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RIEMANN-ROCH THEOREMS FOR HIGHER ALGEBRAIC K-THEORY

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In [1] and [2] Baum, Fulton and MacPherson, generalizing the celebrated Grothendieck-Riemann-Roch theorem, proved that given a category $\mathcal{V}$ of quasi-projective schemes there is a natural transformation called the Todd class of functors (covariant for proper morphisms) between K_0', the homology algebraic K-theory of coherent sheaves and any of the standard homology theories. Here we announce generalizations of the results of [1] and [2] to Quillen's higher algebraic K-theory [8] which may help to illuminate the relationship between algebraic K-theory and more ordinary cohomology theories.

The statements of our theorems depend on defining global analogues of Quillen's construction of Chern classes for the K-theory of a ring [3], [9]. We can use any of the standard cohomology theories defined on $\mathcal{V}$, such as étale or crystalline cohomology or even the Chow ring. All of these theories can be realized for each $X \in \mathcal{V}$ as the hypercohomology of a graded complex or pro-complex Γ_j^*, $j \in \mathbf{Z}$, of sheaves on the Zariski site of X. All of these theories have Chern classes for representations of sheaves of groups and there exist universal classes

$$C_i \in \mathrm{H}^{di}(X, GL(\mathcal{O}_X), \Gamma_i^*) \qquad (d = 1 \text{ or } 2).$$

Using Brown's generalized cohomology "with supports" of simplicial sheaves [6], and the functor $\mathbf{Z}_\infty$ of [5] instead of the "+" construction one can mimic in the category of simplicial sheaves the methods of [3] and [9] to obtain Chern classes for all $p > 0$

$$C_{i,p}^Y; K_p^Y(X) = K_p(X, X - Y) \longrightarrow \mathrm{H}_Y^{di-p}(X, \Gamma_i^*)$$

whose domains are the relative K-groups, defined so as to force a Quillen-style localization sequence. One can show that these classes coincide for $p = 0$ with those of Iversen [7]. For $p > 0$ they are group homomorphisms and are compatible with products in the way described by Bloch [3], hence one can define a Chern character with supports, which is a ring homomorphism

$$ch^Y\colon \bigoplus_{p\geq 0} K_p(X, X - Y) \longrightarrow \bigoplus_{i,p\geq 0} \mathrm{H}_Y^{di-p}(X, \Gamma_i^*) \otimes \mathbf{Q}.$$

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THEOREM 1. *For $Y \in V$ define the Γ-homology groups $H_i(Y, \Gamma_j)$ to be $\mathbf{H}_Y^{dn-i}(X, \Gamma_{n-j}^*)$ where $Y \subset X$ and X is smooth of dimension n. Then there is a natural transformation of covariant functors on the category of proper morphisms in V, with domain the K'-theory of coherent sheaves* [8] ,

$$\tau_* = \bigoplus_{p \geq 0} \tau_p : \bigoplus_{p \geq 0} K'_p(Y) \longrightarrow \bigoplus_{p, i \geq 0} H_{di+p}(Y, \Gamma_i) \otimes \mathbf{Q}.$$

τ_* satisfies the following conditions:

(i) *For all $X \in V$ and $\alpha \in K_p(X)$, $\beta \in K'_q(X)$,*

$$\tau_{p+q}(\alpha \cap \beta) = ch_p(\alpha) \cap \tau_q(\beta).$$

(ii) *If $X, Y \in V$ and $\alpha \in K'_p(X)$, $\beta \in K'_q(Y)$ then $\tau(\alpha \boxtimes \beta) = \tau(\alpha) \boxtimes \tau(\beta)$*

($\boxtimes$ = *external product in K'-theory or Γ-homology*).

(iii) *If j: $U \longrightarrow X$ is an open immersion in V and $\alpha \in K'_p(X)$ then $\tau_p(j^*\alpha) = j^*\tau_p(\alpha)$.*

(iv) If *$X \in ob(V)$ is smooth then the structure sheaf $\mathcal{O}_X$ defines an element $[\mathcal{O}_X]$ of $K'_0(X)$* and $\tau_0[\mathcal{O}_X] = \mathrm{Td}(X) \cap \eta_X$, *where* $\mathrm{Td}(X)$ *is the classical Todd class and $\eta_X \in H_{dn}(X, \Gamma_n)$ is the homology cycle class of X (n = dimension of X).*

The construction of τ_* and the proof of Theorem 1 are adapted from the methods of [1]. However the extension to higher K-theory does involve more detailed "functorial" constructions than are necessary for K_0. Similar methods to those of Theorem 1 and the paper [2] can be used to prove

THEOREM 2. *On the category of quasi-projective algebraic varieties over* $\mathbf{C}$ *there is a natural transformation*

$$\tau_* = \bigoplus_{p \geq 0} \tau_p : \bigoplus_{p \geq 0} K'_p \longrightarrow \bigoplus_{p \geq 0} KU_p^{LC}$$

covariant for proper morphisms, where KU_p^{LC} is the '$L - C$' or 'locally compact' homology topological K-theory associated to the spectrum BU. There is also a natural transformation

$$\tau^* : K^* \longrightarrow KU^*.$$

These maps τ_*, τ^* satisfy the "module", "product" and presheaf properties analogous to (i), (ii) and (iii) of Theorem 1.

