EMBEDDINGS OF $(n - 1)$-SPHERES IN EUCLIDEAN n-SPACE

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1. Introduction. The program described in this report revolves around the basic embedding question of geometric topology: *under what conditions are two embeddings f_1 and f_2 of a space X in a space Y equivalent in the sense that there exists a self-homeomorphism F of the ambient space Y for which $Ff_1 = f_2$?* This article focuses, in particular, on information and questions concerning embeddings of the $(n - 1)$-sphere in Euclidean n-space E^n, a specific problem that serves as a convenient abbreviation for discussing the broader category of embeddings of $(n - 1)$-manifolds in n-manifolds, and the article emphasizes a comparison between information about codimension one embeddings in high-dimensional n-manifolds, high usually requiring n to be at least 5, with the extensive collection of known information pertaining to embeddings of 2-manifolds in 3-manifolds.

First, we fix some indispensible notation and conventions. We use B^n to denote the standard n-cell in E^n consisting of all points in E^n having norm $\leqslant 1$ and S^{n-1} to denote the standard $(n - 1)$-sphere, also called the boundary ∂B^n of B^n, consisting of all points in E^n having norm $= 1$; we call a space homeomorphic to B^n or S^{n-1} an *n-cell* or an *$(n - 1)$-sphere*, respectively. For $1 \leqslant k < n$ we presume E^k to be included naturally in E^n as the subset whose final $(n - k)$ coordinates each equal 0, thereby determining a standard k-cell B^k and a standard $(k - 1)$-sphere S^{k-1} in E^n as well. We use E_+^k to denote the upper half space in E^k consisting of all points having kth coordinate $\geqslant 0$. We say that a k-cell or a $(k - 1)$-sphere X in E^n is *flat* if there exists a homeomorphism of E^n to itself that takes X to the standard object of its type. In this language, our fundamental concern is the question: *under what conditions is an $(n - 1)$-sphere in E^n flat?* Equivalently, *under what conditions is an embedding of S^{n-1} in E^n equivalent to the inclusion $S^{n-1} \to E^n$?*

Flatness questions for spheres and cells form the prototype of questions concerning local matters. Let e denote an embedding of an m-manifold M (a metric space locally homeomorphic to either E^m or E_+^m) in the interior of an n-manifold N (henceforth to be written as Int N). One says that e is *locally flat at $x \in M$* (and that $e(M)$ is *locally flat at $e(x)$*) if there exists a

neighborhood U_x of $e(x)$ in N and there exists a homeomorphism h of U_x onto E^n such that (i) $h(U_x \cap e(M)) = E^m$ in case $x \in \text{Int } M$ or (ii) $h(U_x \cap e(M)) = E^m_+$ in case $x \in \partial M$ (the boundary of M). Negatively, one says that $e(M)$ is *wild* at $e(x)$ if and only if $e(M)$ fails to be locally flat at $e(x)$. Standard parlance reduces the phrase "$e(M)$ is locally flat at each of its points" to "$e(M)$ is locally flat".

Obviously trivial in case $n = 1$, flatness theory in case $n = 2$ essentially was resolved early in this century by the work of Schoenflies [S1] showing (nontrivially) that all cells and spheres in E^2 are flat. It was redirected for $n > 3$ in 1924 by J. W. Alexander's examples of 2-spheres in E^3 [A1], [A2] illustrating the existence of wildness (and also correcting his own announcement to the contrary that all 2-spheres in E^3 are flat). A lull during the next quarter century was interrupted by the appearance of a brief but influential handbook of some wild cells and spheres in E^3, published by Fox and Artin [F1]. Once broken, the lull quickly was dissolved in the ensuing flood of results about the flatness (or tameness) of objects in E^3 generated by Harrold, Moise, Bing, Burgess, and their students, among whom J. W. Cannon should be mentioned for his relatively recent and exceptionally significant contributions. That history is reviewed in an extensive survey by Burgess and Cannon [B27] and in a shorter current report by Burgess [B26].

