least $s$ such that $\varphi_{e,s}^{A_s}(x)$ converges, $f$ is majorized by a recursive function $g$, so $B \leqslant_{tt} A$ since we can define $i$ such that $\varphi_i^C(x) = \varphi_e^C(x)$ if this converges in less than $g(x)$ steps, $\varphi_i^C(x) = 0$ otherwise; for every $C$, $\varphi_i^C$ is total; and $B = \varphi_i^A$. So $B \leqslant_{tt} A$. By symmetry it follows $A \equiv_T B \Rightarrow A \equiv_{tt} B$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial (use joins in the last one).

(4) $\Rightarrow$ (1). If there is a greatest *tt*-degree, let $A$ be of greatest *m*-degree in it. Suppose $f \leqslant_T \mathbf{a}$ is not recursively majorized (and is increasing, with no loss of generality). Define

$$e \in B \Leftrightarrow \varphi_e(e) \text{ converges in less than } f(e) \text{ steps, and } \varphi_e(e) \notin A.$$

Then $B \leqslant_T A$ but $B \nleqslant_m A$ (if $x \in B \Leftrightarrow g(x) \in A$ let $e_0, e_1, \dots$ be indices of $g$ with $e_i > i$; then $e_i \in B \Leftrightarrow g(e_i) \in A \Leftrightarrow \varphi_{e_i}(e_i) \in A$, so $\varphi_{e_i}(e_i)$ must converge in more than $f(e_i) > f(i)$ steps, hence the numbers of steps needed to calculate $\varphi_{e_i}(e_i)$ majorizes $f$). Hence $A <_m A \oplus B$ (since $B \nleqslant_m A$), and $A$ is not the greatest *m*-degree of **a**. □

COROLLARY 3.13 (JOCKUSCH [J2]). *A $T$-degree contains either only one or infinitely many tt-degrees.*

The result can't be improved as in Theorem 3.9, since Martin and Miller [MM] have constructed (with a modification of the construction of a minimal $T$-degree) a nonrecursive hyperimmune-free $T$-degree. So there are nonrecursive $T$-degrees consisting of only one $tt$-degree. From the proof of Theorem 3.12 it follows that if $\mathbf{a}$ is hyperimmune free, so is every $T$-degree below it. Since we know from Theorem 2.5 that $\mathbf{0}'$ contains a hyperimmune set, no $T$-degree above $\mathbf{0}'$ is hyperimmune-free. It's also possible to prove (see [MM]) that no nonrecursive $T$-degree below $\mathbf{0}'$ is hyperimmune-free (for r.e. ones this follows from Theorem 2.5). But there is a hyperimmune-free $T$-degree below $\mathbf{0}''$ (the original construction gives it, as in the case of minimal $T$-degrees), necessarily incomparable with $\mathbf{0}'$.

We know that every r.e. nonrecursive $T$-degree contains infinitely many $tt$-degrees. Something more is true (see [Deg4]).

THEOREM 3.14 (DEGTEV). *Every r.e. nonrecursive $T$-degree contains infinitely many r.e. tt-degrees.*

Actually Degtev exhibits an infinite antichain of r.e. $tt$-degrees. He also proves the existence of an infinite antichain of $tt$-degrees in every $T$-degree with more than one $tt$-degree.

THEOREM 3.15. *If a $T$-degree contains more than one tt-degree, it doesn't contain either a greatest or a maximal tt-degree.*

PROOF. The existence of the greatest $tt$-degree is ruled out by Theorem 3.12. If there is a maximal $tt$-degree, then there is also the greatest (which is obtained as the join of the maximal one with an incomparable $tt$-degree of the same $T$-degree). □

THEOREM 3.16 (JOCKUSCH [J3], KALLIBEKOV [Kal]). *An r.e. $T$-degree can be without greatest or maximal r.e. tt-degrees.*

PROOF. As above, if there is a maximal one then there is the greatest. For greatest elements, let $A$ r.e. be in the greatest r.e. $tt$-degree in the $T$-degree $\mathbf{a}$. Let $G_{tt}(A) = \{e\colon W_e \leqslant_{tt} A\}$ and $G_T(A) = \{e\colon W_e \leqslant_T A\}$. Of course $G_{tt}(A) \subseteq G_T(A)$, but also the opposite inclusion holds, since

$$W_e \leqslant_T A \Rightarrow W_e \oplus A \leqslant_T A \Rightarrow W_e \oplus A \in \mathbf{a} \Rightarrow W_e \oplus A \leqslant_{tt} A \Rightarrow W_e \leqslant_{tt} A.$$

Then $G_T(A) \in \Sigma_3^0$ because $G_{tt}(A) \in \Sigma_3^0$, and by a theorem of Yates [Y3] for $\mathbf{a} < \mathbf{0}'$ this can only happen if $\mathbf{a}'' = \mathbf{0}''$. Hence in all the other cases $\mathbf{a}$ doesn't have a greatest r.e. $tt$-degree. □

Note that of course the $T$-degree $\mathbf{0}'$ has a greatest r.e. $tt$-degree (the $tt$-degree of $K$), although it doesn't have a greatest $tt$-degree.

*Problem* 12. Is $\mathbf{0}'$ the only r.e. nonrecursive $T$-degree with greatest r.e. $tt$-degree?

THEOREM 3.17 (KOBZEV [Ko4]). *There is an r.e. $T$-degree containing a minimal (r.e.) tt-degree.*

PROOF. This is an immediate consequence of the fact that there is an r.e. set whose $tt$-degree is minimal among all the $tt$-degrees, see Theorem 6.5. $\square$

*Problem* 13. Is there an (r.e.) $T$-degree with a least (r.e.) $tt$-degree?

**4. The structure of $m$-degrees.** Remember that all the $m$-degrees below $\mathbf{0}'$ are r.e., so for example a minimal r.e. $m$-degree will also be a minimal $m$-degree, and similar facts.

First of all note that the (r.e.) $m$-degrees are an upper semilattice, since $A \oplus B$ is the least upper bound of $A$ and $B$ (and it is r.e. if $A$ and $B$ are) with respect to $\leqslant_m$ (and also to $\leqslant_{tt}$ and $\leqslant_T$).

When Myhill introduced the notion of maximal set, his hope was that they were so near to cofinite sets (i.e. to the $T$-degree $\mathbf{0}$) to be far enough from $\mathbf{0}'$. This is not so, as we saw in Theorem 2.7, and Martin proved (see [**So1**, Theorem 3.6]) that the $T$-degrees containing maximal sets are in fact those nearest to $\mathbf{0}'$; they are the high r.e. degrees, i.e. the r.e. degrees whose jump is $\mathbf{0}''$. But there is a precise sense in which the maximal sets are near to recursive sets, in the theory of degrees.

THEOREM 4.1 (LACHLAN [**La5**], YOUNG). *Maximal sets have minimal m-degrees.*

PROOF. Let $A$ be maximal and $D$ r.e. nonrecursive such that $D \leqslant_m A$, say via $f$ recursive. Let $S$ be the range of $f$; then $\bar{S} \cap \bar{A}$ is finite and $S$ is infinite (otherwise $D$ is recursive). We want $g$ recursive such that $x \in A \Leftrightarrow g(x) \in D$, so that $A \leqslant_m D$. Let $a \in D$, $b \in \bar{D}$ and simultaneously enumerate $S$ and $A$. Define

$$g(x) = \begin{cases} a & \text{if } x \text{ shows up first in } A, \\ b & \text{if } x \in \bar{S} \cap \bar{A}, \\ \text{the least } y \text{ s.t. } f(y) = x & \text{if } x \text{ shows up first in } S. \end{cases} \quad \square$$

Note that the same holds for $\eta$-maximal sets by the same proof. The previous result is just a special case of the following.

THEOREM 4.2 (LACHLAN [**La5**]). *If $A$ is an r.e. set and* $\mathbf{a}$ *is its m-degree, there is an homomorphism* (*of upper semilattices*) *from the r.e. supersets of $A$* (*under inclusion, modulo finite sets*) *onto the m-degrees below* $\mathbf{a}$.

PROOF. Let $B$ be r.e. and $f$ recursive be such that $B$ = range of $f$. If we define $x \in B^* \Leftrightarrow f(x) \in A$ then the map from $B$ to the $m$-degree of $B^*$ gives the homomorphism. Note that if $B$ = range of $g$ then $f(x) \in A \Leftrightarrow g$(the least $y$ s.t. $g(y) = f(x)) \in A$ and vice versa, so the $m$-degree of $B^*$ does not depend on $f$. Clearly $B^* \leqslant_m A$ via $f$. Now if $C \leqslant_m A$ via $h$ then $C \equiv_m (\text{range } f)^*$. However, if $B \subseteq C$ then $B^* \leqslant_m C^*$; and if $B$ and $C$ differ finitely, $B^* \equiv_m C^*$. $\square$

In the case of $A$ hyperhypersimple we can be more precise.

THEOREM 4.3 (ERSHOV [**E1**]). *If $A$ is hyperhypersimple and* $\mathbf{a}$ *is its m-degree, then*

(1) *every m-degree of an r.e. superset of $A$ is below* $\mathbf{a}$.

(2) *every m-degree below* $\mathbf{a}$ *is represented by an r.e. superset of $A$.*

PROOF. Let $A \subseteq B$ and $B$ r.e.; there is (by Lachlan's characterization of hyperhypersimple sets **[So1**, Theorem 7.1]) a recursive set $R$ such that $A \cup R = B$. Then $x \in B \Leftrightarrow g(x) \in A$, where (if $x_0 \in A$) $g(x)$ is $x_0$ if $x \in R$, and $x$ otherwise. If $\mathbf{b} < \mathbf{a}$, let $B \in \mathbf{b}$ and $x \in B \Leftrightarrow g(x) \in A$. Now if $C = A \cup \text{range } g$ then $C^* = \bar{C} \cup A$ is r.e. Furthermore, $B \equiv_m C^*$ because $B \leqslant_m C^*$ via $g$, and $C^* \leqslant_m B$ via $h$ defined as follows. Let $S$ be recursive such that $C^* = A \cup S$, and $A - S \subseteq g(B)$, and let $b \in B$. Let $h(x) = b$ if $x \in S$ and $h(x) = \mu y\,[g(y) = x]$ otherwise. □

*Problem* 14. Is every simple set $A$ with the two properties of Theorem 4.3 hyperhypersimple?

Degtev **[Deg8]** has proved that there are simple nonhypersimple sets with the first property.

We can give one more result along the previous line. Call an r.e. set $A$ $r$-maximal if $\bar{A}$ is $r$-cohesive, i.e. infinite and no recursive set can split it into two infinite parts. Of course, the same property with r.e. in place of recursive gives the notion of maximal set. Note also that hyperhypersimple sets are such that every r.e. superset is complemented (they form a boolean algebra), whereas for $r$-maximal sets no nontrivial r.e. superset is complemented. The same contrast is reproduced in Theorems 4.3 and 4.4.

THEOREM 4.4 (DEGTEV **[Deg3]**). *If $A$ is $r$-maximal then for every $B$ a nontrivial superset of $A$, $A$ and $B$ are $m$-incomparable.*

PROOF. Nontrivial means that $\bar{B}$ and $B - A$ are both infinite. It's enough to prove that if $B_1$ and $B_2$ are infinite subsets (not necessarily r.e.) of $\bar{A}$ whose symmetric difference is infinite, then $B_1$ and $B_2$ are $m$-incomparable (then, given $B$ as above, apply this to $\bar{A}$ and $\bar{B}$). Suppose e.g. $x \in B_1 \Leftrightarrow f(x) \in B_2$. Then by the properties of $\bar{A}$ there must be an infinite number of $x$'s such that

$$x \in \bar{A} \wedge (\exists y \in \bar{A})[y \neq x \wedge f(y) = x] \wedge (\forall y)[f(y) = x \Rightarrow y \in \bar{A}]$$

(this is because if $x \in \bar{A}$ then $f(y) = x$ for at most finitely many $y$'s in $\bar{A}$ since $f(y) = x$ is a recursive condition; but $B_2 \cap \text{range } f$ must be infinite). Now define inductively a recursive set $R$ this way. Put $x$ in $R$ unless $f(x) < x$ and $f(x)$ is already in $R$, or $x = f(y) \wedge y < x$ and $y$ is already in $R$. The crucial properties of $R$ are that for all $x$

$$(\exists y)[y \neq x \wedge f(y) = x] \Rightarrow \{x, y\} \cap \bar{R} \neq \varnothing, \tag{1}$$

and

$$(\{z : f(z) = x\} \cup \{x, f(x)\}) \cap R \neq \varnothing. \tag{2}$$

It follows that $\bar{R} \cap \bar{A}$ is infinite by (1). Hence, $R \cap \bar{A}$ must be finite by the choice of $A$. Consider the recursive set $S = \{x : f(x) \in R\}$. Now $S \cap A$ and $S \cap \bar{A}$ are both infinite by (1), (2) and $R \cap \bar{A}$ finite, a contradiction.

It's immediate to note that also the converse of Theorem 4.4 holds (let $A$ be r.e. nonrecursive and $B$ be recursive such that $B \cap \bar{A}$ and $\bar{B} \cap \bar{A}$ are infinite: then $A \cup B$ is a nontrivial r.e. superset of $A$ and $A \cup B \leqslant_m A$). Note that from the two previous theorems we have a new proof of this theorem of Lachlan: *if $A$ is hyperhypersimple and $r$-maximal, then $A$ is maximal* (every r.e. superset has $m$-degree below the one of $A$, so there are only trivial r.e.

supersets). We saw that every maximal set has minimal $m$-degree. Does every minimal r.e. $m$-degree contain a maximal set? The answer is no, and can be obtained from the facts that every nonrecursive r.e. $T$-degree contains a minimal r.e. $m$-degree ([**La6**], see also Theorem 5.8) but not always a maximal set. A simple proof of a stronger fact can be given directly.

