to consider situations where spacetime has a topology different from that of $R^4$. This classical approach is in fact rather appropriate for the subject matter.

The book's biggest drawback is its excessively formal character. Whether and how to take over a particular nonrelativistic, macroscopic idealization into relativity is only partially a question of whether the appropriate differential geometric formalism can be set up. To get a real sense of the uses and limitations of some model one also needs to analyze some specific physical situations to which the model is relevant and needs to investigate the model's relation to less phenomenological, more microscopic models. Such discussions are regrettably rare in the book. But within its own framework the book is highly competent. It will remain a useful reference for quite some time.

R. K. SACHS

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*Rational quadratic forms*, by J. W. S. Cassels, London Mathematical Society Monographs No. 13, Academic Press, London-New York-San Francisco, 1978, xvi + 413 pp., $36.50.

The focal point of the book under review, the classification of quadratic forms over $\mathbf{Z}$, can be formulated very simply. If

$$f = \sum f_{ij}x_ix_j \quad \text{and} \quad g = \sum g_{ij}y_iy_j$$

are nondegenerate quadratic forms in $n$ variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively, with coefficients $f_{ij} = f_{ji}$ and $g_{ij} = g_{ji}$ in $\mathbf{Z}$, is it possible to determine whether or not $f$ and $g$ are equivalent over $\mathbf{Z}$, i.e. whether or not there is a linear change of variables

$$y_j = \sum t_{ij}x_i$$

with $(t_{ij})$ an invertible matrix over $\mathbf{Z}$ which will transform $g$ into $f$? This is closely related to the question of describing those integers that are represented by $g$, and to the more general question of which quadratic forms are represented by $g$ over $\mathbf{Z}$. All these problems can, of course, be formulated over any integral domain and not just over $\mathbf{Z}$. In particular, they can be formulated over an arbitrary field where it can be shown, rather simply, that every quadratic form is equivalent to a diagonal form provided the characteristic of the field is not 2. If the field in question is $\mathbf{R}$, then $g$ is equivalent to a diagonal form

$$\sum_1^r x_i^2 - \sum_{r+1}^n x_j^2,$$

and $r$ and $n$ provide a complete set of invariants for equivalence over $\mathbf{R}$. This is Sylvester's Theorem. It is the classification theorem over $\mathbf{R}$. Forms over $\mathbf{R}$ with $r > 0$ and $n > r$ are called indefinite, with $r = n$ positive definite, and so on. Forms over $\mathbf{Z}$ are called indefinite if they are indefinite when viewed over $\mathbf{R}$, and so on. It is important to make the distinction between definite and indefinite since, in so many parts of the theory, these two types of forms behave in such different ways.

The classification problem has its roots in the 19th century. It has a long and chaotic history and it was not until the middle of the 20th century that it was put on a secure foundation. Proofs were incomprehensible, facts were erroneous, and intuitions were misguided. As recently as 1944 it was speculated that the so-called class number for indefinite forms had to be 1, whereas in fact it can be arbitrarily large. If there is any single concept that is responsible for clarifying the whole matter it is the concept of the spinor genus, and this is given the central place in *Rational quadratic forms* which it deserves.

The great liberating force goes back to Hasse's classification of quadratic forms over $\mathbf{Q}$ in 1923. Prior to that time, quadratic forms over $\mathbf{Q}$ relied on the theory over $\mathbf{Z}$ and, because of the difficulties intrinsic to $\mathbf{Z}$, proved to be a highly intricate and elaborate subject. Using the $p$-adic numbers which had just been developed by Hensel, Hasse proved that two quadratic forms over $\mathbf{Q}$ are equivalent if and only if they are equivalent over all the $p$-adic fields $\mathbf{Q}_p$ including $\mathbf{Q}_\infty = \mathbf{R}$, thereby reducing classification over $\mathbf{Q}$ to classification over all $\mathbf{Q}_p$. But classification over $\mathbf{Q}_p$ is easy: if $p < \infty$, there are three invariants that classify forms, the dimension, the determinant, and the so-called Hasse symbol; if $p = \infty$, classification is given by Sylvester's Theorem. So forms are classified over $\mathbf{Q}$, not as easily as over $\mathbf{R}$, but simply enough all the same.