The classical 3-space examples of wild objects indicate that wildness pervades all dimensions $n > 3$ (explanatory constructions appear in §4). The study of flatness in E^n, for large n, lagged behind the 3-dimensional efforts by 10–15 years, not originating until the Mazur and Brown results on the Generalized Schoenflies Theorem late in the 1950's, but developing rapidly throughout the past decade, until at present the high dimensional theory appears to be almost as rich as that of dimension 3. This article, emphasizing interrelationships between these branches of flatness theory, can best be read as a companion to those of Burgess [B26] and of Burgess and Cannon [B27], with those two functioning as the primary reference sources concerning 3-dimensional work and with this one attempting to enact a comparable role concerning high dimensional work.

In 3-dimensional contexts, the concepts of flatness and tameness practically coincide, for then there is little difference between the topological and PL categories; in higher dimensional contexts this is not the case. One says that a (closed) subspace X of a PL n-manifold N is *tame* if there exists a PL homeomorphism h of N to itself such that $h(X)$ is a subpolyhedron; if X is a manifold with a preassigned PL structure, one requires, in addition, that $h|X$ be a PL embedding. Tameness often implies local flatness: if X is a tame m-dimensional submanifold-without-boundary of a PL n-manifold N and if $n - m \neq 2$, then X is locally flat (if $n - m = 1$, an example mentioned here in §4F indicates the necessity of requiring that $h(X)$ be a PL submanifold of N rather than simply a subcomplex of N). On the other hand, Kirby-Siebenmann [K4] have established that local flatness does not imply tameness because manifolds may fail to support PL structures and locally flat embeddings of one PL manifold in another may fail to be approximable by PL embeddings. Hence, ultimately the distinctions are inherently derived from categories to be considered: the concept of tameness belongs to the PL

category; that of flatness, to the topological category.

As prelude to the discussion, we consider the local homotopy properties associated with local flatness. Let A denote a subset of a metric space X, $a \in \text{Cl}(X - A)$ and k a nonnegative integer. We say that $X - A$ is *locally k-connected at a*, written $X - A$ is *k-LC at a*, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each map of S^k into $(X - A) \cap N_\delta(a)$ ($N_\delta(a)$ is defined as the set of points in X whose distance from a is less than δ) extends to a map of B^{k+1} into $(X - A) \cap N_\varepsilon(a)$; synonymously, we also refer to the same property by saying that A is *locally k-co-connected* (in X) *at a*, written A is *k-LCC at a*. We say that $X - A$ is *uniformly locally k-connected*, written *k-ULC*, or that A is *uniformly locally k-co-connected*, written *k-ULCC*, if a $\delta > 0$ exists as above independently of the point $a \in X$. Somewhat similarly, in case $A \subset B \subset X$ and $a \in \text{Cl}(X - A)$, we say that $X - B$ is *locally k-connected in $X - A$ at a*, written $X - B$ is *k-LC in $X - A$ at a*, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each map of S^k into $(X - B) \cap N_\delta(a)$ extends to a map of B^{k+1} into $(X - A) \cap N_\varepsilon(a)$, and we say that $X - B$ is *uniformly locally k-connected* in $X - A$, abgreviated as $X - B$ is *k-ULC in $X - A$*, whenever the corresponding uniform property holds. The stress, at least when confined to statements of results, will fall upon the choice $k = 1$, for which the concept is a (uniform) local version of the relatively familiar concept of simple connectedness (restricted to components).

To apply this language to embedding theory, let e denote an embedding of an *m*-manifold M in an *n*-manifold N such that e is locally flat at $x \in \text{Int } M$. Then $e(x)$ has arbitrarily small neighborhoods U_x in N such that

$$U_x - e(M) \approx E^n - E^m \approx E^m \times (E^{n-m} - \text{origin}).$$

This shows $U_x - e(M)$ to be homotopy equivalent to S^{n-m-1}. Accordingly, in case $m < n - 2$, a necessary condition for $e(M)$ to be locally flat at $e(x)$ is that $e(M)$ be *k-LCC* in N at $e(x)$, $k \in \{0, 1, \ldots, n - m - 2\}$; in case $m = n - 1$, a necessary condition for $e(M)$ to be locally flat at $e(x)$ is that $e(M)$ be *k-LCC* in N at $e(x)$ ($k \geqslant 1$). *For $m \neq n - 2$, the crucial feature arises when $k = 1$: a necessary condition for $e(M)$ to be locally flat at $e(x)$ is that $e(M)$ be 1-LCC in N at $e(x)$.* Furthermore, in order for $e(M)$ to be locally flat at $e(x)$, $x \in \partial M$, $e(M)$ must be *k-LCC* in N at $e(x)$ for each nonnegative integer k.