THEOREM 4.5. *Not every r.e. minimal m-degree contains simple sets.*

PROOF. Let $A$ be a maximal set. Then by the Friedberg splitting theorem [**So1**, Theorem 6.3] there exist $B$ and $C$ two disjoint r.e. nonrecursive sets such that $A = B \cup C$ (see [**So**, Theorem 6.2]). Since $A$ is maximal, $B$ and $C$ are recursively inseparable. The $m$-degree of $B$ doesn't contain simple sets (since if $S$ is simple and $B \leqslant_m S$ via $g$, then $g(C)$ is disjoint from $S$, hence finite and if $x \in R \Leftrightarrow g(x) \in g(C)$, then $R$ is a recursive set separating $B$ and $C$). The $m$-degree of $B$ is minimal, using the following $g$ (with notations as in Theorem 4.1 with $B$ in place of $A$). Simultaneously enumerate $S$, $B$ and $C$ and let

$$g(x) = \begin{cases} a & \text{if } x \text{ shows up first in } B, \\ b & \text{if } x \in \bar{S} \cap \bar{A} \text{ or } x \text{ shows up first in } C, \\ \text{the least } y \text{ s.t. } f(y) = x & \text{if } x \text{ shows up first in } S. \end{cases} \qquad \square$$

Kobzev [**Ko2**] has proved that there are simple nonhypersimple sets with minimal $m$-degree.

Two degrees form a *minimal pair* if **0** is the only degree less than or equal to both of them. For example, two incomparable maximal sets form a minimal pair of $m$-degrees. Another interesting example of a (non-r.e.) minimal pair is the following.

THEOREM 4.6 (JOCKUSCH). *If $A$ is retraceable and immune, the m-degrees of $A$ and $\bar{A}$ form a minimal pair.*

PROOF. Let $A$ be retraceable via a partial recursive function $\psi$, and let $f$ and $g$ be such that $x \in C \Leftrightarrow f(x) \in A \Leftrightarrow g(x) \in \bar{A}$. For all $x$ we have $\{f(x), g(x)\} \cap A \neq \varnothing$. If $f(x)$ and $g(x)$ are sent by $\psi$ into the same element, i.e. there are $m$ and $n$ such that $\psi^m f(x) = \psi^n g(x) = y$ say, then we know that $y \in A$ (because either $f(x) \in A$ or $g(x) \in A$). The set of all such $y$ must be finite since it is r.e. Let $b \in A$ be the smallest element larger than all such $y$.

Assume for a contradiction that $C$ is nonrecursive. Then there is no $x$ such that for all $x$ either $f(x) < x$ or $g(x) < x$ (otherwise $x \in C \Leftrightarrow [f(x) \in A \restriction x$ or $g(x) \in A \restriction x]$, so $C$ is recursive, a contradiction). But then $A - \{0, 1, \ldots, b - 1\}$ is r.e. as follows (contrary to immunity). Fix $z > b$. Choose $x$ such that $f(x), g(x) > z$. Repeatedly apply $\psi$ to $f(x)$ and $g(x)$ until either $\psi$ "sends" $f(x)$ first into $z$ and then into $b$ (i.e. $\psi^p f(x) = z$ and $\psi^q(z) = b$) or $\psi$ "sends" $g(x)$ first into $z$ and then into $b$. Note that at most one case occurs by the choice of $b$, and if $z \in A$ exactly one case occurs. $\square$

We have already noted in the proof of Theorem 3.9 that every nonrecursive r.e. $tt$-degree contains an $A$ as before, so *every nonrecursive r.e. tt-degree contains a minimal pair of m-degrees.*

We turn now to some structural properties of the (r.e.) $m$-degrees.

THEOREM 4.7 (DENISOV [**Den1**]). *If* $\mathbf{a}$ *is an r.e. m-degree such that* $\mathbf{0} < \mathbf{a} < \mathbf{0}'$ *there is an r.e. m-degree* $\mathbf{b}$ *incomparable with* $\mathbf{a}$.

PROOF. Let $A$ be r.e. and in $\mathbf{a}$. We give the strategy to satisfy single requirements. If at step $s$ we attack $\sim(\forall x)[x \in A \Leftrightarrow \varphi_e(x) \in B]$, we look for $n$ and $t$ such that $\varphi_e$ converges (in less than $t$ steps) on $0, \ldots, n$ and either $n \notin A_t \wedge \varphi_e(n) \in B_t$ or $n \in A_t \wedge \varphi_e(n) \notin B_t$. In the first case we don't do anything, hoping that $n \notin A$ (if $n \in A$, we will make another attack); in the second case, we restrain $\varphi_e(n)$ from entering $B$. Note that if $\varphi_e$ is total and if $s$ is a stage in which all the requirements of higher priority have been satisfied, then such an $n$ must exist, otherwise from a certain point on $n \in A \Leftrightarrow \varphi_e(n) \in B_s$ and $A$ is recursive.

If at step $s$ we attack $\sim(\forall x)[x \in B \Leftrightarrow \varphi_e(x) \in A]$, we consider the greatest element of $B_{s-1}$ (let it be $n$) and see if for all $z \leqslant n$ $\varphi_e(z)$ converges (in less than $s$ steps) and $z \in B_{s-1} \Leftrightarrow \varphi_e(z) \in A_s$. If so, we take the first element generated in $K$ and not yet in $B$, and we put it into $B$. If $s$ is a stage in which all the requirements of higher priority have been satisfied, then we will only add a finite number of new elements to satisfy the requirement (otherwise, $B$ differs only finitely from $K$, hence it's creative; and $B \leqslant_m A$, so $A$ itself is creative, contrary to the hypothesis). This means that from a certain point on we are not in the prior hypothesis, so $B \nleqslant_m A$. $\square$

It's interesting to note that the construction gives $\mathbf{b}$ uniformly from $\mathbf{a}$. The corresponding result for r.e. $T$-degree is true, but the finite injury priority method (which has been used previously) only gives a nonuniform proof (as a consequence of Sacks' splitting theorem); to have the uniform result of Yates [**Y2**] the infinite injury method seems necessary. Note that the same proof works for every reducibility caught in between $m$- and $wtt$- (using the fact that only a recursively bounded set of numbers determines the value of the reduction).

Ershov and Lavrov [**EL**] have improved the result, showing that it's always possible to find $\mathbf{b}$ minimal r.e. $m$-degree. From this it follows in particular that *the r.e. minimal m-degrees are not bounded below* $\mathbf{0}'$.

THEOREM 4.8 (LACHLAN [**La1**]). *There is no pair of incomparable r.e. m-degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *with* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$.

PROOF. Let $A$ and $B$ be r.e. sets such that $A \cdot B$ is $m$-complete. (Indeed it suffices to assume that one of $A$ and $B$ is r.e.) We prove that either $A$ or $B$ is $m$-complete (the result follows because if $A \oplus B$ is $m$-complete, so is $A \cdot B$). The idea is to construct $D$ r.e. such that $D \leqslant_m A \cdot B$ and either $K \leqslant_m D \leqslant_m B$ or $K \leqslant_m D \leqslant_m A$. By the recursion theorem we may suppose to have $f$ and $g$ recursive such that $x \in D \Leftrightarrow f(x) \in A \wedge g(x) \in B$. Call $x$ *good at stage* $s$ if $x \notin D_s \wedge g(x) \in B_s$; then $x \in D \Leftrightarrow f(x) \in A$. Call $x$ *good* if $x$ is good at stage $s$ for some $s$. To have $K \leqslant_m A$ it is enough, given an effective enumeration $x_0, x_1, \ldots$ of the good elements, to have $i \in K \Leftrightarrow x_i \in D$, because then $i \in K \Leftrightarrow f(x_i) \in A$.

At stage $s$, if $i \in K_s$ and $x_i$ exists we put $x_i$ in $D$. Also, if $i \in K_s$ and $i$ is not good at $s$, we put $i$ in $D$.

If there are infinitely many good elements $x_i$, then $g(x_i) \in B$ and $x_i \in D \Leftrightarrow f(x_i) \in A$ as noted, and $i \in K \Leftrightarrow x_i \in D$; so $K \leqslant_m D \leqslant_m A$. If there are only finitely many good elements, then for almost every $x$, $x \in D \Leftrightarrow g(x) \in B$ (since if this is false $x \notin D$–otherwise $g(x) \in B$–and $g(x) \in B$, so $x$ becomes good after a while). By the second part of the construction, $K$ and $D$ differ finitely and $D \leqslant_m B$; so $K \leqslant_m B$. □

Note that, as in the analogous (and weaker) result for r.e. $T$-degrees (the nondiamond theorem, see [**So1**, Theorem 14.4]), the proof does not use any priority arguments.

Of course the r.e. $m$-degrees are not dense, since there are minimal r.e. $m$-degrees. In particular density fails toward $\mathbf{0}$. But it holds toward $\mathbf{0}'$.

THEOREM 4.9. *If* $\mathbf{a}$ *is r.e. and* $\mathbf{a} < \mathbf{0}'$, *there is* $\mathbf{b}$ *r.e. such that* $\mathbf{a} < \mathbf{b} < \mathbf{0}'$.

PROOF. If $\mathbf{a}$ is recursive, this is trivial. If not, then by Theorem 4.7 take $\mathbf{c}$ r.e. and incomparable with $\mathbf{a}$, and let $\mathbf{b} = \mathbf{a} \cup \mathbf{c}$; $\mathbf{a} < \mathbf{b}$ obviously, and $\mathbf{b} < \mathbf{0}'$ by Theorem 4.8. □

Kallibekov [**Ka2**] has proved much more than this, namely that if $\mathbf{a}$ is as before then there is an independent set of r.e. $m$-degrees between $\mathbf{a}$ and $\mathbf{0}'$, so that in particular every countable partial ordering is embeddable in the r.e. $m$-degrees between $\mathbf{a}$ and $\mathbf{0}'$.

THEOREM 4.10 (ERSHOV [**E1**]). *The* (*r.e.*) *m-degrees are not a lattice.*

SKETCH OF PROOF. Take any r.e. set $A$ such that $\bar{A}$ is rigid and for some $B \supseteq A$ the following holds, for $C$ and $D$ r.e.,

$$(\forall C)\big(C \cap B \cap \bar{A} = \varnothing \Rightarrow (\exists D)$$
$$\big(D \cap B \cap \bar{A} = \varnothing \wedge \bar{A} \cap D \cap \bar{C} \text{ is infinite}\big)\big).$$

Degtev [**Deg3**] has proved that sets like $A$ and $B$ exist (actually, for every r.e. semirecursive set $A$ with rigid complement there is such a $B$) and that they don't have g.l.b. □

THEOREM 4.11 (ERSHOV [**E1**]). *There is an r.e. m-degree with no minimal predecessor.*

SKETCH OF PROOF. Take $A$ r.e. with rigid complement and such that for every r.e. set $B$, $\bar{A} \cap B$ is not $r$-cohesive. Degtev [**Deg3**] proved that such a set exists. If $C$ is not recursive and $C \leqslant_m A$ via $f$, then $\bar{A} \cap \operatorname{range} f$ is infinite and there is a recursive set $R$ splitting it in two infinite parts. If now $x \in D \Leftrightarrow x \in C \wedge f(x) \notin R$ then $D$ is not recursive, $D \leqslant_m C$ but $C \nleqslant_m D$ (otherwise $\bar{A}$ is not rigid), so $C$ is not minimal. □

Different proofs of the two last results can be obtained using initial segments (see §7).

To finish, we give a property of a different kind.

THEOREM 4.12. *The r.e. m-degrees are not recursively enumerable without repetitions.*

SKETCH OF PROOF. Suppose we have an r.e. sequence $A_n$ of r.e. sets, with only one set for each r.e. $m$-degree; we may suppose that every $A_n$ is not recursive (there are only three recursive $m$-degrees, so we can drop their representatives). With the method of Sacks' splitting theorem, we can find a nonrecursive r.e. set $B$ such that for every $n$, $A_n \nleqslant_T B$, a contradiction. $\square$

We can also obtain the last result using the next fact.

THEOREM 4.13 (YATES [Y2]). *If $A$ is r.e. and different from $\varnothing$ and $\omega$, then the index set $A^* = \{x\colon W_x \equiv_m A\}$ is $\Sigma^0_3$-complete.*

PROOF. It's immediate that $A^* \in \Sigma^0_3$. The only interesting case is for $A$ nonrecursive. Given $B \in \Sigma^0_3$, let $x \in B \Leftrightarrow \exists e \forall y \exists z\, R(x, y, z, e)$ and $h$ recursive such that $\langle i, m\rangle \in W_{h(x)} \Leftrightarrow (\exists e \leqslant i)(\forall y \leqslant m)\exists z\, R(x, y, z, e)$. Then

$$x \in B \Rightarrow (\exists e)\left[(\forall i < e)\left(W^{(i)}_{h(x)} \text{ finite}\right) \wedge (\forall i \geqslant e)\left(W^{(i)}_{h(x)} \text{ infinite}\right)\right],$$

$$x \in \bar{B} \Rightarrow (\forall e)\left(W^{(e)}_{h(x)} \text{ finite}\right)$$

where $P^{(x)} = \{\langle x, y\rangle\colon \langle x, y\rangle \in P\}$. Let

$$\langle e, y\rangle \in C^{(e)}_x \Leftrightarrow (\exists n) \quad \left(n \in W^{(e)}_{h(x)} \wedge y \in A_n\right)$$

where $\{A_n\}_{n\in\omega}$ is a recursive enumeration of $A$. The methods of the thickness lemma (see [So, Theorem 13.1]) give $D_x \subseteq C_x$ such that

$$(\forall e)\left(A \nleqslant_T C^{(e)}_x\right) \Rightarrow A \nleqslant_T D_x,$$

$$A \nleqslant_T C^{(<e)}_x \Rightarrow (\forall i \leqslant e) D^{(i)}_x \subseteq^* C^{(i)}_x$$

where $P^{(<x)} = \bigcup_{i<x} P^{(i)}$. Then

$$x \in B \Rightarrow D_x \equiv_m A, \qquad x \in \bar{B} \Rightarrow A \nleqslant_T D_x;$$

in particular $B \leqslant_T A^*$ and $A^*$ is $\Sigma^0_3$-complete (since $D_x$ is r.e. uniformly in $x$). $\square$

Theorem 4.12 can be obtained from Theorem 4.13 this way: given the $A_n$ as in Theorem 4.12, suppose we have dropped the representative of $\mathbf{0}'$. Then $W_x \leqslant_m K \Leftrightarrow (\exists n)\ (W_x \leqslant_m A_n)$, so $W_x \equiv_m K \Leftrightarrow (\forall n)\ (W_x \nleqslant_m A_n)$ and hence $K^*$ is $\Pi^0_3$, contradicting Theorem 4.12.

