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Harmonic analysis on compact solvmanifolds, by Jonathan Brezin, Lecture Notes in Math., vol. 602, Springer-Verlag, Berlin, Heidelberg, New York, 1977, v + 177 pp.

Harmonic analysis on nilpotent and solvable groups is something of a stepchild in today's mathematical family. Lacking the charisma of semisimple harmonic analysis and the relative docility of abelian harmonic analysis, it receives patronage neither from the élite nor from the masses. Though it enjoys occasional largesse from a variety of donors it must rely for its main livelihood on the benevolent researches of a small number of faithful sympathizers. Even that most famous of nilpotent groups, the cornerstone of the subject, an object of which every modern mathematician should be aware, the Heisenberg group, is rather far from being a household word. And the extremely elegant core of the theory, the method of orbits, probably first adumbrated by Harish-Chandra, but really exposed by A. A. Kirillov, and further substantially developed by Auslander-Kostant and others, probably still counts as specialist's knowledge.[1] This relative obscurity must be considered more a vagary of history than a divine judgement on the subject's intrinsic merits, for nilpotent and solvable harmonic analysis offers problems of depth and classical precedent. The small hardy band mentioned above has been patiently working on some of these. In recent years much of their attention has been directed towards problems of analysis on compact solvmanifolds, that is, on compact quotient spaces $\Gamma \setminus S$ where S is a solvable Lie group and Γ is a discrete subgroup. (If S is nilpotent, then $\Gamma \setminus S$ is called a nilmanifold.) The book under review offers a substantial survey of the work on solvmanifolds done to date. One of the book's strong points, one which makes it a good entryway for those curious about the subject, is the large amount of space devoted to representative examples.

If f is a function (C^∞, L^p, you choose–that's part of the fun) on $\Gamma \setminus S$, then f may be regarded as a function on S invariant under left translation by elements of Γ. That is, $f(\gamma s) = f(s)$ for γ in Γ and s in S. There is a naturally defined action ρ of S on the functions on $\Gamma \setminus S$ by right translation: $\rho(s')f(s) = f(ss')$ for s, s' in S. The basic problem of harmonic analysis in this context is, given a reasonable space of functions, to find the subspaces which are invariant under $\rho(S)$ and are minimal with respect to this property

[1]Kirillov's basic result on nilpotent groups can be stated so succinctly that I can't pass up this opportunity to disseminate it. Let N be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{N}$. Let Ad be the adjoint action of N on $\mathfrak{N}$. Let $\mathfrak{N}^*$ be the vector space dual to $\mathfrak{N}$ and Ad* the action of N contragredient to Ad. Then there is a canonical bijection between the set of equivalence classes of irreducible unitary representations of N and the collection of Ad*N orbits in $\mathfrak{N}^*$.

(the irreducible S-invariant subspaces) and then to express a general f in terms of functions belonging to the irreducible subspaces. If $S = \mathbf{R}$, then up to dilation $\Gamma = \mathbf{Z}$, so functions on $\Gamma \setminus S$ are just periodic functions in the most conventional sense with period 1, and the irreducible subspaces are just the lines generated by the functions $e^{2\pi inx}$. Thus analysis on $\Gamma \setminus S$ is a (considerable) generalization of the well-known theory of Fourier series.

The nicest classes of functions are the smooth functions C^∞ and the square-integrable functions L^2. (There is a unique smooth probability measure on $\Gamma \setminus S$ invariant under right translations by S, and $L^2(\Gamma \setminus S)$ denotes the Hilbert space of functions square integrable with respect to this measure.) On $L^2(\Gamma \setminus S)$, ρ is a unitary representation of S, that is, the operators $\rho(s)$ are unitary on L^2. Unitary representations of groups have received vast amounts of study since World War II (see **[Ma]** for a readable historical sketch) and indeed unitary representation theory can be thought of as abstract (crude? down-to-earth?) harmonic analysis. Thus it is not surprising that much harmonic analysis on solvmanifolds has been concerned with L^2. The author correctly points out, however, that C^∞ harmonic analysis is quite flexible and can be a very useful tool. He also devotes some attention to L^p and continuous functions. However, it may be said that the basic theorem, the starting point of the subject, is an L^2 result, a sibling to the Peter-Weyl Theorem. It is treated in §7 of the book.

THEOREM. *There is a unique decomposition*

$$L^2(\Gamma \setminus S) = \bigoplus_{i=1}^{\infty} H_i$$

of $L^2(\Gamma \setminus S)$ into an orthogonal direct sum of orthogonal subspaces H_i such that

(i) *Any two irreducible subspaces of H_i define equivalent representations of S.*

(ii) *If $i \neq j$ then an irreducible subspace of H_i and an irreducible subspace of H_j define inequivalent representations.*

(iii) *Each H_i is (nonuniquely) the orthogonal direct sum of finitely many irreducible subspaces. (Any two decompositions have the same number of summands.)*

Let σ_i be the isomorphism class of the irreducible submodules of H_i, and let m_i be the number of irreducible summands in a decomposition of H_i. We may write symbolically

$$\rho = \sum_{i=1}^{\infty} m_i \sigma_i.$$

The integer m_i is called the *multiplicity* of σ_i in ρ. The space H_i is called the *σ_i-isotypic component* or the *σ_i (maximal) primary subspace* of $L^2(\Gamma \setminus S)$. Informally, we may state the theorem: ρ decomposes into a discrete sum of irreducible representations, each with finite multiplicity. In the case of Fourier series, σ_i are simply the characters of $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, the m_i are all 1, and the H_i are all one-dimensional. Some of the increased difficulty of general solvable harmonic analysis over the theory of Fourier series is perhaps indicated by the fact that the H_i will typically be infinite dimensional and the m_i will be arbitrarily large.