The next important landmark was a conceptual one due to Witt, in 1937. Instead of working with quadratic forms, Witt suggested that we work with quadratic spaces where, by definition, a quadratic space $V$ over an arbitrary field $F$ of characteristic not 2 is an $n$-dimensional vector space $V$ over $F$ that is provided with a symmetric bilinear form $B$: $V \times V \to F$. In this terminology, equivalence of forms over $F$ is replaced by isometry $V \cong U$ of quadratic spaces. In particular, if we are working over $\mathbf{Q}$, the Hasse principle becomes

$$V \cong U \Leftrightarrow V_p \cong U_p \ \forall p$$

where $V_p$ is the quadratic space obtained from $V$ by extending the scalars from $\mathbf{Q}$ to $\mathbf{Q}_p$. Returning to general fields, the group of isometries of $V$ onto $V$ is called the orthogonal group and is written $O_n(V)$. Several subgroups of importance can be identified within this group, notably the subgroup $O_n^+(V)$ of transformations of determinant 1, the commutator subgroup $\Omega_n(V)$ of the entire group, and a group which I will refrain from defining but which is close to $\Omega_n(V)$ and is written $O_n'(V)$ or $\Theta_n(V)$. (Incidentally, the structures and isomorphisms of these groups have been studied extensively, but that is another story.) Quadratic spaces are good for studying quadratic forms over fields. What about over integral domains? Here it is necessary to modify the concept of a quadratic space to that of a quadratic lattice. For simplicity we consider a principal ideal domain $I$ with a field of quotients $F$. Then a quadratic lattice is, by definition, a module $\Gamma = Ix_1 + \cdots + Ix_n$ on the quadratic space $V$ with $x_1, \ldots, x_n$ a base for $V$. And equivalence of forms over $I$ is replaced by isometry $\Gamma \cong \Delta$ between lattices.

Let us return to the classification problem over $\mathbf{Z}$. Guided by Hasse's results over $\mathbf{Q}$ it is natural to ask if two quadratic forms are equivalent over $\mathbf{Z}$ if and only if they are equivalent over all $\mathbf{Z}_p$ where $\mathbf{Z}_p$ denotes the $p$-adic integers in $\mathbf{Q}_p$ and $\mathbf{Z}_\infty$ denotes all of $\mathbf{Q}_\infty = \mathbf{R}$. In other words, is the Hasse principle true over $\mathbf{Z}$? The answer, in general, is no. But it fails in only a finite way. More specifically, two quadratic forms over $\mathbf{Z}$ are said to be in the same genus if they are equivalent over all $\mathbf{Z}_p$; each genus contains a certain number of equivalence classes over $\mathbf{Z}$; their number is called the class number; the class number can be shown to be finite; if the Hasse principle were true, it would be 1.

All this can be interpreted in the language of quadratic lattices. Consider two quadratic lattices $\Gamma$ and $\Delta$ on $V$ over $\mathbf{Q}$. Then $\Gamma$ and $\Delta$ are said to be in the same class if there is a $\sigma$ in $O_n(V)$ such that $\Gamma = \sigma\Delta$. And they are said to be in the same genus if, for all $p < \infty$, there is a $\Sigma_p$ in $O_n(V_p)$ such that $\Gamma_p = \Sigma_p\Delta_p$ where $\Gamma_p$ and $\Delta_p$ are obtained from $\Gamma$ and $\Delta$ by extending the scalars from $\mathbf{Z}$ to $\mathbf{Z}_p$. Then the class of $\Gamma$ corresponds, essentially, to a class of quadratic forms over $\mathbf{Z}$, while the genus of $\Gamma$ corresponds, essentially, to a genus of quadratic forms over $\mathbf{Z}$.