A very interesting characterization of the $m$-degrees has been found by Ershov (see [E4], although there the results are stated for the theory of enumerations). Call $L$ a $c$-uppersemilattice if it is an uppersemilattice with 0, all its principal ideals (in the usual sense) are countable and $L$ is distributive (i.e. $a \leqslant b \cup c \Rightarrow a = b' \cup c'$ for some $b' \leqslant b$, $c' \leqslant c$). $L$ is universal if it has cardinality of the continuum and for every $c$-uppersemilattice $L'$ of cardinality less than the continuum, every ideal $I$ of $L'$ and every isomorphism $\varphi$ of $I$ onto an ideal of $L$, $\varphi$ can be extended to an isomorphism of $L'$ onto an ideal of $L$. Universal $c$-uppersemilattices are unique up to isomorphism.

THEOREM 4.14 (ERSHOV). *The $m$-degrees are a universal $c$-uppersemilattice.*

The proof of this uses Lachlan's characterization of countable ideals of the $m$-degrees (see Theorem 7.4) and it has many interesting corollaries.

THEOREM 4.15. *For any m-degree* **a**, *the m-degrees greater than or equal to* **a** *are isomorphic to the m-degrees (homogeneity property).*

THEOREM 4.16. **0** *is the only m-degree fixed under every automorphism of the m-degrees.*

More generally, an $m$-degree **a** can be mapped into an $m$-degree **b** by an automorphism iff the $m$-degrees less than or equal to **a** are isomorphic to the $m$-degrees less than or equal to **b**. This is true for **a** and **b** minimal, hence *there is at least a continuum of automorphisms of the m-degrees.* A theorem similar to Theorem 4.14 holds for r.e. degrees as well, and its proof relies on Theorem 7.8 (an effectivization of Theorem 7.14).

**5. The structure of 1-degrees.** As for the $m$-degrees, the 1-degrees below $\mathbf{0}'$ are all r.e. The most striking elementary difference between (r.e.) 1-degrees and (r.e.) degrees of another kind is that the l.u.b. does not always exist for 1-degrees.

THEOREM 5.1 (YOUNG). *If $A$ and $B$ are 1-incomparable simple sets, then their 1-degrees have neither a* g.l.b. *nor a* l.u.b.

PROOF. The proof is similar for the two cases. E.g. let us consider the l.u.b. case; note that $A \oplus B$ is above $A$ and $B$ and it is simple, so any l.u.b. must be simple. Let $D$ be the l.u.b.; we find $z \in \bar{D}$ such that $A \leqslant_1 D \cup \{z\}$ and $B \leqslant_1 D \cup \{z\}$, a contradiction because $D \cup \{z\} <_1 D$ since $D$ is simple (see Theorem 3.7). If $x \in A \Leftrightarrow f(x) \in D$ and $x \in B \Leftrightarrow g(x) \in D$ with $f$ and $g$ one-one, note that we can suppose $\overline{f(A)} = \bar{D}$ and $\overline{g(B)} = \bar{D}$. Since $A$ and $B$ are incomparable, we must have $\bar{D} \cap \overline{\text{range} f} \neq \varnothing$ and $\bar{D} \cap \overline{\text{range } g} \neq \varnothing$. If $z \in \bar{D} \cap \overline{\text{range} f}$ then $A \leqslant_1 D \cup \{z\}$ via $f$ itself. Also, let $t \in \bar{D} \cap \overline{\text{range } g}$ and

$$h(x) = \begin{cases} g(x) & \text{if } g(x) \neq z, \\ t & \text{if } g(x) = z. \end{cases}$$

Then $B \leqslant_1 D \cup \{z\}$ via $h'$.

Similarly, for the g.l.b. case, if $D$ is the g.l.b. we find $z \in D$ such that $D - \{z\} \leqslant_1 A$ and $D - \{z\} \leqslant_1 B$. $\square$

Note that, because of the happy fact that $\mathbf{0}'$ is the same for 1-degrees and $m$-degrees, we obtain many results for 1-degrees from what we already know for $m$-degrees. In particular,

THEOREM 5.2. *If* **a** *is an r.e. 1-degree such that* $\mathbf{0} < \mathbf{a} < \mathbf{0}'$, *there is an r.e. 1-degree* **b** *incomparable with* **a**.

PROOF. See Theorem 4.7. $\square$

THEOREM 5.3 (DEKKER). *The r.e. 1-degrees are not dense.*

PROOF. We know that if $A$ is simple and $z \in \bar{A}$ then $A \cup \{z\} <_1 A$. Using Myhill's theorem (see [**Ro**, Theorem 7.6]) on the coincidence between 1-degrees and recursive isomorphism types, between $A \cup \{z\}$ and $A$ there can't be any other 1-degree, since if $C = A \cup \{z\} \leqslant_1 B$ via $f$ and $B \leqslant_1 A$ via $g$, then $gf(\bar{C}) = \bar{A}$ (by simplicity of $A$). $\square$

THEOREM 5.4. *There is no pair of incomparable r.e.* 1*-degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *with* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$.

PROOF. Let $A$ and $B$ be r.e. sets and $C$ be their l.u.b.; since $A \oplus B$ is above $A$ and $B$, it must be $C \leqslant_1 A \oplus B$. If $A \in \mathbf{a}$ and $B \in \mathbf{b}$ and $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$ then $K \leqslant_1 C$. In particular $K \leqslant_m A \oplus B$, so by Theorem 4.8 $K \leqslant_m A$ or $K \leqslant_m B$ and, because of the properties of $K$, $K \leqslant_1 A$ or $K \leqslant_1 B$. $\square$

THEOREM 5.5. *If* $\mathbf{a}$ *is r.e. and* $\mathbf{a} < \mathbf{0}'$, *there is* $\mathbf{b}$ *r.e. such that* $\mathbf{a} < \mathbf{b} < \mathbf{0}'$.

PROOF. Let $A$ be r.e. and in $\mathbf{a}$, and $B$ r.e. incomparable to it (in the case in which $\mathbf{a}$ is nonrecursive); then $A <_1 A \oplus B$ and, as in Theorem 5.4, $A \oplus B <_1 K$. So let $\mathbf{b}$ be the 1-degree of $A \oplus B$. $\square$

We know that density does not hold, but that it does hold toward $\mathbf{0}'$. What about density toward $\mathbf{0}$?

THEOREM 5.6 (LACHLAN **[La2]**). *There is a minimal r.e.* 1*-degree.*

PROOF. Let $A$, $B$ and $C$ be as in Theorem 4.5. There it's proved that if $D$ is r.e. nonrecursive and $D \leqslant_m B$, then $B \leqslant_m D$. To prove that $B \leqslant_1 D$ actually holds it's enough to take $D_1$ and $D_2$ infinite recursive sets such that $D_1 \subseteq D$ and $D_2 \subseteq \overline{D}$ ($D_2$ exists because from above we already know that $D$ is in the same $m$-degree of $B$ and thus, by Theorem 4.5, it's not simple), and use them instead of the fixed $a$ and $b$ in the definition of $g$.

If $x$ shows up first in $B$, let $g(x)$ be the least element of $D_1 - \{g(0), \ldots, g(x-1)\}$.

If $x \in \overline{S} \cap \overline{A}$ or $x$ shows up first in $C$, let $g(x)$ be the least element of $D_2 - \{g(0), \ldots, g(x-1)\}$.

If $x$ shows up first in $S$, we'd like to take as $g(x)$ the least $y$ such that $f(y) = x$, but it could be $y \in \{g(0), \ldots, g(x-1)\}$. When this happens, $y$ must have been defined along one of the two clauses above (since $f(y)$ is unique), so either $y \in D_1 \subseteq D$ or $y \in D_2 \subseteq \overline{D}$; it's enough to take as $g(x)$ a new element of $D_1$ or $D_2$, respectively. $\square$

As a consequence we have the natural examples.

COROLLARY 5.7. *Both halves of a Sacks' splitting of a maximal set have minimal* 1*-degree, and the two halves form a minimal pair of* 1*-degrees.*

We only quote here a recent result of Degtev **[Deg7]**.

THEOREM 5.8 (DEGTEV). *Every r.e. nonrecursive T-degree contains an r.e. minimal m-degree consisting of only one* 1*-degree.*

The following are nice corollaries of the preceding theorem.

(a) Every r.e. nonrecursive $T$-degree contains an r.e. minimal $m$-degree **[La6]**.

(b) Every r.e. nonrecursive $T$-degree contains an r.e. minimal 1-degree.

(c) Every r.e. nonrecursive $T$-degree contains an r.e. $m$-degree consisting of a single 1-degree (Theorem 3.5).

To see how part (b) follows from the theorem let $A$ be in the minimal $m$-degree, $B$ be nonrecursive and $B \leqslant_1 A$; then $A \leqslant_m B$ because $A$ has minimal $m$-degree, so $A \equiv_m B$ and hence $A \equiv_1 B$ because the $m$-degree is actually a 1-degree; so $A \leqslant_1 B$.

THEOREM 5.9. *There is an r.e.* 1-*degree with no minimal predecessor.*

PROOF. Take $A$ simple and use the facts that every r.e. nonrecursive set $B$ such that $B \leqslant_1 A$ is simple, and hence $B$ is not minimal since $B \cup \{z\} <_1 B$ when $z \in \overline{B}$. $\square$

THEOREM 5.10. *The r.e.* 1-*degrees are not recursively enumerable without repetitions.*

PROOF. The proof of Theorem 4.12 doesn't work because there are infinitely many recursive 1-degrees, but since $\mathbf{0}'$ is the same for 1-degrees and $m$-degrees the other proof of it works; from an enumeration without repetitions of the r.e. 1-degrees, we get an enumeration (with repetitions) of the $m$-degrees below $\mathbf{0}'$ and deduce that $K^* \in \Pi_3^0$, as before. $\square$

*Problem* 15. Classify $\{x\colon W_x \equiv_1 A\}$ for $A$ r.e. infinite and coinfinite.

**6. The structure of *tt*-degrees.** The $tt$-degrees give rise to the structure which is the nearest to the one of $T$-degrees, among those we have studied. So it's interesting to see if the methods that are used for the study of $T$-degrees are still valid in this context.

THEOREM 6.1 (MCLAUGHLIN **[McL]**). *There are two r.e. disjoint sets, $A$ and $B$, such that $A \nleqslant_{tt} A \cup B$.*

PROOF. We give the strategy to satisfy the single requirement $\sim(\forall x)(x \in A \Leftrightarrow A \cup B \vDash \sigma_{\varphi_e(x)})$. Pick a witness $x_e \notin A_s \cup B_s$, and suppose that $\varphi_e(x_e)$ is defined. Let $A_{s+1} \cup B_{s+1} = A_s \cup B_s \cup \{x_e\}$. To decide if $x_e \in A_{s+1}$ or $x_e \in B_{s+1}$, see if $A_{s+1} \cup B_{s+1} \vDash \sigma_{\varphi_e(x_e)}$ or not; if yes, let $x_e \in B_{s+1}$; if not, let $x_e \in A_{s+1}$. Restrain the numbers used negatively in the computation. $\square$

In the case of $T$-degrees, *if $A$ and $B$ are disjoint r.e. sets then $A \cup B \equiv_T A \oplus B$*, so if $\mathbf{a}$ and $\mathbf{b}$ are the $T$-degrees of $A$ and $B$, $\mathbf{a} \cup \mathbf{b}$ is the $T$-degree of $A \cup B$. This is useful for example to deduce the decomposition theorem for r.e. degrees from Sacks' decomposition theorem for r.e. sets. *The result is not true anymore for tt-degrees*, since for $A$ and $B$ as in the previous theorem $A \nleqslant_{tt} A \cup B$ but, by definition, $A \leqslant_{tt} A \oplus B$; so $A \cup B \not\equiv_{tt} A \oplus B$.

*If $A$ and $C$ are r.e. and for every $x$, $x \in A_{s+1} - A_s \Rightarrow (\exists y \leqslant x)(y \in C_{s+1} - C_s)$ then $A \leqslant_T C$. This is called the permitting method*, since we construct $A$ by putting elements in it only when they are permitted by elements of $C$, and it's applied when we know how to construct an r.e. set $A$ with certain properties and we want to push $A$ below a given r.e. set $C$. It is clear that $A \leqslant_T C$. To see if $x \in A$ it's enough to look at the stages in which something less than or equal to $x$ is enumerated in $C$, and see if $x$ is enumerated in $A$ at some of those stages. *The permitting doesn't give in general $A \leqslant_{tt} C$*; let $A$ and $B$ be as in the previous theorem, and $C = A \cup B$. We have $x \in A_{s+1} - A_s \Rightarrow x \in C_{s+1} - C_s$ (since $A$ and $B$ are disjoint) but $A \nleqslant_{tt} C$.

