is hard to take it seriously. One is more likely to think of a certain Swiss hotelier and the German word for seat. Though these are colorful words they bear no relation to the concepts they represent. Extremes of brevity are reached in such statements as "After it has been established that $GVF \subset ST$ and $PSA = ST$ it is natural also to ask whether $GVF = ST$" (p. 596).

Fully three quarters of the chapter are devoted to a discussion of RITZ fractions, that is regular C-fractions or continued fractions of the form $K(a_n z/1)$. RITZ-fractions are soon specialized to the S-fractions of Stieltjes, here all $a_n > 0$, and z is replaced by $1/z$, S-fractions are studied in their relation to positive symmetric functions, functions expressible as Stieltjes transforms $\int_0^\infty d\psi(t)/(z + t)$, as well as to the moment problem. A sketch of the theory of Stieltjes integrals as well as inclusion of proofs of the Montel and Vitali theorems help in making the material accessible to readers of modest preparation.

The computational aspects of the subject are always kept in mind. Not only are many examples considered and worked out, but also if there is a more constructive as well as a more existential approach to a topic, the former is usually chosen. It is thus not surprising that a good deal of emphasis is placed on the quotient-difference algorithm (treated in Chapter 7 in the first volume) which was introduced by Rutishauser in 1954. The q.-d. scheme can be used to compute the coefficients of the RITZ expansion of a formal power series. It is also used in giving a solution to the problem, proposed and solved by Hurwitz, of finding necessary and sufficient conditions for a polynomial with real coefficients to have all of its roots in $R(w) < 0$. (The problem can be solved by means of terminating RITZ fractions.)

In conclusion the author must be congratulated on having written an eminently readable account of a series of interesting topics. This is a book one wants to browse in.

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Convex analysis and measurable multifunctions, by C. Castaing and M. Valadier, Lecture Notes in Math., no. 580, Springer-Verlag, Berlin, Heidelberg, New York, 1977, vii + 278 pp., $11.00.

A "multifunction" Γ from X to Y is simply a map from X into the set $\mathcal{P}(Y)$ of all subsets of Y. This has also been called a "correspondence", or a "multi-valued mapping" by other authors. Whatever the name, the concept is quite elementary, so much so that it is not clear at a glance that there is anything to be learned from it. For instance, it is a straightforward exercise in general topology to define continuity for compact-valued multifunctions from one metric space to another. The set $\mathcal{K}(Y)$ of all compact nonempty subsets of Y is endowed with the Hausdorff metric:

$$\delta(K_1, K_2) = \max\left\{ \sup_{x_1 \in K_1} d(x_1, K_2), \sup_{x_2 \in K_2} d(x_2, K_1) \right\}$$

and Γ should be continuous if and only if it is continuous as a map from X

into $\mathcal{K}(Y)$. Incidentally, most properties of the space Y (completeness, compactness, Hausdorff countability) carry over to $\mathcal{P}(Y)$ (see [20]).

To get more interesting answers, one must ask better questions. During the sixties, there was an outbreak of such questions, from widely different areas of applied mathematics: mathematical programming, control theory, mathematical economics, partial differential equations. I shall try to classify subsequent developments under four headings: measurable selections, Liapunov's theorem, differential inclusions, generalized gradients. But I would like first to recall some facts about continuous selections.

A. Continuous selections. Topologists realized quite early that continuity was too strong a requirement to be of much use, and split it up between upper and lower semi-continuity. A multifunction Γ is lower semi-continuous (l.s.c.) if, whenever $\Gamma(x_0)$ meets some open subset $U \subset Y$, so does $\Gamma(x)$, for x close enough to x_0. It is upper semi-continuous (u.s.c.) if, whenever $\Gamma(x_0)$ is contained in some open subset $U \subset Y$, so is $\Gamma(x)$, for x close enough to x_0. It is usually required that $\Gamma(x)$ be compact, or at least closed, for all x. With that understanding, any u.s.c. multifunction has closed graph, the converse being true when Y is compact.

The first interesting question was answered by Michael: when does there exist a *continuous* function $\gamma: X \to Y$ such that $\gamma(x) \in \Gamma(x)$, all $x \in X$. Such a mapping is called a continuous selection from Γ. Michael [19] has proved that a continuous selection exists whenever X is paracompact (for instance, metrizable), Y is a Banach space, and Γ is l.s.c. with closed convex nonempty values. Since I am already so far off the track, I cannot resist mentioning a beautiful result by Lazar (see [17]): if X is a Choquet simplex and Y a Banach space, any linear (i.e. $\alpha\Gamma(x) + \beta\Gamma(y) \subset \Gamma(\alpha x + \beta y)$ for $\alpha, \beta \geqslant 0$, $x, y \in X$, and $\alpha + \beta = 1$) l.s.c. multifunction Γ from X to Y has a linear continuous selection.