The next building block in the theory is, therefore, a complete desciption of the genus, and this is the same as classification over $\mathbf{Z}_p$. This is known and, for odd primes $p$, it is easy. Suffice it to say, by way of example, that any two quadratic forms of the same dimensions, with coefficients in $\mathbf{Z}_p$ and with unit determinants, are equivalent if and only if these determinants are essentially equal. Classification over $\mathbf{Z}_2$ is due to Jones in 1944 and Pall in 1945.

Finally, $\Gamma$ and $\Delta$ are said to be in the same spinor genus if there is a $\sigma$ in $O_n(V)$ and a $\Sigma_p$ in $O_n'(V_p)$ for all $p < \infty$ such that $\Gamma_p = \sigma_p\Sigma_p\Delta_p$ where $\Gamma_p$ and $\Delta_p$ are as before, and $\sigma_p$ is the extension of $\sigma$ from $V$ to $V_p$. The concept of spinor genus is due to Eichler, with subsequent development by Kneser. The partitioning of quadratic lattices on $V$ into spinor genera is finer than the genus but not as fine as the class. The following fundamental theorems are true.

THEOREM. *The number of spinor genera in a genus is always finite and is a power of* 2.

THEOREM. *In the indefinite case with $n > 3$, the spinor genus coincides with the class.*

Using the second of these theorems, together with additional properties of the spinor genus, it is possible to classify a wide variety of indefinite quadratic forms over $\mathbf{Z}$. There is no classification to speak of in the definite theory, but the following interesting theorem is true.

THEOREM. *Let $f(x)$ be a positive definite integral form in $n > 4$ variables. Then there is an integer $N$ with the following properties: Let $a > N$ be an integer which is primitively represented by $f(x)$ over $\mathbf{Z}_p$ for all primes $p$; then $a$ is primitively represented by $f$ over $\mathbf{Z}$.*

This, then, is the skeleton of the theory developed in the first eleven chapters of *Rational quadratic forms*. The theory remains valid if $\mathbf{Q}$ and $\mathbf{Z}$ are replaced by an algebraic number field and its ring of algebraic integers, but

Cassels has refrained from working in this generality. The deep analytic work of Siegel giving quantitative versions of the representation theory of forms over $\mathbf{Z}$ is mentioned in passing, as is Tamagawa's adelic version of the Siegel theory. The last three chapters of the book have a distinctly different flavor from the first eleven. Chapter twelve is concerned with the existence of a canonical form in each class of positive definite forms. Here the forms can be over $\mathbf{R}$ but the transformations involved must be over $\mathbf{Z}$. Minkowski reduction is defined and it is shown that, generally speaking, there is essentially one Minkowski reduced form in each class. Positive definite forms in $n$-variables can be interpreted in $\mathbf{R}^{n(n+1)/2}$ by looking at their entries on and above the main diagonal of their associated matrices, and the geometry of the sets obtained (for example, of the set that corresponds to all reduced forms) is studied. Siegel domains are introduced and used. Chapter thirteen studies the integral points of orthogonal groups, i.e. the groups of integral automorphs of quadratic forms. For example, we have the following result when $n = 2$.

THEOREM. *The group $O^+(f)$ of proper integral automorphs of a primitive indefinite anisotropic binary form $ax^2 + 2bxy + by^2$ consists of the $T = \pm T_0^{\nu}(\nu \in \mathbf{Z})$ where $T_0$ is of infinite order. The $T \in O^+(f)$ are the matrices*

$$T = \begin{pmatrix} t - bu & -cu \\ au & t + bu \end{pmatrix}$$

*where $t$ and $u$ satisfy certain specified conditions.*

Chapter fourteen is on Gauss' theory of composition of binary forms which is concerned with putting a natural group structure (it turns out to be a finite abelian group) on "primitive classes of quadratic forms of discriminant $D$". It is related to ideal theory in quadratic extensions of $\mathbf{Q}$.