Recall that **[Ro**, Theorem 10.5] every r.e. nonrecursive $T$-degree contains a recursively inseparable pair of r.e. sets.

THEOREM 6.2. *There are r.e. nonrecursive tt-degrees containing no recursively inseparable pair of r.e. sets.*

PROOF. Take the $tt$-degree of a hypersimple set and apply Theorem 2.4. □

Coming back for a moment to the argument studied in §2, we know that

(a) not every r.e. nonrecursive $m$-degree contains a simple set,

(b) not every r.e, nonrecursive $tt$-degree contains a hypersimple set, but every r.e. nonrecursive $T$-degree does.

It would be nice if every r.e. nonrecursive $tt$-degree contained a simple set. Degtev **[Deg4]** proved that this is not true for simple nonhypersimple sets, and Jockusch proved the same for simple sets.

THEOREM 6.3 (JOCKUSCH). *Not every r.e. nonrecursive tt-degree contains a simple set.*

PROOF. We want to construct $A$ r.e. nonrecursive and, for all $e$, an r.e. set $V_e$ s.t.

$$A \equiv_{tt} W_e \Rightarrow V_e \text{ infinite and } V_e \cap W_e \text{ finite.}$$

In particular, for $A = W_e$ we will have that $V_e$ is contained in $\bar{A}$ (except for a finite part) so $A$ is not simple. The requirements for this are, for all $f$ and $g$ recursive and all $e$,

$$\left.\begin{array}{ll}(\forall x) & (x \in A \Leftrightarrow W_e \vDash \sigma_{f(x)}) \\ (\forall x) & (x \in W_e \Leftrightarrow A \vDash \sigma_{g(x)})\end{array}\right\} \Rightarrow V_e \text{ infinite, } V_e \cap W_e \text{ finite.}$$

We give the strategy for a single requirement. Let $\{Z_i\}_{i\in\omega}$ be a partition of $\omega$ into infinite recursive sets. We allow only elements of $\bigcup_{i\geqslant e} Z_i$ to satisfy the requirements for $e$. At stage $s$ we simply assume that $\bigcup_{i>e} Z_i \subseteq A$, i.e. instead of $A_s$ we consider (in analogy to what we have done in Theorem 2.4) $A_s^* = A_s \cup \bigcup_{i>e} Z_i$. If in the construction we will use some element of $A_s^*$ which is not in $A_s$, we will put it into $A$. Hence the crucial column is $Z_e$.

(1) We first see if we can spoil $W_e \leqslant_{tt} A$ via $g$, i.e. if at some stage $s$ for some $x$ we have $x \in W_{e,s} \wedge A_s^* \nvDash \sigma_{g(x)}$, we put in $A_{s+1}$ the elements of $A_s^*$ used in the computation. If nothing from $Z_e$ and used enters $A$ afterwards, then $W_e \nleqslant_{tt} A$ via $g$.

(2) If no such $x$ exists, then we try to use the definition of $W_e$ in terms of $A$ (i.e. $x \in W_e \Leftrightarrow A \vDash \sigma_{g(x)}$) to build $V_e \subseteq \overline{W_e}$ (note that this really gives $V_e \cap W_e = \emptyset$, but with the other requirements in the game we will only have $V_e \cap W_e$ finite). We therefore define $x \in V_e \Leftrightarrow (\exists s)(A_s^* \nvDash \sigma_{g(x)})$ so that $V_e$ is r.e. and disjoint from $W_e$ (since we are not in Case 1). If in the end $V_e$ is infinite, we have satisfied the requirement.

(3) If $V_e$ is finite, we spoil $A \leqslant_{tt} W_e$ via $f$. We want to diagonalize using an element $u \in Z_e - A_s$. Let $z > \max V_e$.

Note that $w \geqslant z \Rightarrow w \notin V_e \Rightarrow (\forall s)(A_s^* \vDash \sigma_{g(w)}) \Rightarrow A \vDash \sigma_{g(w)}$. We want to preserve the computations also when $w < z$, so we first choose $u$ greater than all the elements used in $\sigma_{g(w)}$ for some $w < z$. The idea is now to have $u \in A \not\Leftrightarrow W_e \vDash \sigma_{f(u)}$. To make sure that the value we read on the right-hand side is correct, we take $t_0$ greater than all the elements used in $\sigma_{f(u)}$ and for all $w < t_0$ we preserve the computation relative to $\sigma_{g(w)}$. Since $W_e \leqslant_{tt} A$ via $g$, this freezes $W_e$ up to $t_0$ and insures that the output of $W_e \vDash \sigma_{f(u)}$ is the correct one.

Then we put $u$ in $A$ iff $W_e \nvDash \sigma_{f(u)}$.

This is the idea. What we practically do is to wait for a stage $t > s$ such that $W_{e,t} \nvDash \sigma_{f(u)}$ and

$$w < z \Rightarrow \left(w \in W_{e,t} \Leftrightarrow A_t^* \vDash \sigma_{g(w)}\right), \qquad z \leqslant w < t_0 \Rightarrow w \in W_{e,t}.$$

Then we have apparently $W_e \leqslant_{tt} A$ up to $t_0$, but we spoil $A \leqslant_{tt} W_e$ by putting $u$ into $A$. □

We now begin the study of (r.e.) $tt$-degrees. Since there exist minimal $T$-degrees below $\mathbf{0}'$ and no r.e. $T$-degree is minimal (not even in the r.e. $T$-degrees alone, by Sacks' splitting theorem), for $T$-degrees the r.e. degrees and the degrees below $\mathbf{0}'$ are distinct, and their theories are not elementarily equivalent.

THEOREM 6.4. *There are tt-degrees below* $\mathbf{0}'$ *without r.e. sets.*

PROOF. Let's derive the result from the following theorem of Cooper: there are two r.e. sets $A$ and $B$ such that the $T$-degree of $A - B$ doesn't contain r.e. sets. In fact $A - B \leqslant_{tt} K$ since $A$ and $B$ are r.e. We define $B \subseteq A$ such that for all $x$, $m$, $n$,

$$A - B \neq \varphi_m^{W_x} \quad \text{or} \quad W_x \neq \varphi_n^{A-B}.$$

We give the strategy for a single requirement. Let $B_s \subseteq A_s$ and pick the witness $a \notin A_s$. Wait until a stage $t$ in which $\varphi_{m,t}^{W_{x,t}}(a) = 0$ (if it never comes, then $A - B \neq \varphi_m^{W_x}$). We want to satisfy the other condition. Note that the value of $\varphi_m^{W_x}(a)$ at stage $t$ can change only if there is some $u \in W_x - W_{x,t}$ used in the computation. For every used $u \notin W_{x,t}$ we see if $\varphi_n^{A_t - B_t}(u) = 0$ (otherwise one of them witnesses $W_x \neq \varphi_n^{A-B}$). If yes, let $a \in A_{t+1}$ and restrain all other numbers used to compute $\varphi_n^{A_t - B_t}(u) = 0$ for all such $u$'s. If no one of those $u$ is in $W_x$, $\varphi_m^{W_x}(a)$ at stage $t$ is final so (since we put $a$ into $A$ but not in $B$) $A - B \neq \varphi_m^{W_x}$. Otherwise, let $u$ and $t_0$ be such that $u \in W_{x,t_0} - W_{x,t}$; we'd like to have $u$ as witness of $W_x \neq \varphi_n^{A-B}$. We knew that $\varphi_n^{A_t - B_t}(u) = 0$ (at stage $t$) and now $u \in W_{x,t_0}$. The problem is that now $a \in A_{t+1}$ and this could have changed the computation. But it's enough to put $a$ in $B$ so that (if nothing else happened) $A - B$ looks the same now as it was at stage $t$. □

*Problem* 16. Are the theories of the ordering of $tt$-degrees below $\mathbf{0}'$ and of r.e. $tt$-degrees elementarily equivalent?

THEOREM 6.5 (DEGTEV [**Deg5**], MARCHENKOV [**Ma1**]). *There are r.e. tt-degrees minimal among all the tt-degrees.*

SKETCH OF PROOF. KOBZEV [**Ko4**] has proved that the $tt$-degree of an $\eta$-maximal semirecursive set has the property. □

It's interesting to note that $\eta$-maximal sets have minimal $m$-degrees (this is the obvious extension of Theorem 4.1, see [**E3**]) and $\eta$-maximal semirecursive sets have minimal $tt$-degrees. Since Degtev [**Deg9**] has proved that an r.e. set with r.e. minimal $tt$-degree cannot be high, this together with the Kobzev result quoted previously gives a new proof of the fact that $\eta$-maximal semirecursive sets are a class of $T$-incomplete r.e. sets.

Because of the existence of r.e. minimal $tt$-degrees, the analogue of Sacks' splitting theorem doesn't hold for r.e. $tt$-degrees (of course it's still true that every r.e. nonrecursive set is the union of two r.e. $tt$-incomparable disjoint sets, but we saw in the remarks after Theorem 6.1 why this doesn't imply the result for $tt$-degrees). There is one particular case of the splitting theorem which happens to be true.

THEOREM 6.6 (THE DIAMOND THEOREM). *There are two r.e. incomparable $tt$-degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$ *and* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$.

PROOF. Suppose we have two r.e. disjoint sets $A$ and $B$ such that $A \cup B = K$. In this case $A \cup B \equiv_{tt} A \oplus B$ since $A \cup B \leqslant_{tt} A \oplus B$ is true in general $(x \in A \cup B \Leftrightarrow \{2x, 2x+1\} \cap A \oplus B \neq \varnothing)$ and $A \oplus B$ is r.e. so $A \oplus B \leqslant_{tt} K = A \cup B$. So it's enough to construct $A$ and $B$ nonrecursive as before, and such that for all $f$ and $g$ recursive

$$\left.\begin{array}{ll} (\forall x) & (x \in C \Leftrightarrow A \vDash \sigma_{f(x)}) \\ (\forall x) & (x \in C \Leftrightarrow B \vDash \sigma_{g(x)}) \end{array}\right\} \Rightarrow C \text{ recursive.}$$

The proof is in the style of Sacks' splitting theorem, and the strategy for a single negative requirement as before is this: if at a certain stage $s$ we find an $x$ such that $(\forall y < x)[A_s \vDash \sigma_{f(y)} \Leftrightarrow B_s \vDash \sigma_{g(y)}]$ but $A_s \vDash \sigma_{f(x)} \Leftrightarrow B_s \nvDash \sigma_{g(x)}$ then we try to preserve this situation on one side, restraining (e.g. out of $A$) the elements used negatively in the computations (relative to $A$). If the $B$-side does not change after $s$ then this will work. Otherwise, we will attack the condition once again and preserve a longer initial segment of the computation on the $A$-side. Note that if such an $x$ does not exist, then $C$ is recursive; to see if $x \in C$ it's enough to see if $A_s \vDash \sigma_{f(x)}$ at a stage $s$ after which there is no injury anymore. □

As it is well known, the corresponding result for r.e. $T$-degrees is false; see [**Sol**, Theorem 14.4]. It follows in particular from the theorem that there is an r.e. minimal pair of $tt$-degrees. This is true for $T$-degrees too (see [**Sol**, Theorem 14.1]), but the proof is much more difficult because once a computation is destroyed it can become undefined, whereas $tt$-reductions are total. It is possible to give natural examples of $tt$-minimal pairs.

THEOREM 6.7 (DEGTEV [**Deg4**]). *Every pair of $T$-incomparable coretraceable hypersimple sets is a $tt$-minimal pair.*

PROOF. It's enough to prove that if $A$ is coretraceable hypersimple and $B$ is nonrecursive, then $B \leqslant_{tt} A \Rightarrow B \equiv_T A$. Let $x \in B \Leftrightarrow A \vDash \sigma_{f(x)}$ and $\bar{A} = \{a_0, a_1, \ldots\}$ in order of magnitude. Since $\bar{A}$ is retraceable, knowing $a_n \in \bar{A}$ we know $\bar{A}$ exactly up to $a_n$ (using the retracing function). To have $A \leqslant_T B$ we majorize $\bar{A}$ recursively in $B$. We suppose we have $a_n$ and find, recursively in $B$, a number $\geqslant a_{n+1}$ (we won't have exactly $a_{n+1}$ for the next step, but we know that it's a number between $a_n$ and the bound found for $a_{n+1}$, so for every one of these numbers we will apply the procedure and then we will take the biggest one of the bounds so found). Given $x$, we can suppose that we use only elements $> a_n$ in $\sigma_{f(x)}$. Note that there exists an $x$ such that $x \in B \wedge \omega \nvDash \sigma_{f(x)}$ or $x \in \bar{B} \wedge \omega \vDash \sigma_{f(x)}$ (otherwise $B$ would be recursive since $x \in B \Leftrightarrow \omega \vDash \sigma_{f(x)}$). Since $x \in B \Leftrightarrow A \vDash \sigma_{f(x)}$, this means that an element $\geqslant a_{n+1}$ of $\bar{A}$ has been used in $\sigma_{f(x)}$, and it's enough to take the greatest element used. $\square$

Another consequence of Theorem 6.5 is of course the nondensity of r.e. $tt$-degrees. Density holds towards $\mathbf{0}'$.