Evidently the first questions to answer are, given an S and a Γ, how does one determine the Σ_i, the m_i, and the H_i? We will refer to these questions loosely as "the multiplicity problem." As one gains an understanding of it, one can go on to ask more refined questions: Given H_i, how does one break it up into irreducible subspaces? Are some decompositions preferred in relation to the structure of S and of Γ? What is the analytic nature of the projections onto the H_i? Do they preserve differentiability? continuity? L^p? Are there bases of the irreducible subspaces having good properties and allowing refined decompositions? An so on from renaissance to rococo. But even the short list here goes beyond what is presently known, even for fairly easy examples.

The book under review is divided into 3 main parts. The first and largest discusses most of the above questions for some concrete examples. The middle part discusses an inductive procedure for attacking the multiplicity problem. That is, it discusses the problem, given $S_1 \subseteq S$ normal, with $\Gamma \cap S_1 \backslash S_1$ compact, how can we pass from the knowledge of the σ_i, m_i and H_i for $(\Gamma \cap S_1) \backslash S_1$ to the same knowledge for $\Gamma \backslash S$? The third part presents much of what is known to date concerning the multiplicity problem for nilmanifolds and solvmanifolds.

How much is known? In a limited sense, the multiplicity problem has been solved fairly completely. Precisely, procedures have been described for divesting the problem of computing multiplicities of its harmonic analytic content and reducing it to difficult Diophantine computations. Also, various formulas exist for the projections onto the primary subspaces. For nilmanifolds these answers have been known for some years; for general solvmanifolds the answers are not yet published in definitive form, but the basic phenomena are known, and preprints and summaries have appeared. In place of the aspects of the general problem not broached by the author, representative computations are given in §6. Of course, one can ask for better answers, or at least better control over the answers, qualitative behavior, etc., and here there is room for much work. However it can be said there is no basic mystery about the general nature of the multiplicities, as there certainly still is in the case of compact quotient spaces of semisimple groups.

The most conspicuous general feature (aesthetically pleasing but frustrating from a computational viewpoint) of the multiplicities, clearly shown by the examples in the book, is their arithmetic or Diophantine nature. That this should be so is perhaps not too surprising when one reflects [Ms] that many solvmanifolds $\Gamma \backslash S$ arise by taking S to be the real points of a solvable algebraic group S defined over $\mathbf{Q}$, and Γ an arithmetic subgroup. (All nilmanifolds are of this sort.) In the study of such solvmanifolds, it is natural and appropriate to simultaneously consider the associated adèle group $S_{\mathbf{A}}$ and adèlic homogeneous space $S_{\mathbf{Q}} \backslash S_{\mathbf{A}}$. The adèlic situation was considered in [Mo], the first paper to explicitly study harmonic analysis on general nilmanifolds, but with some exceptions has been largely neglected since. This neglect within the subject is no more deserved than the neglect of the subject from outside, for the adèlic formulation not only adds richness and structure to the theory, but is directly useful, adds insight, and ultimately leads to simpler derivations. The author fortunately does not ignore the adèles. He introduces them in §4 in connection with the "product formula" for multiplicities. He also proves the basic "multiplicity one" theorem for $L^2(S_{\mathbf{Q}} \setminus S_{\mathbf{A}})$ for his examples. However, he does not sufficiently exploit them in this reviewer's opinion. Particularly, in his proof of the product formula he uses them only in a roundabout cumbersome way, while they could be brought to bear directly and incisively. (A technical point: in the discussion of the product formula for the hyperbolic case, class number one should be assumed, but the author does not do so explicitly.) Also the general conditions for multiplicity one are not discussed; it can fail if certain first Galois cohomology classes are nontrivial. See [Co] for an example. Also the adèlic viewpoint might have simplified and clarified §6, especially the role and structure of the oscillator or Weil representation for finite rings.

Such, in brief, is the content of the book. What of the form? The book is very carefully written (not to say proofread–there are numerous typos of a harmless sort, some amusing). The author does his best to communicate the subject as he understands it. He provides many tactical and motivational asides, so even though the material is fairly technical, the reading is usually tolerable. The author's care and his down-to-earth approach will be appreciated by those wishing to learn the subject. In the first part, he really does a very good job of presenting a lot of material with minimal prerequisites. On the negative side, one feels sometimes the forest is being lost for the trees, as for example in the failure to distinguish before §10 between true solvmanifolds and the much simpler but important subclass of nilmanifolds. It would be nice to have a redo of the same material at a much higher level of sophistication. (This is perhaps too much to ask of one author.) Also somewhat dampening is the author's very pessimistic attitude toward his subject, the more so because in several spots where the author paints in his darkest palette, as in §6 and §8, some technical improvements could make the picture rosier. However as a painstaking introduction to a subject that deserves more attention and offers considerable potential for development, and for its attractive examples, (not to mention the fact that it is the only place you can read about a considerable portion of its content) the book is a valuable contribution.

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