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*Decomposition of random variables and vectors*, by Ju. V. Linnik and I. V. Ostrovskiĭ, Translations of Mathematical Monographs, Vol. 48, American Mathematical Society, Providence, R. I., 1977, ix + 380 pp., $38.80. (Translated from the Russian, 1972, by Israel Program for Scientific Translations)

In the Foreword to his book of 1960, Linnik described decomposition of (probability) laws as "a field which in relation to mathematics employed lies between the theory of probability and the theory of functions of a complex variable", nowadays one would add "and of several complex variables." This field stands isolated from the mainstream of probability theory and is largely ignored. Thus it behooves us to be specific about its main concepts, problems, and representative results with their dates, in order to point out the evolution of the field before and after the crucial Linnik book which described its state as of 1960.

Until recently the central problem of probability theory was, and in large part still is, that of behaviour of sums of independent random variables. The inverse problem of decomposition of random variables, more precisely of their laws, was born–with its concepts, problems, and methods–during its heroic period 1934–1938 thanks to P. Lévy, Cramér, Hinčin, and Raikov. For twenty years it attracted little attention except mainly during 1947–1951 when Cramér, Lévy and Dugué produced various examples and counterexamples and Dugué introduced "ridge functions" which were to play an important role in factorizations of analytic characteristic functions. Thereafter Linnik's deep results, the impact of his book of 1960, and his personal influence attracted to the field a number of bright young mathematicians, especially Ostrovskiĭ–the joint author of the book under review. It presents the most exhaustive survey there is of the present state of the field.

The law $\mathfrak{L}$ of a random variable $X$ is described by its distribution or by its distribution function $F$ or by its characteristic function $f$, all with same affixes if any. The set of all laws is metrized by the Lévy metric $d(F_1, F_2)$. If $X = X_1 + X_2$ is sum of independent random variables $X_1$ and $X_2$, then its law $\mathfrak{L} = \mathfrak{L}_1\mathfrak{L}_2$ is described by the convolution, or composition, $F = F_1 * F_2$ or by the product $f = f_1 f_2$. $\mathfrak{L} = \mathfrak{L}_1\mathfrak{L}_2$ is "decomposable," or "factorizable," into "components" $\mathfrak{L}_1$ and $\mathfrak{L}_2$ if neither $\mathfrak{L}_1$ nor $\mathfrak{L}_2$ is degenerate. Decomposability is in fact if not in terminology a property of types of laws. P. Lévy (1937) produced "indecomposable," i.e. nondecomposable nondegenerate laws. On the other hand there are "infinitely divisible" or "infinitely decomposable" laws, "i.d." laws for short: $\mathfrak{L} = \mathfrak{L}_n^n$ for every integer $n > 0$. They were introduced by de Finetti (1929) and the characteristic functions of those with finite second moments were described explicitly by Kolmogorov (1932). The general form of i.d. characteristic functions was obtained by P. Lévy (1934) as a consequence of his exhaustive sample analysis of decomposable processes, i.e. of processes with independent increments. Hinčin (1937) gave a direct purely analytical proof of it: $f$ is i.d. iff $f = e^{\psi}$ with $\psi = (\alpha, \Psi)$ given by

$$\psi(t) = i\alpha t + \int_{-\infty}^{+\infty} h(x, t)\, d\Psi(x),$$

where $\alpha \in R$, $\Psi$ is a nondecreasing function of bounded variation, and

$$h(x, t) = \left(e^{ixt} - 1 - \frac{ixt}{1 + x^2}\right)\frac{i + x^2}{x^2}\left(= -\frac{t^2}{2} \quad \text{at } x = 0\text{–by continuity}\right).$$

The factorization idea and the superficial resemblance of indecomposable laws to prime numbers led to naming the field "Arithmetic of laws," a name which is fast disappearing. At present, decomposition theory abounds in counterexamples to every similarity to arithmetic and to offhand conjectures. It contains a large number of examples of decomposition properties of specific laws and of more or less restrictive classes of laws, but only a few relatively general results.

