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Proof theory, by Kurt Schütte, Grundlehren der Mathematischen Wissenschaften, Band 225, Springer-Verlag, Berlin, Heidelberg, New York, 1977, xii + 302 pp., $34.10.

This is the long-awaited translated revision of the author's *Beweistheorie* [8]. As Schütte says in his preface, it was originally intended to be the second edition of that book "but in fact has been completely rewritten". Even so the amount of fresh material is quite impressive. The obvious comparisons to be made are with the first edition of [8] and with Takeuti's *Proof theory* [12] previously reviewed in this journal [4]. These are the main candidates for contemporary texts on the subject which essay some comprehensiveness. Very briefly, there is much to admire and to welcome in this new book, both with respect to choice of material and to manner of presentation. As to the latter, we have (as expected from the author) meticulous attention to details, careful definitions, complete but compact proofs, and assimilable organization into units and subunits. However, I cannot recommend the book unreservedly, for reasons to be gone into below.

There are three parts: A. Pure Logic, B. Systems of Arithmetic, and C. Subsystems of Analysis. The subject matter of A is by now fairly standard, but the formal framework in terms of *positive* and *negative parts* is not, even though it had been introduced in [8]. The idea is that A is a positive (negative) part of F if it occurs in F in such a way that the truth of A (falsity of A) implies the truth of F. The simplest example is given by $F = (A_1 \wedge \cdots \wedge A_n \rightarrow B_1 \vee \cdots \vee B_m)$ where each A_i is a negative part of F and each B_j a positive part; similarly for $F = (\neg A_1 \vee \cdots \vee \neg A_n \vee \cdots \vee B_m)$. The syntactic definition of these notions is given in terms of *P-forms* $\mathcal{P}[*]$ and *N-forms* $\mathfrak{N}[*]$ so that A is a positive part of F if $F = \mathcal{P}[A]$ for a P-form $\mathcal{P}$ (similarly for negative parts and N-forms). It is also necessary for setting up the logical systems to consider *NP-forms* $\mathfrak{Q}[*_1, *_2]$ which are N-forms in $*_1$ and P-forms in $*_2$. These notions are supposed to permit combining the advantages of Hilbert-style inferential systems with those of Gentzen-style sequential systems. In the former, as here, one infers individual formulas while in the latter one infers "sequents" $\Gamma \supset \Delta$ where $\Gamma = \{A_1, \ldots, A_n\}$, $\Delta = \{B_1, \ldots, B_m\}$ are sequences or sets of formulas; $\Gamma \supset \Delta$ holds under the same conditions as $A_1 \wedge \cdots \wedge A_n \rightarrow B_1 \vee \cdots \vee B_m$.

It is of interest to compare the two systems of axioms and rules for the classical predicate calculus CP, taken here with basic logical symbols $\perp$, $\rightarrow$, $\forall$. In Schütte's formulation these are: $(Ax.\text{I})\mathfrak{Q}[A, A]$, $(Ax.\text{II})\mathfrak{N}[\perp]$, $(\rightarrow \text{R1})\mathfrak{N}[A \rightarrow \perp]$, $\mathfrak{N}[B] \vdash \mathfrak{N}[A \rightarrow B]$, $(\forall\text{R1})\mathcal{P}[F(a)] \vdash \mathcal{P}[\forall x F(x)]$ (when a is not free in the conclusion), and $(\forall\text{R2})\forall x F(x) \rightarrow \mathfrak{N}[F(t)] \vdash \mathfrak{N}[\forall x F(x)]$. In Gentzen's formulation these are: $(Ax.\text{I})'(\Gamma, A \supset \Delta, A)$, $(Ax.\text{II})'(\Gamma, \perp \supset \Delta)$, $(\rightarrow \text{R1})'(\Gamma \supset \Delta; A)$, $(\Gamma, B \supset \Delta) \vdash (\Gamma, A \rightarrow B \supset \Delta)$, $(\rightarrow \text{R2})'(\Gamma, A \supset \Delta, B) \vdash (\Gamma \supset \Delta, A \rightarrow B)$, $(\forall\text{R1})'(\Gamma \supset \Delta, F(a)) \vdash (\Gamma \supset \Delta, \forall x F(x))$, (when a is not free in the conclusion), and $(\forall\text{R2})'(\Gamma, F(t) \supset \Delta) \vdash (\Gamma, \forall x F(x) \supset A)$. In addition, Gentzen uses structural rules (permutation, contraction), but such are not needed when Γ, Δ are taken to be sets and we read $\Gamma, F = \Gamma \cup \{F\}$. It is to be noted that there is no rule corresponding to ($\rightarrow$R2)′ in Schütte's system; the obvious analogue would simply be $\mathcal{P}[A \rightarrow B] \vdash \mathcal{P}[A \rightarrow B]$. Finally, the result that his system is closed under the rule $\mathcal{P}[A], (A \rightarrow B) \vdash \mathcal{P}[B]$ corresponds to the closure of the second system under the *cut-rule*

$$(\Gamma \supset \Delta, A), \quad (\Gamma', A \supset \Delta') \vdash (\Gamma, \Gamma' \supset \Delta, \Delta'),$$

which is of course the key feature of Gentzen's proof theory.