Two other remarks dealing with matters of definition: by an *ε-set*, where ε is a positive number, we mean a set of diameter less than ε; and for a subset A of E^n, by an *ε-map* of A into E^n we mean a map that moves no point as much as ε.

What follows is an expanded rendition of an earlier survey [**D9**], the major difference being inclusion here of treatments of sewings of crumpled cubes in §8 and of Ancel and Cannon's work on the Locally Flat Approximation Theorem and its consequences in §5. Besides these, the contents are organized as follows: §3 sets forth results known to imply flatness of $(n - 1)$-spheres in E^n; §4 provides some examples of wild spheres; §5 concentrates on general properties of topologically embedded spheres and of crumpled *n*-cubes (that is, spaces homeomorphic to the closure of a component of the complement of an $(n - 1)$-sphere topologically embedded in S^n); §6 gives a characterization

of the compact sets that lie on flat spheres, new for higher dimensions and corresponding precisely to that in dimension 3 (discovered also by Ancel and Cannon); §7 indicates why a new concept of pairwise taming set acts as the appropriate high dimensional analogue to the 3-dimensional taming set concept; finally, §9 makes application of an inflating technique, related to the identity sewing of a crumpled cube to itself, in order to construct some wild spheres with unusual geometric properties.

2. Collateral subjects and the fundmantal technique. To study topological embeddings, besides knowing some algebraic-topological duality theory, one must be acquainted with piecewise linear (PL) theory, for in it one encounters the technique most instrumental for developing sophisticated flattening results–the concept of engulfing. It functions as the means for expanding an open subset of an n-manifold to enclose, or engulf, a subpolyhedron P of the manifold, provided certain dimension, connectivity and finiteness conditions are satisfied, among which invariably is the demand that the codimension of P (defined as n-dim P) be at least 3. The literature reveals several diverse kinds of engulfing results, no one of which pertains to all possible situations. The radial engulfing theory formulated by Connell [**C15**] and improved by Bing [**B10**] finds frequent applications to embedding problems, often in modified form aligned with methods of Bryant and Seebeck [**B22**], [**B23**].

Although engulfing techniques do shed information about n-manifolds without any restriction on n, the full power of the process is engaged only when $n \geqslant 5$, because only then is there room enough to control top dimensional portions of the space. This limitation to $n \geqslant 5$ explains why, in Burgess' phrase, E^4 is an outcast among Euclidean spaces, with so few of the results described throughout this survey being known for $n = 4$.

T. B. Rushing's book [**R1**] develops a substantial overview of the important embedding techniques, It also functions as the only comprehensive reference organizing data about the local flatness of embedded manifolds.

In addition to PL theory, the other collateral area germane to embeddings is decomposition theory. For studying decompositions of E^n ($n > 2$), no reference as ideal as Rushing's text exists, but the survey articles of Armentrout [**A7**], [**A8**] provide a valuable perspective. While the topic encompasses no general technique so indispensible as engulfing, it persistently touches flattening theory through the interweaving of results about decompositions and results about flatness, a connection explicitly noted in and reinforced by Cannon's work on taming relations [**C6**] and further strengthened by Ancel and Cannon's recent solution of the Locally Flat Approximation Theorem [**A3**].

3. Characterizations of flat spheres.

3A. *General flatness results.* Here we state several general flatness theorems, proved without resorting to engulfing methods, that are valid at least in dimensions $n \geqslant 4$. The result crucial for opening up the territory was Brown's version of the Generalized Schoenflies Theorem:

3A.1. THEOREM [**B14**]. *Any bicollared $(n - 1)$-sphere in E^n is flat.*

An $(n - 1)$-sphere Σ in E^n is *bicollared* if there exists an embedding f of

$S^{n-1} \times (-1, 1)$ in E^n such that $\Sigma = f(S^{n-1} \times 0)$.