THEOREM 6.8 (KALLIBEKOV). *If* $\mathbf{a} < \mathbf{0}'$ *is an r.e. tt-degree, there is an r.e. tt-degree* $\mathbf{b}$ *such that* $\mathbf{a} < \mathbf{b} < \mathbf{0}'$.

Actually Kallibekov **[Ka2]** proved much more: if $\mathbf{a}$ is as above, there is an independent set of r.e. $tt$-degrees between $\mathbf{a}$ and $\mathbf{0}'$, so every countable partial ordering is embeddable in the r.e. $tt$-degrees between $\mathbf{a}$ and $\mathbf{0}'$. It is known that for every r.e. $T$-degree between $\mathbf{0}$ and $\mathbf{0}'$ there is an r.e. $T$-degree incomparable with it. The result is true of r.e. $tt$-degrees too, but it doesn't follow from the one for $T$-degrees, since there are r.e. $tt$-degrees $\mathbf{a}$ $T$-complete such that $\mathbf{0} < \mathbf{a} < \mathbf{0}'$ (e.g. let $\mathbf{a}$ be the $tt$-degree of a $T$-complete hypersimple set).

THEOREM 6.9 (DENISOV). *If* $\mathbf{a}$ *is an r.e. tt-degree such that* $\mathbf{0} < \mathbf{a} < \mathbf{0}'$, *there is an r.e. tt-degree* $\mathbf{b}$ *incomparable with* $\mathbf{a}$.

We can ask if, as in the case of r.e. $m$-degrees, it's always possible to choose $\mathbf{b}$ to be a minimal r.e. $tt$-degree. The answer is no, because Marchenkov **[Ma2]** has proved that *the r.e. minimal tt-degrees are bounded below* $\mathbf{0}'$. Both Theorems 6.8 and 6.9 are corollaries of the following.

THEOREM 6.10 (DEGTEV **[Deg9]**). *If $A$ and $B$ are r.e. sets such that $A \nleqslant_{tt} B$ and $A <_{tt} K$, there is an r.e. set $C$ such that $B <_{tt} C$ and $A$ and $C$ are incomparable.*

PROOF. Theorem 6.9 follows taking $B$ recursive.

Theorem 6.8 follows (for $B$ nonrecursive) by taking $A$ incomparable with $B$. To prove Theorem 6.10, first of all we want $B \leqslant_{tt} C$; this is easily achieved by letting $2x$ enter $C$ if $x \in B$. All the other requirements will only do something on the odd numbers. Our goal is to have $A \nleqslant_{tt} C$, $C \nleqslant_{tt} A$ and $C \nleqslant_{tt} B$. Let $\{Z_i\}_{i\in\omega}$ be a partition of the odd numbers into infinite recursive sets.

To satisfy $C \nleqslant_{tt} A$ and $C \nleqslant_{tt} B$, given $e$, we put at stage $s$ into $C$ the $i$th element of $Z_e$ if $i$ is in $K_s$ and is less than the maximum of the length of agreement between $C_s$ and the $e$th truth-table reduction relative to $A_s$ or $B_s$ (i.e. less than the greatest $x$ such that for every $y < x$ is $y \in C_s \Leftrightarrow A_s \vDash \sigma_{\varphi_e(y)}$ with $\varphi_e(y)$ computed in less than $s$ steps, and similarly for $B_s$). Note that if $C \leqslant_{tt} A$ or $C \leqslant_{tt} B$ via $\varphi_e$, then at least one of the lengths of agreement goes to infinity, so we code $K$ into $Z_e \cap C$ and this is impossible because then $K \leqslant_{tt} C$, hence $K \leqslant_{tt} A$ or $K \leqslant_{tt} B$ against the hypothesis (the last one because otherwise $A \leqslant_{tt} B$).

To satisfy $A \nleqslant_{tt} C$, given $e$, at stage $s$ we look for the least $x$ such that $(\forall y \leqslant x)(y \in A_s \Leftrightarrow C_s \vDash \sigma_{\varphi_e(y)})$ and something in $\bigcup_{i>e} Z_i \cap \bar{C}_s$ is used in $\sigma_{\varphi_e(x)}$, and put the odd elements used in $\sigma_{\varphi_e(x)}$ and which are in $\bigcup_{i>e} Z_i$ into

$C$. If $A \leqslant_{tt} C$ via $\varphi_e$ and $e$ is minimal, then $A \leqslant_{tt} B$ because given $x$ we can test $C \models \sigma_{\varphi_e(x)}$; for the even elements we are reduced to $B$ (since $2x \in C \Leftrightarrow x \in B$), and the odd elements in $\bigcup_{i>e} Z_i$ used in any computation are in $C$. So we only need to know $\bigcup_{i \leqslant e} Z_i \cap C$, which is (by the minimality of $e$) a finite set. □

THEOREM 6.11 (DEGTEV **[Deg5]**, KOBZEV **[Ko5]**). *The (r.e.) tt-degrees are not a lattice.*

SKETCH OF PROOF. The two sets considered in Theorem 4.10 are already the wanted counterexample. Degtev **[Deg5]** has proved that if an r.e. set is $tt$-reducible to them, it's already $m$-reducible to them, and Kobzev **[Ko5]** has proved that below them (in the sense of $tt$-reducibility) there are only r.e. sets. □

*Problem* 17. Does every pair of r.e. sets have g.l.b. in the r.e. $tt$-degrees iff it has g.l.b. in the $tt$-degrees (and, if yes, do the two g.l.b. coincide)?

For $T$-degrees the answer (to both questions) is yes (Lachlan).

THEOREM 6.12 (DEGTEV, KOBZEV). *There is an r.e. tt-degree with no minimal (r.e.) predecessor.*

SKETCH OF PROOF. Apply **[Deg5]** and **[Ko5]** to the set considered in Theorem 4.11. □

Degtev **[Deg9]** has recently proved that for every r.e. nonrecursive $tt$-degree $\mathbf{a} < \mathbf{0}'$ there is an r.e. $tt$-degree $\mathbf{b}$ with no minimal r.e. predecessor which is incomparable with $\mathbf{a}$, thus combining Theorem 6.9 and (part of) 6.12.

THEOREM 6.13. *The r.e. tt-degrees are not recursively enumerable without repetitions.*

PROOF. As in Theorem 4.12. □

THEOREM 6.14. *If $A$ is r.e. then $\{x\colon W_x \equiv_{tt} A\}$ is $\Sigma^0_3$-complete.*

PROOF. Simply check that the set is $\Sigma^0_3$, since the proof of Theorem 4.13 already gives a reduction of every $\Sigma^0_3$ set to it. □

For lack of a better place, we quote here a last connection between $tt$- and $m$-reducibility: if $A$ and $B$ are maximal sets and $A \equiv_{tt} B$, then $A \equiv_m B$ (Kobzev, see **[Ko2]**). Since Lerman **[Le]** has proved that every $T$-degree containing a maximal set contains infinitely many incomparable $m$-degrees of maximal sets, it follows that every such $T$-degree contains infinitely many incomparable $tt$-degrees of maximal sets.

**7. Elementary equivalence and undecidability.** What we have done before allows us already to say that

THEOREM 7.1. *The orderings of r.e. T-degrees, tt-degrees, m-degrees and 1-degrees are pairwise nonelementarily equivalent.*

PROOF. 1-degrees differ from all the others because they aren't an upper semilattice. $m$-degrees differ from $tt$-degrees and $T$-degrees because there aren't incomparable $m$-degrees with l.u.b. $\mathbf{0}'$. $tt$-degrees differ from $T$-degrees because there are r.e. minimal *tt*-degrees. Of course all these differences can easily be expressed in the first order language of the theory of partial orderings (with least and greatest elements). □

Before turning to the question of decidability for any one of the prior theories, we have to quote some more results. We will not prove them, since the proofs require cumbersome constructions, so we feel free to quote the strongest results. Let's start with the *T*-degrees.

THEOREM 7.2 (LACHLAN, LEBEUF). *For every partially ordered set P such that*

(a) *P is countable,*

(b) *P has a least and a greatest element,*

(c) *any two elements of P have an l.u.b.,*

*there is a T-degree* **a** *such that the T-degrees less than or equal to* **a** *are order-isomorphic to P.*

This result (see **[LL]**) is just the latest one in a series of results on initial segments for *T*-degrees, whose three most important steps were the construction of a minimal *T*-degree (Spector), the theorem where *P* is a countable distributive lattice (Lachlan) and an arbitrary finite lattice (Lerman). Since the conditions (a), (b) and (c) are clearly necessary for initial segments with greatest element, the result is the final one for this kind of initial segment. Rubin **[Ru]** has many results on initial segments without greatest elements, e.g. he constructs an initial segment whose order type is $\aleph_1$.

THEOREM 7.3. *Every countable partial ordering with greatest element which is realizable as initial segment of the T-degrees is (simultaneously) realizable as initial segment of the tt-degrees.*

The result is achieved by making the top *T*-degree hyperimmune-free (see **[NS]**. Actually this is really automatic in the standard proof of Theorem 7.2). So every *T*-degree below the top one is hyperimmune-free itself, and hence consists of a single *tt*-degree.

For *m*-degrees the situation is not so liberal.

THEOREM 7.4 (LACHLAN **[La3]**). *Given a set P partially ordered by* $\leqslant_P$, *there exists an m-degree* **a** *such that the m-degrees less than or equal to* **a** *(and different from* $\{\phi\}$ *and* $\{\omega\}$*) are isomorphic to P iff*

(a) *P has a least and a greatest element* (0 *and* 1),

(b) *every finite subset of P containing* 0 *and* 1 *and closed under l.u.b. is a finite distributive lattice, i.e. if* $0 \neq 1$ *there exists a sequence* $\{P_n\}_{n\in\omega}$ *of finite distributive lattices such that for every n*

- $P_n \subseteq P_{n+1}$,
- 0 *and* 1 *are in* $P_n$ *and are the least and the greatest elements of it,*
- *if* $x, y \in P_n$ *and z is the l.u.b. of x and y in* $P_n$, *then z is the l.u.b. of x and y in* $P_m$, *for every* $m \geqslant n$ *and* $P = \bigcup_{n\in\omega} P_n$.

In particular the finite initial segments of the *m*-degrees are exactly the finite distributive lattices (with least and greatest element). Lachlan **[La4]** has also proved that every countable partial ordering that can be represented as an initial segment of the $m$-degrees, can be simultaneously represented as an initial segment of the ($tt$- and) $T$-degrees. In particular there is a set whose $T$-degree and $m$-degree are minimal.

For 1-degrees the situation is not quite clear.

THEOREM 7.5 (LACHLAN **[La2]**). *Every partial ordering which is realizable as an initial segment of the $m$-degrees, is (simultaneously) realizable as an initial segment of the* 1*-degrees, but not conversely.*

*Problem* 18. Characterize the countable initial segments of the 1-degrees.

Some necessary conditions are known (see **[La2]**), e.g. every finite initial segment is a lattice of a certain kind, but the situation seems complicated since some nondistributive finite lattices are not isomorphic to initial segments, but some others are.

A consequence of the previous results is

THEOREM 7.6. *The orderings of $T$-degrees, $tt$-degrees, $m$-degrees and* 1*-degrees are pairwise nonelementarily equivalent (except possibly $T$-degrees and $tt$-degrees).*

PROOF. 1-degrees differ from all the others because they aren't an upper semilattice. $m$-degrees differ from $tt$-degrees and $T$-degrees because they don't admit nondistributive finite lattices as initial segments. □

*Problem* 19. Are the orderings of $T$-degrees and $tt$-degrees distinct (nonelementarily equivalent)?

Now we turn to the question of decidability.

THEOREM 7.7 (LACHLAN). *The theories of the orderings of $T$-degrees, $tt$-degrees, $m$-degrees and* 1*-degrees are undecidable and nonaxiomatizable.*

SKETCH OF PROOF. Every finite distributive lattice is embeddable as an initial segment, and the sets of first order sentences true of no distributive lattice and true of some finite distributive lattice are recursively inseparable.

The only problem may arise for 1-degrees, since what we called initial segments are really initial segments above $\mathbf{0}$, the 1-degree of the infinite coinfinite recursive sets. But $\mathbf{0}$ is definable, since so are $\{\varnothing\}$ and $\{\omega\}$ (the only two minimal 1-degrees in the real sense) and $\mathbf{0}$ is the least 1-degree above both of them. □

Nerode and Shore **[NS]** have recently proved that the four prior theories are recursively isomorphic to each other and to second order arithmetic.

We end by looking at the r.e. degrees. Since no r.e. $T$-degree is minimal, there is no proper initial segment for r.e. $T$-degrees.

*Problem* 20. Characterize the finite (countable) initial segments of the r.e. $tt$-degrees.

THEOREM 7.8 (LACHLAN **[La5]**). *Given a set $P$ partially ordered $\leqslant_p$, there exists an r.e. $m$-degree $\mathbf{a}$ such that the $m$-degrees less than or equal to $\mathbf{a}$ are isomorphic to $P$ iff*

(a) *$P$ has a least and a greatest element* (0 *and* 1),

(b) *every finite subset of $P$ containing* 0 *and* 1 *and closed under l.u.b. is a finite "effective" distributive lattice, i.e. if* $0 \neq 1$ *there exists a sequence* $\{P_n\}_{n \in \omega}$ *like the one in Theorem* 7.4 *such that if* $\leqslant_n$ *is the order of* $P_n$ *then*

- *the relation* $x \in P_n$ (*of* $x$ *and* $n$) *is r.e.*,
- *the relation* $x \leqslant_n y$ (*of* $x, y$ *and* $n$) *is* $\Pi^0_2$,
- *the operations of union and intersection in* $P_n$ *are recursive* (*uniformly in* $n$).