One may distinguish several kinds of results. I. The celebrated Lévy-Cramér theorem and its consequences. Its seminal role, the importance of normal laws in Probability and Statistics and its use in extending the usual normal limit theorems, ought to lead to the inclusion of the theorem and of its consequences into Probability theory. II. Hinčin's fundamental decomposition theorem and various indecomposability examples, most of which require only the analytical apparatus of a graduate probability course, may interest probability students. III. Properties of laws with no indecomposable components, or "$I_0$-laws", which require delicate and intricate investigations of analytic characteristic functions and, especially, of entire ones. IV. Generalizations to the multidimensional case and more abstract spaces as well as to "$\alpha$-factorizations."

I. In 1934 P. Lévy stated without proof that normal laws are decomposable into normal components only. In 1935 he deduced normal decomposition "stability": If a law is approximately normal so are its components, "approximately" being defined in terms of the Lévy metric. Furthermore, he extended his normal convergence theorem omitting the usual asymptotic negligibility requirement. In 1936 Cramér proved the Lévy decomposition conjecture, and in 1937 Raikov proved a similar decomposition of Poisson type laws into Poisson type components only. Both proofs used Hadamard factorization theorem for entire functions. In 1955 Linnik combined both results: If a law is composed of a normal law and a Poisson type one, so are all its components. Normal decomposition stability was extended to all decompositions by Linnik (1960). The following property is primarily due to him: If $F_n = F_{n1} * F_{n2}$ and $d(F_n, F) \to 0$ as $n \to \infty$, then $\max_{j=1,2} \inf_{G \in K_F} d(F_{nj}, G) \to 0$, where $K_F$ denotes the set of all components of $F$. Meanwhile Sapogov (1951, 1959) obtained an estimate of normal decomposition stability. It was shown to be the best possible of its kind by Maloševskiĭ (1968), who wrote the book's chapter on stability. Together with Nikitin he wrote the next chapter on limit theorems without asymptotic negligibility condition. There one finds the extended Poisson convergence theorem (Mačis, 1967), the extended Lindeberg-Feller normal convergence one (Zolotarev, 1968), and others. Not included are Zolotarev extensions of general limit theorems (1970).

II. *The fundamental decomposition theorem* (Hinčin, 1937/38) says that every law can be represented as a composition of at most two laws, one law consisting of a countable number of indecomposable ones and the other one, or $I_0$-law, having no indecomposable components. This decomposition is not necessarily unique. Example: The uniform distribution on $[-1, +1]$ with $f(t) = (\sin t)/t$ has a set of distinct decompositions (into indecomposable components) of the power of the continuum.

The $I_0$-laws are i.d. so that any law whose characteristic function has at least one zero has at least one indecomposable component. The $I_0$-laws form a proper subset of i.d. laws: The geometric distribution with $f(t) = (1-p)/(1-pe^{it})$, $0 < p < 1$, is i.d. but its components are indecomposable. Every i.d. law is composed of a countable number of $I_0$-laws.

*Indecomposable laws.* The set of all indecomposable laws is a $G_\delta$-set dense in the complete metric space of all laws; in fact, the set of all purely discontinuous indecomposable laws is dense in this space. See Parthasarathy, Rao, and Varadhan (1962).

EXAMPLES. Let the "spectrum" $S(F)$ of a law be the set of all points of increase of its distribution function $F$. Call a set $A \subset R$ "decomposable" if $A = B + C$ where $B$ and $C$ have each at least two points; otherwise $A$ is "indecomposable." If for every two distinct pairs $(x, y)$ and $(x', y')$ of points of $A$, $x - y \neq x' - y'$, then $A$ is indecomposable.

If $S(F)$ is indecomposable and inf $S(F)$ or sup $S(F)$ is finite, then the law with $F$ is indecomposable. For every perfect set $\mathfrak{S}$ there is an indecomposable law with $S(F) = \mathfrak{S}$. Similarly for every compact set $\mathfrak{S}$, and there are laws which are not discrete if $\mathfrak{S}$ is uncountable. Arcsine law is indecomposable while absolutely continuous.