From his result that locally collared implies collared (see also Connelly [C16] for a direct, alternative proof), Brown also obtained a corollary revealing that the global problems reduce to the local problems.

3A.2. THEOREM [B15]. *Each locally flat $(n-1)$-sphere in E^n is flat.*

Historically the distinctions between the cases $n = 3$ and $n > 3$ were first displayed by Cantrell's doctoral dissertation, ultimately refined into the following result:

3A.3. THEOREM [C7]. *If Σ is an $(n-1)$-sphere in E^n $(n \geqslant 4)$ and p is a point of Σ such that Σ is locally flat at each point of $\Sigma - p$, then Σ is flat.*

Contrast with the $n = 3$ case is provided by the classical Fox-Artin examples [F1] of wild 2-spheres in E^3 that are locally flat modulo one point.

Since wildness and flatness are inherently local properties, it might seem that Theorem 3A.3 would readily extend to imply that codimension one manifolds can have no isolated wild point. Such an extension is less automatic than one initially suspects, for Cantrell's arguments depend too strongly on the specific topological type of $\Sigma - p$ to permit this localization directly. Using engulfing techniques (requiring $n \geqslant 5$) Černavskiĭ initially derived an important result leading to this extension, and later Kirby [K2] and Černavskiĭ [C10] both discovered an alternative approach, in the spirit of Cantrell's, that applied in all dimensions.

3A.4. THEOREM [C10], [K2]. *If B_1 and B_2 are flat $(n-1)$-cells in E^n such that $B_1 \cap B_2 = \partial B_1 \cap \partial B_2$ is an $(n-2)$-cell that is flat in both boundaries, then $B_1 \cup B_2$ is flat.*

Theorem 3A.4 readily implies the expected strengthened form of Cantrell's One Point Theorem (3A.3).

3A.5. COROLLARY [C10], [K2]. *If Σ is an $(n-1)$-sphere in E^n $(n \geqslant 4)$ and W^* is the set of points at which Σ is wildly embedded, then W^* contains no isolated point.*

3A.6. COROLLARY [C10], [K2]. *If Σ is an $(n-1)$-manifold in the interior of an n-manifold $(n \geqslant 4)$ and W^* is the set of points at which Σ fails to be locally flat, then W^* is empty or uncountable.*

With an elegant geometric construction Kirby [K1] sharpened the description of the wild set.

3A.7. THEOREM. *Suppose Σ is an $(n-1)$-manifold in an n-manifold N $(n \geqslant 4)$ such that Σ is locally flat modulo a Cantor set $X \subset \Sigma$, where X is tame both as a subset of Σ and as a subset of N. Then Σ is locally flat.*

Despite the apparent improvement upon 3A.5 given by 3A.7, the most powerful among the results just listed is the one concerning the union of flat cells (3A.4), for there exist relatively short proofs of 3A.7 based on 3A.4 (detailed first by Harrold and Seebeck [H1] and later by Benson [B2]).

Studying the local peripheral structure surrounding the embedded $(n-1)$-

manifold, Harrold and Seebeck [H1] introduced a weak local flatness property. Given a point p of an $(n - 1)$-sphere Σ in E^n, they say that Σ is *locally weakly flat at p* if there exist arbitrarily small n-cells B such that $p \in \text{Int } B$, $B \cap \Sigma$ is an $(n - 1)$-cell, and there exists a homeomorphism h of B onto B^n sending a neighborhood in Σ of $\partial B \cap \Sigma$ into $B^n \cap E^{n-1}$. In proving the ensuing theorem, they show that the set of wild points reduces to the situation covered by 3A.7.

3A.8. THEOREM [H1]. *If the $(n - 1)$-sphere Σ in E^n is locally weakly flat (at each of its points), then Σ is flat.*

Cantrell and Lacher [C8] have applied Theorem 3A.4 to obtain another general flatness theorem.