In particular the finite initial segments of the r.e. $m$-degrees are still exactly the finite distributive lattices.

THEOREM 7.9 (LACHLAN [**La2**]). *The finite initial segments* (*with greatest element*) *of the r.e.* 1-*degrees are exactly the finite distributive lattices* (*with least and greatest elements*).

So for r.e. 1-degrees the situation is better than for general 1-degrees. As usual, every finite distributive lattice can be simultaneously realized as an initial segment of the r.e. $m$-degrees and 1-degrees [**La2**].

*Problem* 21. Characterize the countable initial segments of the r.e. 1-degrees.

As in Theorem 7.7 we obtain

THEOREM 7.10 (LACHLAN). *The theories of the orderings of r.e. $m$-degrees and* 1-*degrees are undecidable and nonaxiomatizable.*

*Problem* 22. Are the theories of the orderings of r.e. $t$-degrees and $T$-degrees decidable?

Let us quote, to finish, a last result. V′yugin [**V**] has proved that Theorem 7.8 is true not only if we consider initial segments, but also in general for segments above an r.e. $m$-degrees $\mathbf{b} < \mathbf{0}'$, in the sense that given $P$ as in the theorem there exists an r.e. $m$-degree $\mathbf{a}$ such that for all $m$-degrees $\mathbf{c}$, $\mathbf{c} < \mathbf{a} \Rightarrow \mathbf{c} < \mathbf{b}$ or $\mathbf{b} < \mathbf{c}$ and the $m$-degrees between $\mathbf{b}$ and $\mathbf{a}$ are isomorphic to $P$. This gives the usual undecidability result for the r.e. $m$-degrees above $\mathbf{b}$.

**8. Where we end: variations on the theme of *tt*-reducibility.** In this final chapter we introduce two reducibilities that have been studied less than the others we have been concerned with previously. Both are variations of the concept of *tt*-reducibility. They are

(a) *bounded truth-table reducibility*: $A \leqslant_{btt} B$ if there is a recursive function $f$ and a number $n$ such that for all $x$, $x \in A \Leftrightarrow B \models \sigma_{f(x)}$ and $\sigma_{f(x)}$ uses at most $n$ elements ($n$ is called the norm of the *btt*-reduction).

(b) *Weak truth-table reducibility*: $A \leqslant_{wtt} B$ if there is a number $e$ and a recursive function $f$ such that $A = \varphi_e^B$ and $\varphi_e^B(x)$ uses the oracle $B$ to answer questions only about elements less than $f(x)$.

Note that $m$-reducibility is an example of *btt*-reducibility with norm 1. So

$$A \leqslant_m B \Rightarrow A \leqslant_{btt} B \Rightarrow A \leqslant_{tt} B.$$

If $A \leqslant_{tt} B$ via $g$ then we only use–to see if $x \in A$–the oracle $B$ on elements less than 1 + the maximum element used in $\sigma_{f(x)}$. So

$$A \leqslant_{tt} B \Rightarrow A \leqslant_{wtt} B \Rightarrow A \leqslant_T B.$$

Now $\leqslant_{wtt}$ is called weak truth-table reducibility because it is like $tt$-reducibility in the sense that we have the bound on the elements used by the oracle, but it is weaker in the sense that we don't know in advance how these elements are going to be used. For obvious reasons, $\leqslant_{wtt}$ is also called bounded $T$-reducibility.

As usual we have the notion of degree, in particular of $\mathbf{0}$ (in both cases consisting of all the recursive sets) and $\mathbf{0}'$. We will see that no one of the prior implications among the two new reducibilities and the old ones can be reversed, not even on the r.e. sets.

Let's begin with $\leqslant_{btt}$. In general $\leqslant_m$ and $\leqslant_{btt}$ of norm 1 do not coincide, since always $A \leqslant_{btt} \bar{A}$ but e.g. $K \nleqslant_m \bar{K}$ (otherwise also $\bar{K} \leqslant_m K$ and $\bar{K}$ is r.e.). But they do coincide for r.e. sets.

THEOREM 8.1. *If $A$ and $B \neq \varnothing, \omega$ are r.e. and $A \leqslant_{btt} B$ with norm* 1, *then $A \leqslant_m B$.*

PROOF. We have a recursive function $f$ such that for all $x$ either $x \in A \Leftrightarrow f(x) \in B$ or $x \in A \Leftrightarrow f(x) \in \bar{B}$ (and, given $x$, we know which one of the two cases happens). We want a recursive $g$ such that for all $x$ $x \in A \Leftrightarrow g(x) \in B$. Let $b_0 \in B$ and $b_1 \in \bar{B}$. If $x \in A \Leftrightarrow f(x) \in B$ let $g(x) = f(x)$. If $x \in A \Leftrightarrow f(x) \in \bar{B}$ then one and only one of $x \in A$ and $f(x) \in B$ happens, so we know which one (generate $A$ and $B$ until the answer comes). If $x \in A$ let $g(x) = b_0$. If $f(x) \in B$ let $g(x) = b_1$. □

When $A \leqslant_{btt} B$ it's possible (using a bigger norm) to obtain a $btt$-reduction that uses only a fixed propositional formula (with a fixed number of elements), so that a $btt$-reduction really associates to every $x$ a set $\{b_1^x, \ldots, b_n^x\}$ of elements (for some fixed $n$) such that the answer to $x \in A$ depends only on the truth-value of $b_i^x \in B$ and not on the elements $b_i^x$ themselves. We know that an $m$-complete set is not simple. Post has generalized this as follows: a $btt$-complete set is not simple. Here is the usual strengthening of this fact.

THEOREM 8.2 (KOBZEV [**Ko1**]). *$B \leqslant_{btt} A$ and $B$ part of an r.i. pair of r.e. sets $\Rightarrow A$ nonsimple.*

PROOF. We prove, by induction on the norm $n$, the theorem and a lemma. The result for $n = 1$ comes from Theorems 8.1 and 2.2a; the lemma simply says that an r.e. subset of $\bar{A}$ immune is finite.

For $n > 1$, let $B \leqslant_{btt} A$ and $\{b_1^x, \ldots, b_n^x\}$ be the elements associated with $x$. Since only their membership in $A$ or $\bar{A}$ matters, we have two cases: either for every $x$, $\{b_1^x, \ldots, b_n^x\} \subseteq \bar{A} \Rightarrow x \in B$ or for every $x$, $\{b_1^x, \ldots, b_n^x\} \subseteq A \Rightarrow x \in \bar{B}$. Suppose the first (the other is analogous). If $B$ and $C$ are r.i. then

$$x \in C \Rightarrow \{b_1^x, \ldots, b_n^x\} \cap \bar{A} \neq \varnothing.$$

If $A$ were simple then for some finite set $F \subseteq \bar{A}$ we would have

$$x \in C \Rightarrow \{b_1^x, \ldots, b_n^x\} \cap F \neq \varnothing.$$

(To see this consider those $x$ such that $x \in C \wedge b_n^x \in A$. They are an r.e. set, so by the inductive hypothesis there is a finite set $F_0$ such that $\{b_1^x, \ldots, b_{n-1}^x\} \cap F_0 \neq \varnothing$, when $x$ is as before. But the $b_n^x$'s such that $\{b_1^x, \ldots, b_{n-1}^x\} \cap F_0 = \varnothing$ are an r.e. set $F_1$ contained in $\bar{A}$, hence $F_1$ is finite. Let $F = F_0 \cup F_1$.)

Let $x \in R \Leftrightarrow \{b_1^x, \ldots, b_n^x\} \cap F \neq \varnothing$; $R$ is recursive and $C \subseteq R$. Also, $B \cap R$ and $C$ are r.i. (if $B \cap R \subseteq D$ and $C \subseteq \bar{D}$ then $D \cup \bar{R}$ separates $B$ and $C$). We split $R$ into the $n$ recursive parts

$$x \in R_1 \Leftrightarrow x \in R \wedge b_1^x \in F, \qquad x \in R_2 \Leftrightarrow x \in R - R_1 \wedge b_2^x \in F, \text{ etc.}$$

Now for at least one $i$, $B \cap R_i$ and $C \cap R_i$ are r.i., but $B \cap R_i \leqslant_{btt} A$ with norm $n - 1$ (we dispensed with $b_i^x$ since $b_i^x \in F \subseteq \bar{A}$), and by the inductive hypothesis $A$ is not simple. $\square$

It immediately follows, since there is a $tt$-complete simple set, that $tt$-completeness and $btt$-completeness do not coincide. Young has also proved that $btt$-completeness and $m$-completeness do not coincide. Both these results (and part of Theorem 3.11) are generalized by the following theorem, which is interesting in that it gives some more information on the structure of the r.e. $T$-complete sets.

THEOREM 8.3 (KALLIBEKOV **[Ka2]**). (a) *The complete btt-degree contains infinitely many r.e. m-degrees.*

(b) *The complete tt-degree contains infinitely many r.e. btt-degrees.*

PROOF. (a) We construct a set $\{A_i\}_{i \in \omega}$ of r.e. sets such that for all $i$, $K \leqslant_{btt} A_i$ and for all $i \neq j$, $A_i \nleqslant_m A_j$.

To have $K \leqslant_{btt} A_i$ we want $x \in K \Leftrightarrow \{2x, 2x + 1\} \cap A_i \neq \varnothing$. If at stage $s$, $x \in K_s$ and $\{2x, 2x + 1\} \cap A_{i,s} = \varnothing$ then we put in $A_i$ the first element among $2x$ and $2x + 1$ that is not restrained. If they are both restrained, we leave out of $A_i$ the one restrained by the condition of higher priority, and put in $A_i$ the other.

To have $A_i \nleqslant_m A_j$ via $\varphi_e$, note that by the above $\{2x, 2x + 1\} \cap A_i \neq \varnothing \Rightarrow x \in K$, so choose a recursive subset of $K$; let it be $R_{ije}$ (in such a way that $R_{ije} \cap lmn = \varnothing$ if $\langle i, j, e\rangle \neq \langle l, m, n\rangle$) and let $a_0, a_1, \ldots$ be its enumeration in order of magnitude. We only look at $2a_0, 2a_1, \ldots$; if at a certain state $s$, $\varphi_e(2a_m)$ is defined in less than $s$ steps and $2a_m \notin A_{i,s}$ then

- if $\varphi_e(2a_m) \notin A_{j,s}$ put $2a_m$ into $A_i$ and restrain $\varphi_e(2a_m)$ from entering $A_j$,
- if $\varphi_e(2a_m) \in A_{j,s}$ then restrain $2a_m$ from $A_i$.

The strategy succeeds because only one element is restrained to satisfy the second type of requirement, but we have two elements to satisfy the first type.

(b) Similarly, construct $\{A_i\}_{i \in \omega}$ such that $K \leqslant_{tt} A_i$ and $A_i \nleqslant_{btt} A_j$ if $i \neq j$. Only notational changes are necessary, and we make $x \in K \Leftrightarrow I_x \cap A_i \neq \varnothing$ where $I_0 = \{0\}$, $I_1 = \{1, 2\}, \ldots$ and $I_x$ has $x + 1$ elements. Here, to spoil $btt$-reductions with norm $n$ we have to take care of $n$ elements, and for $n \leqslant x$, $I_x$ has more than $n$ elements. $\square$

Actually Kallibekov [**Ka2**] has proved that if $\mathbf{a}$ is an r.e. *btt*-degree such that $\mathbf{a} < \mathbf{0}'$, there is an independent set of r.e. *btt*-degrees between $\mathbf{a}$ and $\mathbf{0}'$, and this extends part (b) (take $\mathbf{a}$ as the *btt*-degree of a *tt*-complete non-*btt*-complete set). Part (a) is extended by the analogous result on *m*-degrees, quoted after Theorem 4.9 (taking $\mathbf{a}$ as the *m*-degree of a *btt*-complete non-*m*-complete set).

The previous theorem is obviously related to the kind of problems that have been studied in §3. We can also quote in this direction (see [**Ko3**] and [**Deg9**])

THEOREM 8.4 (KOBZEV). *Every r.e. nonrecursive btt-degree contains infinitely many m-degrees (and no maximal m-degree).*

THEOREM 8.5 (DEGTEV). *Every nonrecursive tt-degree contains at least two btt-degrees.*

Recall from Theorems 4.1 and 6.5 that maximal sets have minimal *m*-degrees and $\eta$-maximal semirecursive sets have minimal *tt*-degrees.

THEOREM 8.6 (KOBZEV). (a) *Maximal sets do not have minimal btt-degrees.*
(b) *$\eta$-maximal semirecursive sets have minimal btt-degrees.*

Note that maximal sets do not have minimal *tt*-degrees either, because they are high (see remarks following Theorem 6.5). Marchenkov has proved that every r.e. nonrecursive *tt*-degree contains a minimal r.e. *btt*-degree.

The analogues of Theorems 6.4 and 6.6 hold by the same proof, and this gives elementary differences among r.e. *btt*-degrees and all the other r.e. degrees, except r.e. *tt*-degrees.

Since most of the work here has still to be done, we do not even attempt to formulate specific problems. One general question is of course

*Problem* 23. Is the ordering of r.e. *btt*-degrees elementarily equivalent to the ordering of r.e. *tt*-degrees? Is the ordering of *btt*-degrees elementarily equivalent to the ordering of *tt*-degrees (*m*-degrees)?

We turn now to *wtt*-degrees. Here too it is possible to generalize the fact that hypersimple sets are not *tt*-complete.