III. *$I_0$-laws with normal components.* Let $\mathcal{P}(\Psi)$ be the "Poisson spectrum" of an i.d. law with $\psi = (\alpha, \Psi)$, i.e. the set of nonzero points of increase of $\Psi$; the law has a normal component when $\Psi(+0) - \Psi(-0) > 0$. We say that an i.d. law with normal component is a "Linnik law" if it has a countable Poisson spectrum $\mathcal{P}(\Psi)$ such that for any two of its elements of same sign, one is an integer multiple of the other.

*Linnik* (1958): Every $I_0$-law with normal component is a Linnik law. The proof is very intricate and long (pp. 95–130).

*Linnik* (1959)-*Ostrovskiĭ* (1965): If $\int_{|x| \geqslant y} d\Psi(x) = O(e^{-ay^2})$ for $y \to \infty$ and some $a > 0$, then a Linnik law is an $I_0$-law (with normal component). Ostrovskiĭ simplified Linnik's proof while weakening his sufficiency condition to the stated one; the proof is intricate and long (pp. 145–168). The condition cannot be omitted altogether, for there are Linnik laws which have components which are not even i.d. (Goldberg and Ostrovskiĭ, 1971).

*$I_0$-laws without normal components.* Let us mention only three representative and related results.

*Cramér* (1949)-*Shimizu* (1964): The i.d. law (without normal component) given by

$$\psi(t) = i\alpha t + c\int_a^b (e^{ixt} - 1)\, dx, \qquad 0 \leqslant a < b/2\, (< \infty), c > 0,$$

is not an $I_0$-law. A consequence of a somewhat more general Shimizu result:

Except for the normal law all nondegenerate stable laws have indecomposable components.

*Ostrovskiĭ* (1965): If the Poisson spectrum of an i.d. law without normal component lies in $[a, b]$ with $0 < a < b \leqslant 2a$, then this law is an $I_0$-one.

*Cuppens* (1969): Let an i.d. law with $\psi = (\alpha, \Psi)$ be such that $\int_{R\setminus\{0\}}(1 + x^2)/x^2 \, d\Psi(x) < \infty$ and $\Psi$ has a continuous derivative $\Psi'$ (the law has no normal component). It is an $I_0$-law iff the set $\{x\colon \Psi'(x) > 0\}$ lies in $(a, 2a)$ or in $(-2a, -a)$ where $a > 0$.

IV. *Decompositions of multidimensional laws.* Extensions to the multidimensional case appeared at birth of the decomposition problem. Cramér (1936) did it for the Lévy-Cramér case. Lévy (1937) did it for the i.d. laws representation and in 1948 showed that Wishart distribution–of interest to statisticians,

$$f(t_1, t_2, t_3) = \left[(1 - it_1)(1 - it_3) + t_2^2\right]^{-1/2}$$

is indecomposable while its projections are i.d. Starting in 1965, primarily Ostrovskiĭ and Cuppens extended almost all main results of the one-dimensional case to the multidimensional one. This required and continues to require delicate investigations of analytic characteristic functions in several complex variables. The decomposition problem is, in fact, that of factorizing Fourier transforms of probability measures on euclidean spaces into same kind of factors. This suggests at once that it ought to be studied in full generality within the framework of abstract harmonic analysis. The first results in this direction can be found in Parthasarathy's book (1968).

*α-factorizations.* A characteristic function $f$ is "α-factorized" into two "α-factors" if $f = f_1^{\alpha_1} f_2^{\alpha_2}$ where $\alpha_1$ and $\alpha_2$ are positive. In general, $f_1^{\alpha_1}$ and $f_2^{\alpha_2}$ are not characteristic functions so that there is loss of contact with probability theory. This kind of factorization was introduced by Linnik (1955). It is a purely analytical generalization and emerged as a tool for Darmois-Skitovich characterization of the normal law by means of independence of two linear forms in independent random variables. The characterization problems in mathematical statistics and the various methods used are discussed in the book on the subject by Kagan, Linnik, and Rao (1972, English translation 1973).