3A.9. THEOREM. *If the $(n - 1)$-sphere Σ in E^n admits a curvilinear PL triangulation T such that each closed simplex of T is flat in E^n, then Σ is flat.*

Examples to be described later show that the hypotheses in 3A.9 requiring the triangulation to be PL and the closed simplexes, rather than the open simplexes, to be flat are independently necessary.

3B. *The homotopy-theoretic characterization of flatness.* Most higher dimensional flattening results are based on the homotopy-theoretic criterion for local flatness, established separately by Černavskiĭ [C13] and myself [D4].

3B.1. THEOREM. *If Σ is an $(n - 1)$-manifold topologically embedded in the interior of an n-manifold N ($n > 5$) such that $N - \Sigma$ is 1-LC at each point of Σ, then Σ is locally flat.*

Price and Seebeck [P1] developed the substantial portion of Theorem 3B.1, based upon a conceptually simple but technically intricate engulfing argument, proving the result if, in addition, Σ is connected and is locally flat at some point. Building upon their work, the proof in [D4] also invokes the deep result due to Hsiang and Wall [H5] that any closed n-manifold ($n > 5$) homotopy equivalent to a torus (the Cartesian product of 1-spheres) is homeomorphic to a torus. Černavskiĭ's short and elegant argument, which avoids the technical complexity embodied in [P1], uses an infinite radial engulfing towards ends unlike Price and Seebeck's and eventually employs Siebenmann's Cellular Approximation Theorem [S6], itself based upon the homotopy classification of tori [H5]; the level of complexity could be significantly diminished in the Černavskiĭ procedure if a person could find an elementary (no surgery theory) proof for the following special case of Siebenmann's theorem:

3B.2. *Suppose G is an upper semicontinuous decomposition of E^n into points and countably many cellular sets such that (1) for each $\varepsilon > 0$ only a finite number of elements of G have diameter greater than ε and (2) E^n/G is homeomorphic to E^n. Then the decomposition map π: $E^n \to E^n/G$ can be approximated by homeomorphisms.*

Addendum. R. D. Edwards [E7] has recently discovered a nonsurgical proof of Siebenmann's full theorem.

Antecedents to Theorem 3B.1 include (for $n > 5$) Černavskiĭ's result that

any n-cell 1-ULCC embedded in E^n is flat [C11] and Seebeck's generalization that an $(n - 1)$-sphere Σ 1-ULCC embedded in E^n is flat iff Σ can be homeomorphically approximated by locally flat spheres [S2].

3C. *Applications of the homotopy characterization.* In this portion we describe flatness results derived as applications of Theorem 3B.1 and list some related unanswered questions. Let Σ denote an $(n - 1)$-sphere in E^n and U a component of $E^n - \Sigma$. One says that Σ can be *homeomorphically approximated from U* if for each $\varepsilon > 0$ there exists an ε-homeomorphism h of Σ into U.

3C.1. THEOREM. *An $(n - 1)$-sphere Σ in E^n $(n \geqslant 5)$ is flat if it can be homeomorphically approximated from each component of $E^n - \Sigma$.*

Obtaining this as a corollary to the 1-LCC Criterion reverses the original procedure for the 3-dimensional case, in which Bing first proved Theorem 3C.1 [B4] and from it developed the 1-LCC characterization of local flatness for codimension one submanifolds [B5]. In [S2] Seebeck accomplished a nearly equivalent high dimensional analogue, without appeal to the 1-LCC Property, by proving that an $(n - 1)$-sphere in E^n $(n \geqslant 5)$ is flat if it can be homeomorphically approximated from each component of $E^n - \Sigma$ by locally flat spheres.

Adding either 3B.1 or 3C.1 to the argument of Bryant in [B17] produces a slight improvement of his result.

3C.2. THEOREM [B17]. *Suppose that $\{\Sigma_\alpha\}$ is an uncountable collection of pairwise disjoint $(n - 1)$-spheres in E^n $(n \geqslant 5)$. Then all but countably many of the Σ_α are flat.*

As in the 3-dimensional case, still unanswered is the question of whether the hypothesis for Theorem 3C.1 can be weakened by replacing the homeomorphisms with maps of Σ into $E^n - \Sigma$. Using terminology introduced by Wilder, we say that Σ is *free from a component U* of $E^n - \Sigma$ if for each $\varepsilon > 0$ there exists an ε-map f of Σ into U.