If X is a Noetherian scheme, there is a filtration

$$N^{\cdot}X = \{X^{(0)} = X \supset X^{(1)} \supset \cdots \supset X^{(n)} \supset \cdots\}$$

of its underlying topological space, called the *coniveau* filtration, defined by

$$X^{(k)} = \{x \in X \mid \overline{\{x\}} \text{ has codimension} \geq k \text{ in } X\}.$$

For any simplicial sheaf $F.$ on X, there is a natural exact couple associated to $N^{\cdot}X$ and hence a spectral sequence [8], [3]:

$$E_1^{pq}(F.) = \bigoplus_{x\in X^{(p)}-X^{(p+1)}} H_x^{p+q}(X, F.) \Rightarrow H^{p+q}(X, F.).$$

Given a cohomology theory $\bigoplus_{i\geq 0} \Gamma_i^*$, for each $X \in V$ one obtains maps of coniveau spectral sequences

$$E_r^{pq}(C_i)\colon E_r^{pq}(\mathbf{Z}_\infty B.GL(\mathcal{O}_X)) \longrightarrow E_r^{pq}(K(di, \Gamma_i^*))$$

which on E_1 terms may be written

$\bigoplus C^x_{i,-p-q}$:

$$\begin{array}{ccc}
\bigoplus_{x\in X^{(p)}-X^{(p+1)}} K^x_{-p-q}(X) & \longrightarrow & \bigoplus_{x\in X^{(p)}-X^{(p+1)}} \mathbf{H}_x^{di+p+q}(X, \Gamma_i^*) \\
\downarrow\wr & & \downarrow\wr \\
\bigoplus_{x\in X^{(p)}-X^{(p+1)}} K_{-p-q}(\mathrm{k}(x)) & \overset{\gamma_i^x}{\dashrightarrow} & \bigoplus_{x\in X^{(p)}-X^{(p+1)}} \mathbf{H}^{d(i-p)+p+q}(x, \Gamma^*_{i-p})
\end{array}$$

where the vertical isomorphisms come from duality and Quillen's localization theorem [8].

THEOREM 3. *The map γ_i^x in the diagram above is equal to* $(-1)^p(i-1)!/(i-p-1)!\, C_{i-p,-p-q}$.

This is a consequence of the following Riemann-Roch theorem without denominators and with supports.

THEOREM 4. *Let $j\colon Y \longrightarrow X$ be a closed immersion of smooth schemes in V. Then we have isomorphisms*

$$j_*\colon K_q(Y) \xrightarrow{\sim} K_q(X, X-Y),$$
$$j_!\colon \mathbf{H}^{d(i-p)-q}(Y, \Gamma^*_{i-p}) \longrightarrow \mathbf{H}_Y^{di-q}(X, \Gamma_i^*)$$

where $p = \operatorname{codim}_X(Y)$. For each $k \geq 0$ there exist polynomials $P_k(T_0, \ldots, T_{k-p}; U_1, \ldots, U_{k-p})$ with integer coefficients such that if $\alpha \in K_q(Y)$ and $q \leq 2k$ then

$$C_k^Y(j_*\alpha) = j_!(P_k(rk, C_1, \ldots, C_{k-p}; C_1(N_{X/Y}), \ldots, C_{k-p}(N_{X/Y}))(\alpha)).$$

Here $rk, C_1, \ldots, C_{k-p}$ are the universal Chern classes in $\mathbf{H}^*(Y, GL(\mathcal{O}_Y), \Gamma_*^*)$ and the $C_i(N_{X/Y})$ are the Chern classes of the normal bundle of Y. The polynomial P_k in these classes defines a class in $\mathbf{H}^{dk}(Y, GL(\mathcal{O}_Y), \Gamma_k^*)$ which we apply to α.

Theorem 4 is proved by using the "deformation to the normal bundle" construction of [1] to reduce to the case of the zero section $Y \subset N_{X/Y}$ followed by explicit computations. The same methods may be used to prove Theorems 3 and 4 in the case where X is any connected regular one-dimensional scheme and Γ^* is étale cohomology, in which case $j_!$ is defined by Grothendieck's absolute purity theorem (SGA5, I §5.1). In the case $X = \mathrm{Spec}(\mathcal{O}_S)$, $\mathcal{O}_S$ the ring of S-integers in a global field, this extends more ad-hoc results of Soulé in [9].

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