Beautiful though they are, these theorems do not meet the needs of applied mathematicians, because multifunctions arising from applications very rarely are lower semi-continuous. Consider, for instance, a function ϕ on $X \times Y$, continuous with respect to y; assuming Y to be compact, define $\Gamma(x)$ as the set of points in Y where $\phi(x, \cdot)$ attains its minimum. In other words, $\Gamma(x)$ is the set of solutions of the optimization problem $\inf_{y \in Y}\phi(x, y)$, depending on the parameter $x \in X$; this is a very usual situation in applied mathematics. Even with very strong continuity or regularity assumptions on ϕ, with respect to both variables, the multifunction Γ will not be lower semi-continuous; however, continuity is enough to make it upper semi-continuous (see [4] for an excellent treatment; believers in catastrophe theory might find [9] to their liking).

The upshot is that applied mathematicians, at a very early stage, have abandoned all hope for continuous selections, and have turned to measurable selections (see, however, [13] for an alternative approach).

B. Measurable selections. Two early results on measurable selections are recognized as seminal in optimization and control literature. The first one is due to von Neumann [21] the second to Fillipov [11], and both are called lemmas, although the original proofs were quite intricate, because they were used as tools for some other purpose. Two separate kinds of results have flown from them, and they have merged but recently (see [24]). The first type of measurable selection theorem imposes some very strong measurability condition on Γ (typically, that its graph be a borelian subset of $X \times Y$), but nothing at all on the values $\Gamma(x)$ (except that they be nonempty). The second type has a weaker measurability condition on Γ (typically, that Γ be borelian as a map from X to $\mathcal{K}(Y)$), but requires $\Gamma(x)$ to be compact-valued.

Measurable selection theorems are of particular importance to control theory. Systems in engineering or economics are usually modelled by differential equations

$$dx/dt = f(t, x(t), u(t)), \quad \text{in } R^n, \tag{$\mathcal{E}$}$$

where $x(t)$ is the state of the system and $u(t)$ the control, to be chosen at each time t in some prescribed set U. The set $\Gamma(t, x) = f(t, x, \mathfrak{U})$ certainly has physical relevance, as the set of all permissible velocities at time t, in state x, so that one might consider it more realistic to replace the original differential equations ($\mathcal{E}$) by the so-called differential inclusion:

$$dx/dt \in \Gamma(t, x(t)). \tag{$\mathcal{I}$}$$

The equivalence of ($\mathcal{E}$) and ($\mathcal{I}$) is not a trivial matter, and certainly requires a measurable selection theorem.

C. Differential inclusions. Differential inclusions, such as ($\mathcal{I}$), can be studied for their own sake, independently of any control system they might arise from. The question then is raised: when does the initial-value problem:

$$dx/dt \in \Gamma(t, x(t)), \qquad x(0) = x_0 \in R^n \tag{$\mathcal{I}_0$}$$

have a solution? I mean a local solution, on some time interval $[-\varepsilon, +\varepsilon]$; uniqueness, of course, is not to be expected.

The first answer is: when Γ is lower semi-continuous and the $\Gamma(x)$ convex, closed, nonempty; for then Michael's theorem applies, there is a continuous selection $f(t, x)$ in $\Gamma(t, x)$, and any solution of $dx/dt = f(t, x)$ will do. However, this is useless, because the multifunctions encountered in practice are not likely to be lower semi-continuous. We need a statement for upper semi-continuous right-hand sides.

Such a statement can be given as follows: assume that the multifunction $\Gamma(T, x)$ is measurable with respect to t, u.s.c. with respect to x, and satisfies some boundedness condition; assume moreover that the values $\Gamma(t, x)$ are all convex, compact, nonempty. Then the initial-value problem ($\mathcal{I}_0$) has a local solution.