3C.3. *Question.* Is an $(n - 1)$-sphere Σ in E^n flat if it is free from each component of $E^n - \Sigma$?

According to work of Bryant and Lacher [B20] and myself [D10], flatness can be detected if maps of Σ into $E^n - \Sigma$ are tied to the inclusion by a homotopy instantaneously ranging into $E^n - \Sigma$, first proved in case $n = 3$ by Hempel [H2]. For precision, given a component U of $E^n - \Sigma$, one says that Σ *can be deformed into U* if there exists a map H of $\Sigma \times I$ into E^n such that $H(s \times 0) = s$ and $H(s \times t) \in U$ for each $s \in \Sigma$ and $t \in (0, 1]$.

3C.4. THEOREM [B20], [D10]. *An $(n - 1)$-sphere Σ in E^n $(n \geqslant 5)$ is flat if it can be deformed into each component U of $E^n - \Sigma$.*

3C.5. COROLLARY. *An $(n - 1)$-sphere Σ in E^n $(n \geqslant 5)$ is flat if there exist a flat sphere S in E^n and a map f of E^n onto itself such that $f|S$ is a homeomorphism of S onto Σ and $f(E^n - S) = E^n - \Sigma$.*

A similar result, allowing structural singularities of another sort, has been devised by Bryant, Lacher and Smith [B21]. The relevant definition goes: a

compact set X in E^n has a *manifold mapping cylinder neighborhood* if there exists a map ψ of a closed (possibly nonconnected) $(n-1)$-manifold M onto X for which the mapping cylinder Z_ψ embeds naturally in E^n as a closed neighborhood of X.

3C.6. THEOREM [B21]. *An $(n-1)$-sphere Σ in E^n $(n \geqslant 5)$ is flat if it has a special manifold mapping cylinder neighborhood, namely, one for which the mapping cylinder Z_ψ is obtained from a Serre fibration $\psi: M^{n-1} \to \Sigma$, where each component of M^{n-1} is simply connected.*

Examples to be mentioned in §4 will clarify the need for the special hypothesis concerning the mapping cylinder neighborhoods. To circumvent it, Bryant and Lacher have a variation to 3C.6 juxtaposing mapping cylinder neighborhoods and freeness.

3C.7. THEOREM [B20]. *An $(n-1)$-sphere Σ in E^n $(n \geqslant 5)$ is flat if it has a manifold mapping cylinder neighborhood and it is free from each component of $E^n - \Sigma$.*

A property reminiscent of locally weakly flat was formulated by Burgess in [B24], declaring that an $(n-1)$-sphere Σ in E^n can be *locally spanned from a component* U of $E^n - \Sigma$ if for each $\varepsilon > 0$ and each $p \in \Sigma$ there exist $(n-1)$-cells D and R such that (1) $p \in \operatorname{Int} D \subset \Sigma$, (2) $\operatorname{Int} R \subset U$, (3) Bd $D = $ Bd R, and (4) $\operatorname{diam}(D \cup R) < \varepsilon$. Expressed specifically in 3-dimensional terms, the argument given in [B24] serves equally well in higher dimensions to establish the following result:

3C.8. THEOREM [B24]. *An $(n-1)$-sphere Σ in E^n $(n \neq 4)$ is flat if it can be locally spanned from each component of $E^n - \Sigma$.*

Several questions related to the work in [B24] have not been settled. For instance, Burgess weakened the definition of "locally spanned from U", saying that an $(n-1)$-sphere Σ in E^n can be *locally spanned in a component* U of $E^n - \Sigma$ if for each $\varepsilon > 0$ and $p \in \Sigma$ there exists an $(n-1)$-cell D of diameter less than ε, with $p \in \operatorname{Int} D \subset \Sigma$, such that for each $\alpha > 0$, U contains an $(n-1)$-cell R satisfying (1) diam $R < \varepsilon$ and (2) there is a homeomorphism of Bd D to Bd R moving points of Bd D less than α. In addition, one says that Σ can be *uniformly locally spanned in U* if the above properties hold for all sufficiently small disks D in Σ.