In writing a book like this it is necessary to make some decisions on the philosophy to be used right from the start. First, should the language be that of forms, or of matrices, or of vector spaces and lattices. Second, should the approach be elementary assuming as little as possible from the reader. And third, should things be done over $\mathbf{Q}$ and $\mathbf{Z}$, or should they be done over algebraic number fields. The first decision really applies to the first eleven chapters alone, i.e. up to the development of the spinor genus. When it comes to classes and genera, the choice between forms, matrices and lattices is really a matter of taste, my taste being lattices first, forms second, and matrices third. It is possible to develop spinor genera using forms, and Watson originally did this when he discovered the spinor genus independently of Eichler, but the ideas are forced and unnatural and, without trying to be dogmatic, it ceases to be a matter of taste. Spaces are right, forms are wrong. The approach of Cassels is basically geometrical, i.e. via spaces, with an intermingling here and there of forms. Personally I would have preferred a little more geometry, but the mix is fine. Next there is the matter of being elementary. The only nonelementary fact that is assumed is Dirichlet's theorem on primes in an arithmetic progression. Other than this, all that is required of the reader is a knowledge of the simplest facts about linear algebra, group theory, and $\mathbf{Z}$ and $\mathbf{Q}$. A few basic results from the geometry of numbers are proved in the book. Even the $p$-adic numbers are constructed. Finally, there is the decision on whether to work over $\mathbf{Q}$ or, more generally, over algebraic number fields. The author chose **Q**. This has its pros and it has its cons. On the positive side, it enables the approach to be more elementary, the proofs are more concrete, there is no need to use results from class field theory which is difficult enough to understand let alone to develop, and it makes the material more accessible to mathematicians in other areas–group theory, combinatorics, topology, differential geometry–who in the past have found **Z** and **Q** good enough for their purposes. On the negative side it must be said that these same mathematicians are beginning to find **Z** and **Q** too specialized, that it is not as simple as the author suggests to extend things from **Q** to algebraic number fields, and that the reader who has mastered the subject over **Q** will be faced with a psychological barrier in having to go over it all again over an algebraic number field. My advice to the novice who intends to work in quadratic forms is, in fact, to start out over number fields.

So much for overall philosophy. Some other points should also be mentioned. Cassels emphasizes the effectiveness of the results whenever he can. This is a welcome feature of the book although, on one occasion, I found his explanation inadequate and unconvincing. Next, at the very end of the book he shows how the use of Dirichlet's theorem can be replaced by some elementary, but nontrivial, theory. He also shows that the folklore on the equivalence between the geometric and the form approach to spinor genera is true, a service to the expert, but incomplete and confusing to others. The author's development of Minkowski reduction and composition theory is clearly done and to be recommeded. My overall disappointments include a certain vagueness that is all too often covered by a wave of the hand, and an incompleteness that leaves you with the feeling that you have not been brought to the frontiers of research. Whether or not the decision to work over **Q** is a disappointment will depend on what you intend to use the book for.

The audience for *Rational quadratic forms* will be those mathematicians who wish to apply the arithmetic theory of quadratic forms and either want to learn the subject or have a good reference source for theorems over **Z**; students who wish to work in the theory; and specialists who are interested in seeing the subject from a somewhat different perspective. The ultimate questions are whether to buy the book; and, having bought it, whether to read it; and, in reading it, whether one will enjoy it. My answer to the first of these questions is yes; to the second, yes if you are just interested in **Z** or if you are looking for a different perspective; my answer to the third question is that I did.

O. T. O'MEARA

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*Compact right topological semigroups and generalizations of almost periodicity*, by J. F. Berglund, H. D. Junghenn and P. Milnes, Lecture Notes in Math., vol. 663, Springer-Verlag, Berlin-Heidelberg-New York, 1978, x + 243 pp., $12.00.

This monograph in lecture-notes' clothing (hereafter referred to as BJM) has something in it for everyone: Semigroups $S$ and the backchat between