The proof is quite similar to the existence proof for differential equations with continuous right-hand side: one can either use a fixed-point theorem, or follow the Peano approximation procedure. The convexity assumption has particular significance. Without it, the theorem is false (try solving $dx/dt = \Gamma(t)$, with Γ defined by $\Gamma(t) = +1$ for $t < 0$, $\Gamma(t) = -1$ for $t > 0$, and $\Gamma(t) = \{-1, +1\}$ for $t = 0$). It is needed at several stages of the proof, and implies that the set of all solutions $x(\cdot)$ to ($\mathcal{I}_0$) is compact in $\mathcal{C}^0([-\varepsilon, +\varepsilon])$. It is all the more remarkable that Fillipov should have been able to trade convexity for continuity of Γ with respect to both variables ([12]; see [1] for another proof).

The end of that line of investigation lies in stating necessary condition for optimality when a payoff function, usually $J(x(T))$ or $\int_0^T j(t, x(t), \dot{x}(t))\,dt$, is defined on the trajectories of $(\mathcal{G}_0)$. This would yield a Pontrjagin maximum principle, independent of the particular representation $f(t, x(t), u(t))$ chosen for the permissible velocities, and applicable to problems too irregular to be tractable by classical means (for instance, problems with constraints on the state $x(t)$ in R^n). Clarke [8] has begun fulfilling this program.

D. Liapunov's theorem. Let now X be endowed with a σ-algebra $\mathcal{Q}$ and a finite positive measure μ; we assume this measure to have no atoms, i.e. no sets $A \in \mathcal{Q}$, $\mu(A) > 0$, such that for any subset $B \subset A$, $B \in \mathcal{Q}$, either $\mu(B) = \mu(A)$ or $\mu(B) = 0$. For instance, the Lebesgue measure would fit the case, whereas the Dirac measure would not.

Consider a measurable multifunction Γ from $(X, \mathcal{Q})$ to R^n, all values $\Gamma(x)$ to be compact, convex, nonempty, and uniformly bounded. Denote by $\ddot{\Gamma}(x)$ the set of extreme points in $\Gamma(x)$, so that $\Gamma(x)$ is the closed convex hull of $\ddot{\Gamma}(x)$. Finally, denote by $\mathbb{S}(\Gamma)$ the set of all measurable selections of Γ (similarly, $\mathbb{S}(\ddot{\Gamma})$ for $\ddot{\Gamma}$):

$$\mathbb{S}(\Gamma) = \{\phi \in L^\infty(X; R^n) | \phi(x) \in \Gamma(x) \quad \forall x\}.$$

It is clear that $\mathbb{S}(\Gamma)$ is convex, closed, bounded in all L^p spaces, and hence compact in the weak-* topology $\sigma(L^\infty, L^1)$. It is not so obvious that $\mathbb{S}(\ddot{\Gamma})$ is exactly the set of extreme points in $\mathbb{S}(\Gamma)$, so that by the Krein-Milman theorem, $\mathbb{S}(\Gamma)$ is the closed convex hull of $\mathbb{S}(\ddot{\Gamma})$ for $\sigma(L^\infty, L^1)$. As a matter of fact, we have even better: for any $\phi \in \mathbb{S}(\Gamma)$, $k \in N$, and $f_1, \ldots, f_k \in L^1(X; R^n)$, some $\psi \in \mathbb{S}(\ddot{\Gamma})$ can be found with the property that:

$$\int_X \langle \phi, f_i \rangle\, d\mu = \int_X \langle \psi, f_i \rangle\, d\mu \quad \text{for } 1 \leqslant i \leqslant k.$$

The Liapunov convexity theorem (see [18] for a short proof) is essentially the case when $\Gamma(x)$ is just the constant interval $[0, 1]$ in R. Note that, if the f_i, $1 \leqslant i \leqslant k$, are kept fixed, the sets

$$\left\{\int_x \langle \phi, f_i \rangle\, d\mu | \phi \in \mathbb{S}(\Gamma)\right\} \subset R^k,$$

$$\left\{\int \langle \psi, f_i \rangle\, d\mu | \psi \in \mathbb{S}(\ddot{\Gamma})\right\} \subset R^k,$$

are equal; it is obvious that the first one is convex (because $\mathbb{S}(\Gamma)$ is), so the second one has to be convex too (although $\mathbb{S}(\ddot{\Gamma})$ is not). The lesson is that integrating with respect to a nonatomic measure will produce some degree of convexity where it might be sorely lacking.

These results have deep significance for applied mathematics. In control theory alone, I could mention at least two areas which depend heavily on these ideas: studying the bang-bang principle (see [15]), and understanding relaxed (or chattering) controls (see [10]). New applications have recently been found in partial differential equations (homogeneization theory). However, I shall confine myself to mathematical economics, where the Liapunov convexity theorem is of such importance that a discrete version has been evolved (Shapley-Folkman [23]; see [10] for another proof).