THEOREM 8.7. *$B \leqslant_{wtt} A$ and $B$ part of an r.i. pair of r.e. sets $\Rightarrow A$ nonhypersimple.*

PROOF. We refer to the proof and notations of Theorem 2.4. If $B = \varphi_e^A$ with bound $f$, then use the fact

$$x \in B \wedge y \in C \Rightarrow \varphi_e^A(x) = 1 \wedge \varphi_e^A(y) = 0$$

to deduce that if

$$x \in B \wedge y \in C \wedge \varphi_e^{A^*}(x) = \varphi_e^{A^*}(y)$$

there must be something wrong in $A^*$ below $\max\{f(x), f(y)\}$, etc. $\square$

It follows in particular that *wtt-completeness and T-completeness are distinct*, since there is a hypersimple $T$-complete set. This will be generalized in Theorem 8.13.

THEOREM 8.8 (LACHLAN **[La7]**). *tt-completeness and wtt-completeness are distinct.*

A criterion for *wtt*-completeness may be obtained from the Martin-Lachlan criterion for $T$-completeness (see **[Sol**, Theorem 5.2]). The idea of the proof is the same as the one used in Theorem 2.1.

THEOREM 8.9 (SOLOVE'V). *An r.e. set $A$ is wtt-complete iff there are two recursive functions $h$ and $g$ such that if $\varphi_e$ and $A$ agree up to $h(x)$, then $W_{g(e,x)} \neq W_x$.*

PROOF. If $A$ is *wtt*-complete, then $K \leqslant_{wtt} A$ and since $W_x \leqslant_1 K$ uniformly in $x$, $W_x \leqslant_{wtt} A$ uniformly in $x$, i.e. there are $t$ and $f$ such that $W_x = \varphi^A_{t(x)}$ and $\varphi^A_{t(x)}(z)$ uses the oracle $A$ only for elements less than $f(x, z)$. Fix $a$ and let $h(x) = f(x, a)$; we only use $A$ up to $h(x)$ to know if $a \in W_x$. Let $g$ be such that $z \in W_{g(e,x)} \Leftrightarrow \varphi^{\varphi_e}_{t(x)}(z) = 0$; then if $\varphi_e$ and $A$ agree up to $h(x)$ we have $a \in W_{g(e,x)} \Leftrightarrow \varphi^A_{t(x)}(a) = 0 \Leftrightarrow a \notin W_x$ and $W_{g(e,x)} \neq W_x$.

Let now $h$ and $g$ be as before. Given $B$ r.e. we want $B \leqslant_{wtt} A$ (so $A$ is *wtt*-complete). Let $\varphi_{t(x)}$ be the characteristic function of $A_{\mu s(x \in B_s)}$ when $x \in B$ (totally undefined otherwise). By the recursion theorem there is $f$ recursive such that $W_{f(x)} = W_{g(t(x),f(x))}$. If $n$ is the least $s$ such that $A$ and $A_s$ agree up to $hf(x)$ ($n$ depends on $x$, recursively in $A$) then $x \in B \Leftrightarrow x \in B_n$ (if $x \in B$ and $x \notin B_n$ then $n \leqslant \mu s(x \in B_s)$ and by definition of $n$ and $t$ it follows that $A$, $A_{\mu s(x \in B_s)}$ and $\varphi_{t(x)}$ agree up to $hf(x)$, so by the hypothesis on $g$, $W_{g(t(x),f(x))} \neq W_{f(x)}$, a contradiction) so $B \leqslant_T A$. And since we only need knowledge of $A$ up to $hf(x)$ to obtain $n$, $B \leqslant_{wtt} A$. $\square$

Often, proofs of results for (r.e.) $T$-degrees give already the needed bound and hence the corresponding results for (r.e.) *wtt*-degrees. Here are two useful facts that transfer from $T$-reducibility to *wtt*-reducibility but not, as we saw, to *tt*-reducibility.

– *if $A$ and $B$ are disjoint r.e. sets, $A \cup B \equiv_{wtt} A \oplus B$.* Obviously $A \cup B \leqslant_{wtt} A \oplus B$; for the other direction it's enough to prove e.g. $A \leqslant_{wtt} A \cup B$. To see if $x \in A$, ask if $x \in A \cup B$; if not, $x \notin A$; otherwise generate $A$ and $B$ until $x$ is found in one or the other (if $x \in B$ then $x \notin A$ because $A \cap B = \varnothing$). Here the bound is the identity function.

– *the permitting method preserves wtt-reducibility.* If $x \in A_{s+1} - A_s \Rightarrow (\exists y \leqslant x)\ (y \in C_{s+1} - C_s)$, then the only elements of $C$ we need to know to answer the question $x \in A$ are those less than or equal to $x$, so the bound is the identity once again.

Since Yates **[Y1]** has proved, using the permitting method, that every nonzero r.e. $T$-degree contains a simple nonhypersimple set, the same result holds for *wtt*-degrees by the last observation. In particular the analogue of Theorem 6.3 is false for *wtt*-degrees.

Other examples of results that hold automatically (because of their proofs for the case of $T$-degrees) for r.e. *wtt*-degrees are the *nondiamond theorem* and the existence of *minimal pairs* [**So1**, Theorems 14.4 and 14.1]. Sometimes the restriction to *wtt*-reducibility makes proofs a lot easier than in the case of $T$-reducibility, as it occurs in the next result.

THEOREM 8.10 (LADNER-SASSO [**LS**]). *Density and splitting can be combined for r.e. wtt-degrees, i.e. for every r.e.* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a} < \mathbf{b}$ *there are r.e.* $\mathbf{b}_0$ *and* $\mathbf{b}_1$ *such that* $\mathbf{a} < \mathbf{b}_i < \mathbf{b}$ $(i = 0, 1)$ *and* $\mathbf{b}_0 \cup \mathbf{b}_1 = \mathbf{b}$.

PROOF. Let $A \in \mathbf{a}$, $B \in \mathbf{b}$: $A <_{wtt} B$. If we simply split $B$ into $B_0$ and $B_1$ then by the observation above $B_0 \oplus B_1 \equiv_{wtt} B$. To have $\mathbf{a} < \mathbf{b}_i$ it is enough to take $\mathbf{b}_i$ as the *wtt*-degree of $A \oplus B_i$ $(i = 0, 1)$. This also gives $\mathbf{b}_0 \cup \mathbf{b}_1 = \mathbf{b}$. To have $\mathbf{b} \nleqslant \mathbf{b}_i$ (and hence $\mathbf{b}_i \nleqslant \mathbf{a}$) we have the requirements

$$B \nleqslant_{wtt} A \oplus B_0 \quad \text{and} \quad B \nleqslant_{wtt} A \oplus B_1.$$

If $\{b_s\}_{s \in \omega}$ is a one-one enumeration of $B$, at stage $s$ we put $b_s$ into one and only one of $B_0$ and $B_1$, and decide in which one with the following strategy (for a single requirement). To have e.g. $B \neq \varphi_e^{A \oplus B_0}$ with bound function $\varphi_i$, we do the construction in such a way that when the requirement is *not* satisfied, then $B \leqslant_T A$ with bound function $\varphi_i$, so $B \leqslant_{wtt} A$ contradicting the hypothesis. Suppose $B = \varphi_e^{A \oplus B_0}$ with bound function $\varphi_i$; given $x$ we search for a stage $s$ in which $B_s(x) = \varphi_{e,s}^{A_s \oplus B_{0,s}}$ and such that $A$ and $A_s$ are the same up to $\varphi_i(x)$ (we find $s$ recursively in $A$ and only using $A$ up to $\varphi_i(x)$). Since $\varphi_e^{A \oplus B_0}(x)$ only uses values of $A$ or $B_0$ up to $\varphi_i(x)$, the only important thing to have $\varphi_{e,s}^{A_s \oplus B_{0,s}}(x) = \varphi_e^{A \oplus B_0}(x)$ (and hence $B_s(x) = B(x)$) is to avoid having something less than $\varphi_i(x)$ enter $B_0$ after stage $s$.

The construction is this. For any stage $t$ define $f(t)$ as the maximum of $\varphi_{i,t}(y)$ for $y$ less than or equal to the length of agreement between $B_t$ and $\varphi_{e,t}^{A_t \oplus B_{0,t}}$, and let $g(t) = \max_{t' \leqslant t} f(t')$. Then $b_t$ does not enter $B_0$ if it is less than $g(t)$.

Now $g$ is nondecreasing and in the prior hypothesis ($B = \varphi_e^{A \oplus B_0}$ with bound $\varphi_i$) $g$ is unbounded (otherwise a finite oracle is enough for $\varphi_e$, and $B$ is recursive). Hence, given $x$ search for $s$ as before such that $\varphi_i(x) \leqslant g(s)$; then for $t \geqslant s$ $b_t$ does not enter $B_0$ at stage $t$ if it is less than $g(t) \geqslant g(s) \geqslant \varphi_i(x)$. $\square$

Note that the proof of the splitting theorem for r.e. $T$-degrees requires a strong form of the finite injury priority method (with no recursive bound on the number of injuries), and the proof of the density theorem requires the infinite injury priority method, but this is not all.

THEOREM 8.11. *The orderings of r.e. T-degrees, wtt-degrees and tt-degrees are pairwise nonelementarily equivalent.*

PROOF. r.e. *wtt*-degrees differ from r.e. *tt*-degrees because the splitting theorem does not hold for the latter (see remarks after Theorem 6.5). They differ from r.e. $T$-degrees too, because Lachlan [**La8**] has proved that the combination of density and splitting doesn't hold for r.e. $T$-degrees. $\square$

THEOREM 8.12 (LADNER-SASSO [**LS**]). *An r.e. T-degree contains either one or infinitely many r.e. wtt-degrees.*

PROOF. If it contains two comparable r.e. *wtt*-degrees, apply the density theorem. If it contains two incomparable r.e. *wtt*-degrees, each one of them is comparable with their join, so apply density again. □

Actually both cases can happen (they both happen below any given r.e. nonrecursive $T$-degree; see **[LS]**). It's easy to modify the proof of Theorem 3.16 to have that if an r.e. $T$-degree $\mathbf{a}$ contains only one r.e. *wtt*-degree, then $\mathbf{a}'' = \mathbf{0}''$. The case for $\mathbf{a} = \mathbf{0}'$ follows from the existence of $T$-complete sets that are not *wtt*-complete and Theorem 8.12.

THEOREM 8.13. *The complete T-degree contains infinitely many r.e. wtt-degrees.*

*Problem* 24. Does the complete *wtt*-degree contain infinitely many r.e. *tt*-degrees?

Certainly it does not contain only one r.e. degree, and this is true in general.

THEOREM 8.14 (P. F. COHEN **[Co]**). *No r.e. wtt-degree contains only one r.e. tt-degree.*

We quote the next theorem not for its intrinsic interest but because it gives rise to an interesting situation.

THEOREM 8.15 (LADNER-SASSO **[LS]**). *Every r.e. nonrecursive wtt-degree* $\mathbf{a}$ *has the anticupping property, i.e. there is an r.e.* $\mathbf{b}$ *such that* $\mathbf{0} < \mathbf{b} < \mathbf{a}$ *and for every r.e.* $\mathbf{c} < \mathbf{a}$, $\mathbf{b} \cup \mathbf{c} < \mathbf{a}$.

Here too the proof is not hard. It's known that for some r.e. $T$-degree $\mathbf{a}$ the anticupping property holds (e.g. for $\mathbf{0}'$) but not for all (see **[So1**, §15**]**); dealing with the anticupping property for r.e. $T$-degrees has never been a simple task, but Theorem 8.15 easily implies a nice existence theorem. Note that if an r.e. $T$-degree $\mathbf{a} \neq \mathbf{0}$ contains only one r.e. *wtt*-degree, then it has the anticupping property; let $\mathbf{b}$ be as in the theorem, and $A \in \mathbf{a} \wedge B \in \mathbf{b}$; if $C \leqslant_T A$ and $B \oplus C \equiv_T A$ then $B \oplus C \equiv_{wtt} A$, so $A \equiv_{wtt} C$ and $A \equiv_T C$. The result quoted after Theorem 8.12 then implies that below any given r.e. nonrecursive $T$-degree there is an r.e. degree with the anticupping property. Of course, Theorem 8.15 is again an elementary difference between r.e. $T$-degrees and *wtt*-degrees, and part (a) of the next result gives another one (see **[So1**, §14**]**).

THEOREM 8.16. (a) *Every r.e. wtt-degree* $\mathbf{a} < \mathbf{0}'$ *is the g.l.b. of two incomparable r.e. wtt-degrees (P. F. Cohen* **[Co]**).

(b) *Not every pair of (r.e.) wtt-degrees has g.l.b., so the (r.e.) wtt-degrees are not a lattice (Ladner-Sasso* **[LS]**).

Since $\leqslant_{wtt}$ lies between $\leqslant_{tt}$ and $\leqslant_T$, Theorem 7.3 completely characterizes the countable initial segments of *wtt*-degrees. For a similar reason, from **[La4]** it follows that every countable partial ordering that can be represented as initial segment of the $m$-degrees can be simultaneously represented as an initial segment of *btt*-degrees.

*Problem* 25. Characterize the initial segments of (r.e.) *btt*-degrees.

As a consequence of the prior embeddability results, we have

THEOREM 8.17 (NERODE-SHORE [NS]). *The theories of the orderings of btt-degrees and wtt-degrees are undecidable, nonaxiomatizable and recursively isomorphic to second order arithmetic.*

*Problem* 26. Are the theories of the orderings of r.e. *btt*-degrees and *wtt*-degrees decidable?

We just mention that a notion of *bounded weak truth-table reducibility* ($\leqslant_{bwtt}$) can be defined in the obvious way. Of course

$$A \leqslant_{btt} B \Rightarrow A \leqslant_{bwtt} B \Rightarrow A \leqslant_{wtt} B$$

but $\leqslant_{tt}$ and $\leqslant_{bwtt}$ are incomparable, even on r.e. sets (see **[Ko6]**). Also, *bwtt*-completeness and *btt*-completeness coincide (see **[La7]**), so this notion is not of any help in the study of r.e. $T$-complete sets.