There are three recent books by authors who made significant contributions to decomposition theory and which overlap with the book under review. Lukăcs, *Characteristic functions* (2nd ed., 1970) consists of an almost exhaustive survey of characteristic functions including their factorizations and contains a lucid introduction to the main results–in the one-dimensional case. Ramachandran's book on *Advanced theory of characteristic functions* (1967) is a penetrating introduction to the methods and results of decomposition theory–in the one-dimensional case. Cuppens' book (1975) is devoted to *Decomposition of multivariate distributions*, and this brings us to a first criticism of the Linnik-Ostrovskiĭ book: It is regrettable that Lévy's "quite profound results" obtained in his investigation of $I_0$-membership i.d. laws with no normal component and with finite Poisson spectrum are not given therein–if only in an Appendix; in Cuppens' book some of Lévy's results are given in an extended and generalized form. Also the title is slightly misleading: In decomposition theory random variables and vectors figure only in terms of their laws, and the theory, while its origin is probabilistic, is purely analytical. However, those defects – at least in the eyes of the reviewer –are of very little importance. For the book ought to be considered as a classic–the best of its kind. It is well written and very instructive.

The total impression about the state of the theory is somewhat disturbing. The ingenuity and power of the methods and the great wealth of results still leave the basic problem unsolved: Find applicable general criteria so that, given a law one can find all its components, and, in particular, find whether it is an indecomposable or an $I_0$-law. It is hoped that the Linnik-Ostrovskiĭ book will serve as a catalyst for further search in this direction.

The untimely death of Linnik was a great loss for mathematics and for those who knew him.

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*Completeness and basis properties of sets of special functions*, by J. R. Higgins, Cambridge Tracts in Mathematics, no. 72, Cambridge Univ. Press, Cambridge, London, New York, Melbourne, 1977, x + 134 pp., $19.95.

The questions considered in this book arise from our wanting to represent a given function as a linear combination of particularly interesting or useful auxiliary functions–for example, the eigenfunctions of a boundary value problem. In this setting the idea has been traced back to Daniel Bernoulli, who used the expansion as a formal device; it was Fourier who showed that (sometimes) the formal solution is really a solution. There are natural questions to ask about Fourier series (apart from their use in solving eigenvalue problems): Does the series converge? Does it converge to the function we got it from? If so, is it the only series of its kind that represents that function? A collection of functions $\varphi_n$ such that every function $f$ (in a suitable class) has a unique expansion $\Sigma a_n\varphi_n$ that converges (in a suitable topology) to $f$ is called a basis. This notion, when formulated in abstract terms, can be considered in any Banach space, or even in more general spaces; a given set $\{\varphi_n\}$, regarded as abstract elements, may or may not form a base depending on which space they are taken to belong to. Thus for example the trigonometric functions $\{e^{inx}\}$ form a basis in $L^2$ (periodic functions of integrable square) but not in $C$ (continuous functions under uniform convergence). The trigonometric functions also form an orthogonal set, but this is only a feature that is convenient for computing the coefficients in the expansion, not an essential part of the idea of a basis. Most of the familiar separable Banach spaces turn out to have bases, but we know (only since 1973) that there are separable Banach spaces that have no bases [5].

A similar idea entered mathematics in a different way and beginners sometimes confuse it with the idea of a basis. In abstract terms, a set $\{\varphi_n\}$ of elements of a Banach space is called total if every element of the space can be represented as the limit of a sequence of finite linear combinations of the $\varphi_n$, i.e. as $\lim_{N\to\infty}\Sigma_1^N a(k, N)\varphi_k$ rather than as $\lim_{N\to\infty}\Sigma_1^N a(k)\varphi_k$. This is the