Roughly speaking, the fundamental theorem of mathematical economics states that it is possible for the market to equate supply and demand by a proper choice of prices. It is proved under stringent convexity assumptions, that may not be met in practice. On the other hand, in practice the number of consumers and (to a lesser degree) of producers is very large. Aumann [3] has shown how to trade one for the other: he represents the (supposedly infinite) set of all agents by the interval $[0, 1]$ with Lebesgue measure, and uses a Liapunov-type theorem to do without the convexity assumptions. This approach has been very successful in understanding the economics of perfect competition, where the influence of any individual agent on overall prices can be considered negligible (see [16]).

E. Generalized gradients. Let f be a real function defined on a subset $\operatorname{dom} f$ of some Banach space X. Generalized gradients have been defined when f is convex l.s.c. (in which case $\operatorname{dom} f$ is convex), and when f is locally lipschitzian (in which case $\operatorname{dom} f = X$). Their common feature is that the set $\partial f(x_0)$ of generalized gradients of f at the point x_0 is convex and closed in X^* (see [10], [22] and [6], [7]). Thus, we have a natural multifunction ∂f from $\operatorname{dom} f$ into X^*, with convex closed values.

It would certainly be unfair to classify the study of generalized gradients as a branch of multifunction theory. They exhibit features, and require methods, of their own, particularly in the convex case. The equation $dx/dt \in -\partial\phi(x)$, for instance, with ϕ a convex l.s.c. function, is really a nonlinear partial differential equation of parabolic type, with the heat equation as a special case; curiously enough, the solution to the initial-value problem is unique, although the right-hand side is multivalued (see [5]).

There is some overlapping, however, Differential equations of the form $dx/dt \in -\partial\phi(x) + \Gamma(x)$, with $\Gamma(x)$ an u.s.c. multifunction with compact convex values, have been used for modelling planning procedures in economies with public goods ([14]); there seems to be a bright future for them in other areas of applied mathematics (see [2]). On quite another tack, convex analysis has developed methods for solving optimization problems in Banach spaces. In practice, these will be L^p spaces or Sobolev spaces, and the functionals involved will be in integral form. The usual calculus of variations, for instance, seeks to minimize the functional

$$F(x(\,)) = \int_0^T f\left(t, x(t), \frac{dx}{dt}(t)\right) dt$$

over some path space. Convex analysis will provide conditions for optimality in terms of F, and they will have to be translated in terms of f to become practicable. Relating integral concepts (such as ∂F) to pointwise concepts (such as ∂f) is an essential step in these methods, and typically requires a measurable selection theorem.

Conclusion. I shall now pay some lip service to the role of reviewer and say a few words about the book at hand. Castaing and Valadier are leading experts on the subject, and their book is long overdue. It is a technical and complete exposition of the theory in sections B, C, D, and the part of section E concerning convex integrals. The statements are precise to the point of ponderousness; there are no frills, no brightening up things for the reader (except for the misprints; I recommend the one on page 37, where the authors "pretend" the Hausdorff metric), no motivations nor applications outside mathematics. The authors' idea of an application is to prove Strassen's theorem, without actually stating it (the reader is expected to know or to guess), or to define the conditional expectation of a random closed convex set.

This is a book for the expert. Right at the beginning, the reader is expected to know what a multifunction is (no definition is provided), and as it goes on, he will be expected to know much more. Indeed, it is doubtful whether one could actually read the book without a complete set of Bourbakis close at hand, if only for the vocabulary (Dunford-Schwartz will not do). The authors' statement that "the only necessary prerequisite for an intelligent reading is a good knowledge of analysis" has the same ring of truthfulness as the famous claim of Bourbaki, that reading his treatise should require no particular knowledge of mathematics. Nor is functional analysis the main ingredient. Measure theory lends its flavour throughout, as it should be with any book dealing with measurable multifunctions. The book culminates in a decomposition theorem for the dual space of L_E^∞ (E a Banach space), two proofs of which are provided in the last chapter. This kind of result does crop up in the detailed study of convex integrands, but it certainly is measure theoretic in nature.

For all its defects this is the best reference on the subject. It contains a wealth of material which previously was very hard to find, scattered away in minor French publications, and makes rewarding, if hard, reading. It is a pity, though, that no attempts were made at historical perspective, or to broaden the scope. An introductory book on the subject, in the spirit of [4], remains to be written; meanwhile, the beginner is referred to [25].

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