It may be a matter of first exposure and/or taste, but I do not find it natural to work in Schütte's system despite its elegance. But as evidence that this is not just a personal prejudice it can be mentioned that the system has not caught on at all in the literature (though **[8]** has been an important reference since its publication 18 years ago). And, it seems, there are good reasons to prefer Gentzen's sequential (or natural deduction) systems. First, one can visibly isolate the role of each of the logical operators at each point of the derivation. Secondly, hardly any change has to be made to treat intuitionistic logic–one simply restricts to sequents $(\Gamma \supset \Delta)$ where Δ contains at most one formula. By contrast, Schütte has to introduce a new (but variant) system involving implications $(A_1 \rightarrow (\ldots \rightarrow (A_n \rightarrow B)\ldots))$ and the notion of *right* and *left part* for the intuitionistic predicate calculus IP.

In sampling some of the details of the proof-theoretical arguments throughout the book, I kept finding myself translating the set-up into more familiar terms. The trouble is that one must deal with these at every turn, so the process becomes rather trying. Of course if the work had been presented within the sequential framework there would have been greater duplication of Takeuti's **[12]**. But that would not have been so extensive as to lessen its value. Perhaps newcomers to the subject will take more readily to the positive/negative parts approach (but with the possible prospect of later disorientation with respect to the more standard formulations).

Among the things done in Part A is the interpolation theorem for IP and thence for CP. This and other applications of cut-elimination for CP and IP had also been treated in **[12]**. Another topic dealt with in both books is the cut-elimination theorem for simple type theory ("Takeuti's conjecture") as established by the semantical methods of Prawitz and Takahashi. The groundwork for that result had been laid by Schütte himself in a 1960 paper[1] where he showed equivalence of the conjecture with extendibility of every partial valuation to a total valuation. More recent work by Girard **[6]** should have been mentioned; among other things this separates in a simple and informative way a class of sentences for which derivability implies cut-free derivability by elementary means. It is amusing historically that Schütte treats simple type theory under Pure Logic. Russell had thought that all of mathematics could be derived from logic in the guise of an evident theory of properties or classes of higher type. Takeuti had hoped to establish constructively the consistency of classical analysis. Indeed consistency is a trivial consequence of the cut-elimination theorem for type theory, but no construc-

[1]Reference to this and some other papers mentioned in this review are to be found in the author's Bibliography.

tive proof in the ordinary sense of the word is known nor does any seem likely. Perhaps the placement of the subject in Part A reflects the disappointment in both hopes.

Part B begins with an excellent introduction to systems of notation based in a natural way on hierarchies of normal functions of ordinals, in this case for the ordinals up to Γ_0. For number theory only the ordinals below ε_0 are needed; the system of notations up to Γ_0 is required only when predicative analysis is taken up later. However, it is hardly more trouble to do the work all at once as done by the author. There is an extension to a much more substantial system of notations whose presentation is wisely delayed to the last chapter of the book.

What is particularly welcome in the exposition here is that the ordinal-theoretic base is frankly explained and accepted as the source of the notation systems, rather than suppressed as was the case in [**8**]. But the author's requirements for a constructive proof theory are also satisfied. He says (p. 87, §14): "It is clear from §13 that the inductive definition of the $<$-relation corresponds to that in the interpretation we gave of the ordinal terms as notations for ordinals $< \Gamma_0 \ldots.$ Now the ordinal terms are to be understood without reference to their interpretations and simply as formal strings of symbols for which we have a constructively defined $<$-relation. All the theorems about ordinal terms in this section will be proved constructively using only the above inductive definition. In this way we lay a constructive foundation for the proof theoretic applications . . . ". So, our conscience is assuaged. (The supposed need for a constructive development is argued briefly in the introduction, but no more convincingly than by Takeuti in his book; cf. the discussion in [**4**].)