To finish, let us draw a picture of the relationships among the reducibilities we have studied (lack of arrows means lack of implications).

$$\begin{tikzcd}
 & A \leqslant_{btt} B \Rightarrow A \leqslant_{tt} B \Rightarrow A \leqslant_{wtt} B \arrow[dr, Rightarrow] & \\
A \leqslant_1 B \Rightarrow A \leqslant_m B \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & A \leqslant_T B. \\
 & A \leqslant_Q B \arrow[ur, Rightarrow] &
\end{tikzcd}$$

We haven't been too explicit in §2 in treating the relationships among $\leqslant_Q$ and the other reducibilities, so here are the missing links (for r.e. sets):

- $A \leqslant_m B \Rightarrow A \leqslant_Q B$ since $m$ reducibility uses single questions on $B$;
- $A \leqslant_Q B \not\Rightarrow A \leqslant_{wtt} B$ otherwise every $Q$-complete set is *wtt*-complete;

but there are $Q$-complete hypersimple sets (e.g. a $T$-complete semirecursive hypersimple set) that are not *wtt*-complete (because hypersimple). This is interesting because it says that in a $T$-reduction we may use the oracle only for single questions, without having necessarily a recursive bound on the size of these questions.

- $A \leqslant_{btt} B$ (of norm 2) $\not\Rightarrow A \leqslant_Q B$. Let $\langle x, y\rangle \in A \Leftrightarrow x \in B \vee y \in B$;

then $A \leqslant_{btt} B$ of norm 2 by definition, for any $B$. We construct a $E$ such that $A \nleqslant_Q B$ using this strategy (for a single requirement); to spoil the $Q$-reduction $z \in A \Leftrightarrow W_{\varphi_e(z)} \subseteq B$ choose $x_e$ and $y_e$ (distinct) as witnesses, and enumerate $W_{\varphi_e(z)}$ where $z = \langle x_e, y_e\rangle$. If at stage $s$ we find (for the first time) $W_{\varphi_e(z),s} \nsubseteq B_s$ then pick $u_e \in W_{\varphi_e(z),s} \wedge u_e \notin B_s$; since $x_e$ and $y_e$ are distinct, one of them (say $x_e$) is distinct from $u_e$. Put $x_e$ in $B$ (so $z \in A$) and restrain $u_e$. If we never find such a stage, then $W_{\varphi_e(z)} \subseteq B$ but then $z \notin A$ since neither $x_e$ nor $y_e$ go into $B$. The intuitive reason why a definition like the one of $A$ works is that $A$ is defined from $B$ in a disjunctive way, whereas $Q$-reducibility uses a conjunctive request on $B$ ($W_{\varphi_e(z)} \subseteq B$).

## BIBLIOGRAPHY

**[Be]** O. V. Belegradek, *m-degrees of the word problem*, Sibirsk Math. J. **19** (1978), 867–870.

**[Co]** P. F. Cohen, *Weak truth-table reducibility and the pointwise ordering of* 1-1 *recursive functions*, Doctoral Dissertation, Univ. of Illinois, Urbana, Illinois, 1975.

**[Da1]** M. Davis, *The undecidable*, Raven Press, New York, 1965.

**[Da2]** ______, *Unsolvable problems*, Handbook of Math. Logic (Barwise, ed.), North-Holland Amsterdam, 1977, pp. 567–594.

**[Deg1]** A. N. Degtev, *Remarks on retraceable, regressive, and pointwise-decomposable sets*, Algebra and Logic **9** (1970), 390–395.

**[Deg2]** ______, *Hypersimple sets with retraceable complements*, Algebra and Logic **10** (1971), 147–154.

**[Deg3]** ______, *m-powers of simple sets*, Algebra and Logic **11** (1972), 74–80.

**[Deg4]** ______, *Hereditary sets and tabular reducibility*, Algebra and Logic **11** (1972), 145–152.

**[Deg5]** ______, *tt- and m-degrees*, Algebra and Logic **12** (1973), 78–89.

**[Deg6]** ______, *Partially ordered sets of* 1*-degrees, contained in recursively enumerable m-degrees*, Algebra and Logic **15** (1976), 153–164.

**[Deg7]** ______, *On minimal* 1*-degrees and tt-reducibility*, Sibirsk. Math. Z. **17** (1976), 1014–1022. (Russian)

**[Deg8]** ______, *On m-degrees of supersets of simple sets*, Mat. Zametki **23** (1978), 889–893. (Russian)

**[Deg9]** ______, *Three theorems on tt-degrees*, Algebra i Logic **17** (1978), 270–281. (Russian)

**[Den1]** S. D. Denisov, *On m-degrees of recursively enumerable sets*, Algebra and Logic **9** (1970), 254–256.

**[Den2]** ______, *Three theorems on elementary theories and tt-reducibility*, Algebra and Logic **13** (1974), 1–2.

**[Dek]** J. C. Dekker, *A theorem on hypersimple sets*, Proc. Amer. Math. Soc. **5** (1954), 791–796.

**[E1]** Y. L. Ershov, *Hyperhypersimple m-degrees*, Algebra and Logic **8** (1969), 298–315.

**[E2]** ______, *On inseparable pairs*, Algebra and Logic **9** (1970), 396–399.

**[E3]** ______, *Positive equivalences*, Algebra and Logic **10** (1971), 378–394.

**[E4]** ______, *The upper semilattice of numerations of a finite set*, Algebra and Logic **14** (1975), 159–175.

**[EL]** Y. L. Ershov and I. A. Lavrov, *The upper semilattice* $L(\gamma)$, Algebra and Logic **12** (1973), 93–106.

**[ELTT]** Y. L. Ershov et al., *Elementary theories*, Russian Math. Surveys **20** (1965), 35–105.

**[FeR]** J. Ferrante and C. W. Rackoff, *Computational complexity of logical theories*, Lecture Notes in Math., vol. 718, Springer-Verlag, Berlin and New York, 1979.

**[Fr]** R. M. Friedberg, *Three theorems on recursive enumeration*. I, *Decomposition*. II, *Maximal set*. III, *Enumeration without duplication*, J. Symbolic Logic **23** (1958), 309–316.

**[Go]** R. L. Goodstein, *Recursive analysis*, North-Holland, Amsterdam, 1961.

**[Hi]** P. G. Hinman, *Recursion-theoretic hierarchies*, Springer, 1978.

**[J1]** C. G. Jockusch, Jr., *Semirecursive sets and positive reducibility*, Trans. Amer. Math. Soc. **131** (1968), 420–436.

**[J2]**, ______, *Relationships between reducibilities*, Trans. Amer. Math. Soc. **142** (1969), 229–237.

**[J3]** ______, *Degrees in which the recursive sets are uniformly recursive*, Canad. J. Math. **24** (1972), 1092–1099.

**[JSo]** C. G. Jockusch, Jr. and R. I. Soare, *Post's problem and his hypersimple set*, J. Symbolic Logic **38** (1973), 446–452.

**[Ka1]** S. Kallibekov, *Index sets of degrees of unsolvability*, Algebra and Logic **10** (1971), 198–204.

**[Ka2]** ______, *On degrees of recursively enumerable sets*, Sibirsk. Math. J. **14** (1973), 290–293.

**[Ka3]** ______, *Truth tabular degrees of recursively enumerable sets*, Math. Notes **14** (1973), 958–961.

**[KM]** A. S. Kechris and D. A. Martin, *Infinite games and effective descriptive set theory*, Proc. of London Conf. on Analytic Sets, 1978.

**[Ko1]** G. N. Kobzev, *btt-reducibility*, Algebra and Logic **12** (1973), 107–115.

[Ko2] ______, *On btt-reducibility*. II, Algebra and Logic **12** (1973), 242–248.

[Ko3] ______, *On r-separable sets* (preprint).

[Ko4] ______, *On tt-degrees of* r.e. *T-degrees*, Mat. Sb. **106** (1978), 507–514. (Russian)

[Ko5] ______, *On the semilattice of tt-degrees*, Bull. Acad. Sci. Georgian SSR **90** (1978). (Russian)

[Ko6] ______, *Recursively enumerable bw-degrees*, Math. Notes Acad. Sci. USSR **21** (1977), 473–477.

[Ku] G. L. Kurdjumov, *An algorithm-theoretic method of studying homogeneous random media*, Soviet Math. Dokl. **19** (1978), 166–171.

[La1] A. H. Lachlan, *A note on universal sets*, J. Symbolic Logic **31** (1966), 573–574.

[La2] ______, *Initial segments of one-one degrees*, Pacific J. Math. **29** (1969), 351–366.

[La3] ______, *Initial segments of many-one degrees*, Canad J. Math. **22** (1970), 75–85.

[La4] ______, *Solution to a problem of Spector*, Canad J. Math. **23** (1971), 247–256.

[La5] ______, *Recursively enumerable many-one degrees*, Algebra and Logic **11** (1972), 186–202.

[La6] ______, *Two theorems on many-one degrees of recursively enumerable sets*, Algebra and Logic **11** (1972), 127–132.

[La7] ______, *wtt-complete sets are not necessarily tt-complete*, Proc. Amer. Math. Soc. **48** (1975), 429–434.

[La8] ______, *A recursively enumerable degree which will not split over all lesser ones*, Ann. Math. Logic **8** (1975), 307–365.

[LL] A. H. Lachlan and R. Lebeuf, *Countable initial segments of the degrees of unsolvability*, J. Symbolic Logic **41** (1976), 289–300.

[Lad] R. E. Ladner, *On the structure of polynomial time reducibilities*, J. Assoc. Comput. Mach. **22** (1975), 155–171.

[LS] R. E. Ladner and L. P. Sasso, *The weak truth table degrees of recursively enumerable sets*, Ann. Math. Logic **8** (1975), 429–448.

[Le] M. Lerman, *Turing degrees and many-one degrees of maximal sets*, J. Symbolic Logic **35** (1970), 29–40.

[Ma1] S. S. Marchenkov, *The existence of recursively enumerable minimal truth-table degrees*, Algebra and Logic **14** (1975), 257–261.

[Ma2] ______, *On the congruence of the upper semilattices of recursively enumerable m-powers and tabular powers*, Math. Notes Acad. Sci. USSR **20** (1976), 567–570.

[Ma3] ______, *Tabular powers of maximal sets*, Math. Notes Acad. Sci. USSR **20** (1976), 766–770.

[Ma4] ______, *One class of partial sets*, Math. Notes Acad. Sci. USSR **20** (1976), 823–825.

[Mt1] D. A. Martin, *Borel determinacy*, Ann. of Math. (2) **102** (1975), 363–371.

[Mt2] ______, *Descriptive set theory*, Handbook of Math. Logic, (Barwise (ed.)), North-Holland, Amsterdam, 1977, pp. 783–815.

[MPE] D. A. Martin and M. B. Pour-El, *Axiomatizable theories with few axiomatizable extensions*, J. Symbolic Logic **35** (1970), 205–209.

[MN] G. Metakides and A. Nerode, *Effective content of field theory*, Ann. Math. Logic **17** (1979), 289–320.

[Mi] D. P. Miller, Ph. D. dissertation, University of Chicago, 1981.

[MM] W. Miller and D. A. Martin, *The degrees of hyperimmune sets*, Z. Math. Logik Grundlag Math. **14** (1968), 159–166.

[McL] T. G. McLaughlin, *A theorem on intermediate reducibilities*, Proc. Amer. Math. Soc. **19** (1968), 87–90.

[NS] A. Nerode and R. A. Shore, *Second order logic and first order theories of reducibility ordering*, Proc. of Kleene Symposium, North-Holland, New York (to appear).

[O] P. G. Odifreddi, *Classical recursion theory* (to appear).

[Po] E. L. Post, *Recursively enumerable sets of positive integers and their decision problems*, Bull. Amer. Math. Soc. **50** (1944), 284–316.

[PE] M. B. Pour-El, *Effectively extensible theories*, J. Symbolic Logic **33** (1968), 56–68.

[Ro] H. Rogers, Jr., *Theory of recursive functions and effective computability*, McGraw-Hill, New York, 1967.

**[Ru]** J. M. Rubin, Ph.D. dissertation, Univ. Calif., Berkeley, 1981.

**[Sh]** J. R. Shoenfield, *Undecidable and creative theories*, Fund. Math. **49** (1961), 171–179.

**[So1]** R. I. Soare, *Recursively enumerable sets and degrees*, Bull. Amer. Math. Soc. **84** (1978), 1149–1181.

**[So2]** ______, *Constructions in the recursively enumerable degrees*, CIME Summer School, Bressanone (Italy), 1979 (to appear).

**[Sl]** V. D. Solove'v, *Q-reducibility and hyperhypersimple sets*, Prob. Meth. and Cybern. **10–11** (1974), 121–128. (Russian)

**[V]** V. V. V'yugin, *Segments of recursively enumerable m-degrees*, Algebra and Logic **13** (1974), 361–373.

**[Y1]** C. E. M. Yates, *Three theorems on the degree of recursively enumerable sets*, Duke Math. J. **32** (1965), 461–468.

**[Y2]** ______, *On the degrees of index sets*, Trans. Amer. Math. Soc. **121** (1966), 309–328.

**[Y3]** ______, *On the degrees of index sets*. II, Trans. Amer. Math. Soc. **135** (1969), 249–266.

**[Yo]** P. R. Young, *Linear orderings under one-one reducibility*, J. Symbolic Logic **31** (1966), 70–85.

Dipartimento di Matematica, Università di Torino, Torino, Italy

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801