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BOUNDARY REGULARITY AND EMBEDDED SOLUTIONS FOR THE ORIENTED PLATEAU PROBLEM

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Any fixed C^2 Jordan curve Γ in $\mathbf{R}^3$ is known to span an orientable minimal surface in several different senses. In the work of Douglas, Rado and Courant (see e.g. [3, IV, §4]) the minimal surface occurs as an area-minimizing mapping from a fixed orientable surface of finite genus and may possibly have self-intersections. In the work of Federer and Fleming (see e.g. [4, §5]) the minimal surface, which occurs as the support of an area-minimizing rectifiable current, is necessarily embedded (away from Γ) but was not previously known even to have finite genus. Our work in [7], which establishes complete boundary regularity for the latter surface, thus implies that *there exists an orientable embedded minimal surface with boundary* Γ. In fact:

THEOREM 1. *For any compact orientable $n-1$ dimensional C^2 embedded submanifold N of $\mathbf{R}^{n+1}$, there exists an orientable bounded stable minimal embedded $C^{1,\alpha}$ (for all $0 < \alpha < 1$) hypersurface M with boundary N so that the closure of M in $\mathbf{R}^{n+1}$ equals $M \cup S$ for some compact set $S \subset \mathbf{R}^{n+1} \sim N$ of Hausdorff dimension $\leqslant n-7$.*

Using the existence theory for area minimizing rectifiable currents [4, 5.1] and their interior regularity theory [5, Theorem 1], Theorem 1 follows from our boundary regularity result [7, 11.1]:

THEOREM 2. *If U is an open subset of $\mathbf{R}^{n+1}$, T is an n dimensional absolutely area minimizing locally rectifiable current in U, and ∂T is an oriented embedded C^2 submanifold of U, then, for some open neighborhood V of* spt ∂T *in U, $V \cap$* spt T *is an embedded $C^{1,\alpha}$ hypersurface with boundary for all $0 < \alpha < 1$.*

W. K. Allard [1, §5] has proven such regularity near points on the boundary of the convex hull of spt T. Boundary regularity in $n = 2$ for the *unoriented* problem [4, 5.3.21] (and so the existence of possibly nonorientable embedded minimal surfaces with boundary) also follows form his work. For $k \geqslant 2$, $C^{k,\alpha}$

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smoothness (analyticity) in Theorems 1, 2 for $C^{k,\alpha}$ (analytic) boundaries follows from [8, 1.10]. In proving Theorem 2, we obtain:

THEOREM 3. *Any compact orientable $n-1$ dimensional embedded minimal submanifold of $\mathbf{S}^n = \mathbf{R}^{n+1} \cap \{x: |x| = 1\}$ with boundary $\mathbf{S}^n \cap \{(x_1, \ldots, x_{n+1}): x_n = 0 = x_{n+1}\}$ must be a great hemisphere.*

[2, Theorem A] shows that spt T above may have an $n-7$ dimensional *interior* singular set and that the analogue of Theorem 3 for submanifolds *without boundary* is false. For $n = 2$, Theorem 2 implies:

THEOREM 4. *For any C^2 Jordan curve Γ in $\mathbf{R}^3$, there exists a nonnegative integer G_Γ so that:*

(1) *The Douglas-Courant type, genus g least-area problem* [3, IV, 4.1, 4.4] *for Γ has no solution whenever $g > G_\Gamma$.*

(2) *There exists a Douglas-Courant type genus G_Γ least-area solution for Γ, and any such solution is embedded.*

(3) *The number of such solutions is finite if Γ is $C^{4,\alpha}$.*

There are also *a priori* bounds on G_Γ, the number of solutions, and the absolute value of the Gaussian curvature of any solution.

SKETCH OF PROOF OF THEOREM 2. To obtain regularity near a point $a \in$ spt ∂T, we assume $a = 0$ and first prove that *the support of some oriented tangent cone at* 0 *is contained in a hyperplane.* For $n = 2$, this follows from the monotonicity formula [1, 3.4], interior regularity [5], and the planar nature of geodesics on $\mathbf{S}^2$. For $n > 2$, an inductive argument using linear barriers is required. Letting $H_\pm = \mathbf{R}^n \cap \{(y_1, \ldots, y_n): \pm y_n > 0\}$ and rotating, we assume that for some positive integer m the oriented tangent cone is the sum of m times $H_+ \times \{0\}$ and $m-1$ times $H_- \times \{0\}$, both taken with the usual orientation $\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_m$. Since the case $m = 1$ has been treated by Allard [1, §5], we henceforth assume $m \geqslant 2$.

Using [4, 5.4.2], we now see that the normalized height

$$h(r) = \sup\{|x_{n+1}|/r: (x_1, \ldots, x_{n+1}) \in \text{spt } T,\ |(x_1, \ldots, x_n)| \leqslant r\}$$

has lower limit 0 *as* $r \downarrow 0$. After establishing that $h(r)$ is comparable (except for a boundary curvature term and a slight change in r) with the cylindrical excess $\mathrm{Exc}(T, 0, r)$ of [4, 5.3], we may apply the interior regularity theorem [4, 5.3.14] in vertical circular cylinders which do not meet spt ∂T. From this, one finds C^1 domains $\Omega_\pm \subset H_\pm$ which are mutually tangent at the origin so that over $\Omega_+ \cup \Omega_-$, spt T separates into graphs of real analytic minimal-surface-equation solutions:

$$(1)\quad u_1^+ \leqslant u_2^+ \leqslant \cdots \leqslant u_m^+ \ \text{ on } \Omega_+, \qquad u_1^- \leqslant u_2^- \leqslant \cdots \leqslant u_{m-1}^- \ \text{ on } \Omega_-.$$

Concerning the boundary behavior of each $u_i^\pm$, one may, at this stage, only conclude that

(2) $$\lim_{\Omega_\pm \ni y \to 0} |u_i^\pm(y)| + |Du_i^\pm(y)| = 0.$$

The goal of the middle third of [7] is the specific estimate

(3) $$\limsup_{r \downarrow 0} r^{-1/2} h(r) < \infty.$$

Besides involving many well-known concepts of geometric measure theory (monotonicity, excess, blowing-up) and well-known nonparametric regularity estimates (DeGiorgi-Nash, Schauder), the work here includes a new estimate on the radial derivative of each $u_i^\pm$ and a new comparison between spherical and cylindrical excess.

Using (3), we verify that $\Omega_\pm$, $u_i^\pm$ may be chosen so that

(4) $$\Omega_\pm \textit{ is a } C^{1,1/10} \textit{ domain,} \qquad u_i^\pm \in C^{1,1/4}(\text{Clos } \Omega_\pm).$$

Under conditions (1), (2), and (4), the $C^{1,\alpha}$ Hopf-type boundary point lemma of Finn and Gilbarg [6, Lemma 7] implies that $u_1^+ = \cdots = u_m^+$, $u_1^- = \cdots = u_{m-1}^-$. For a small open ball B about 0, we then subtract off the oriented component, which meets the graph of u_1^+, of the regular points of $B \cap (\text{spt } T) \sim \text{spt } \partial T$ to obtain an area minimizing $S \in \mathcal{R}_n^{\text{loc}}(B)$ with $\partial S = 0$ and spt $S = B \cap$ spt T. The proof is completed by using the interior regularity theorem [4, 5.3.18] which implies that (since $h(r) \longrightarrow 0$ as $r \downarrow 0$) spt S is, near 0, an embedded real analytic minimal submanifold.

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