The proof theory of elementary ("pure") number theory PN is here (unexpectedly) treated via Gödel's Dialectica functional interpretation. This is already to be found in book or monograph form, e.g. Stenlund [**10**] and Troelstra [**13**]. But in both those references normalization of terms for primitive recursive functionals had been accomplished by so-called computability arguments, which do not yield explicit ordinal information. Instead the author follows Howard's assignment to terms of ordinals under ε_0 (apparently the first exposition within a book). The trouble with this is that ordinals play no intrinsic role; the reader confronted for the first time with the use of ordinals in proof theory may well be mystified. Methodologically and pedagogically, this seems to me to be a retrograde step from the first edition [**8**], whose great virtue was the systematic use of infinitary derivations and the canonical assignment of ordinals as lengths to derivations. To be sure, infinitary derivations are used later in the book when dealing with predicative analysis, and elementary number theory resurfaces as a part of that. But the resulting order of topics is unsatisfactory. At the very least there should have been a more extensive description and comparison of the principal methods which have been used to treat elementary number theory since Gentzen's pioneering work. (One gets only 7 lines on p. 147.) In particular, explicit mention should have been made of Tait's paper [**11**] where primitive recursive functionals are interpreted as infinite terms, ordinals again being associated naturally to the latter.

The really novel material is to be found in Part C, Subsystems of Analysis, which forms almost half the book in length. We have here for the first time (in book form) an exposition of a considerable portion of the author's and the reviewer's work on predicativity, for which the principal result given is the exact bound Γ_0 on the autonomously provable ordinals. Along with this is provided considerable information on both finitary and infinitary systems of arithmetic with free predicate variables, arithmetical analysis, ramified analysis, and Δ_1^1-analysis. One point should have been made clear with respect to this last system, namely that only the Δ_1^1-*comprehension rule* (Δ_1^1-CR) is being used, not the Δ_1^1-*comprehension axiom* (Δ_1^1-CA). The difference (much discussed in the literature) is that in the former one takes

$$\forall x[F(x) \leftrightarrow G(x)] \vdash \exists X \forall x[x \in X \to F(x)]$$

while the latter takes

$$\forall x[F(x) \leftrightarrow G(x)] \to \exists X\, \forall x[x \in X \to F(x)],$$

in both cases for $\Sigma_1^1 F$ and $\Pi_1^1 G$. The reducibility of (Δ_1^1-CA) to predicative systems was first established by Friedman **[5]**; this required new arguments, not mentioned here.

The last chapter begins with a detailed development of a hierarchy of ordinal functions using higher number classes in an essential way; naturally attached to this hierarchy is a notation system OT^* which is susceptible of constructive treatment. The exposition here follows a paper by W. Buchholz[1] fairly closely. (Reference is made to the reviewer's idea for this system and Aczel's initial development of it. The important contribution by Bridge **[1]** which bridged Aczel to Buchholz is indicated in the introduction but should also have been mentioned at this point, together with the result of **[1]** by which it superseded the Bachmann-Pfeiffer-Isles hierarchies.) The author uses OT^* to tie down a system PA of Π_1^1-analysis. This employs Takeuti's syntactical transformations which were presented in **[12]**, but replaces the opaque system of ordinal diagrams with the more palatable OT^*. Still it is all heavy going, especially since no rationale is given either for Takeuti's choice of syntactic reductions or for the assignments of ordinals. Mention is made at the end of the book of related work by Pohlers on systems ID_n of iterated inductive definitions, also by means of Takeuti's methods. Subsequent to the publication of this book, Pohlers extended his work directly to transfinitely iterated systems ID_α and computed their provable recursive well-orderings in OT^*; this is to be found in **[7]**. Further significant progress on these systems has been made by Buchholz **[2]** and Sieg **[9]** both of whom provide different but more perspicuous methods and obtain more information. Using the latter together with **[3]** and **[5]** we have at last a chain of understandable reductions of the classical system (Σ_2^1-AC) to a constructive system, together with an evaluation of its least nonprovably recursive ordinal.

One general complaint I have is that the author is too sparing with informal explanations. This allows him to pack a great deal into a relatively compact space. All the details are there, but it will take great determination on the part of a student to learn the subject by working through them step-by-step. As suggested at the conclusion of the review **[4]**, there is still no one best source for a beginner. This book is an important and valuable addition to a growing secondary literature, which is perhaps best absorbed by wandering back and forth from one book to another and thence to the primary sources as quickly as possible.

We wish to add our thanks and compliments to J. N. Crossley for translating this work into (very readable) English.

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Linear estimation and stochastic control, by M. H. A. Davis, Wiley, New York, 1977, xii + 224 pp., $14.95.

In this monograph the author gives a highly readable introduction to two topics, namely the Kalman filter and the stochastic linear regulator problem. These two topics have been called the "bread and butter" of modern stochastic systems theory. They have the fortunate feature that the mathematical techniques used are elegant, while at the same time the results have been quite widely used in engineering and other applications. Rather modest background is needed to read the book. The equivalent of introductory real-analysis, probability, and some familiarity with elementary linear systems theory should suffice.

The linear estimation problem is as follows. Given random variables X, Y_s for $s \in S$, all of zero mean and finite variance, find the approximation $\hat{X}$ to X