3C.9. *Question.* Is an $(n-1)$-sphere Σ in E^n flat if it can be locally spanned in each component of $E^n - \Sigma$? (Unknown for $n = 3$.)

3C.10. *Question.* Is an $(n-1)$-sphere Σ in E^n flat if it can be uniformly locally spanned in each component of $E^n - \Sigma$? (Affirmative answer in case $n = 3$ [B24].)

Further questions arise as these definitions are relaxed to permit singularities in the spanning cells R, that is, to require only that R be the continuous (instead of the homeomorphic) image of an $(n-1)$-cell.

Cannon has a generalization of Theorem 3C.8:

3C.11. THEOREM [C4]. *An $(n-1)$-sphere Σ in E^n $(n \geqslant 5)$ is flat if for each point $s \in \Sigma$ and each $\varepsilon > 0$ there exists an $(n-1)$-sphere S in E^n such that*

$s \in$ Int S, diam $S < \varepsilon$, and each component of $S - (S \cap \Sigma)$ is simply connected.

Frequently the spheres encountered are locally flat at some points, and the focus turns to the set X of points at which Σ is not known to be locally flat. In this case an elementary result, which should be considered part of the folklore, involving the embedding of X in both Σ and E^n can be applied.

3C.12. THEOREM. *An $(n - 1)$-sphere Σ in E^n $(n \geqslant 5)$ is flat if it is locally flat modulo a set $X \subset \Sigma$ such that $\Sigma - X$ is 1-ULC and $E^n - \Sigma$ is 1-ULC in $E^n - X$.*

3C.13. COROLLARY. *An $(n - 1)$-sphere Σ in E^n $(n \geqslant 5)$ is flat if it is locally flat modulo a k-cell $(k \neq n - 3)$ that is twice flat, that is, flat both as a subset of Σ and as a subset of E^n.*

Another apparatus for determining flatness involves piercing properties. Given a subset X of an $(n - 1)$-sphere Σ, we say that Σ can be *pierced along X* if there exist an embedding f of $X \times [-1, 1]$ in E^n such that $f(X \times [-1, 0))$ and $f(X \times (0, 1])$ lie in distinct components of $E^n - \Sigma$ and that $f(X \times 0) = X$; whenever meaningful, we add that Σ can be *pierced along X by locally flat* or *flat objects* (manifolds, cells) if there exist such embeddings f for which the images $f(X \times [-1, 1])$ are locally flat. Obviously, the local flatness of Σ is equivalent to its being pierced along a neighborhood of each point. Less obviously, if Σ can be pierced nicely along enough $(n - 2)$-dimensional sets, Σ must be locally flat.

3C.14. THEOREM [D11]. *An $(n - 1)$-sphere Σ in E^n $(n \geqslant 5)$ is flat if Σ can be pierced along each $(n - 2)$-cell K in Σ, where K is flat relative to Σ, by flat $(n - 1)$-cells.*

3C.15. *Question.* Is an $(n - 1)$-sphere Σ in E^n flat if it can be pierced along each $(n - 2)$-cell in Σ?

What minimal information concerning the pierced sets determines flatness? A tentative answer is offered in terms of the following property. Let p be a point of an $(n - 1)$-sphere Σ, and suppose $\{C_i | i = 1, 2, \ldots\}$ is a collection of $(n - 1)$-cells in Σ, each containing p, such that $C_{i+1} \subset$ Int C_i $(i = 1, 2, \ldots)$. Call the union of p and the sets ∂C_i an *infinite bullseye* (in Σ) centered at p. Burgess [B24] signalled the value of the infinite bullseye, in his solution to 3C.16 for the case $n = 3$.

3C.16. THEOREM [D11]. *An $(n - 1)$-sphere Σ in E^n $(n \geqslant 5)$ is flat if, for each point p in Σ, Σ can be pierced along an infinite bullseye centered at p.*

Benson has used piercing along a special set of potential wild points to establish a flattening theorem.

3C.17. THEOREM [B1]. *If the $(n - 1)$-sphere Σ in E^n $(n \geqslant 5)$ is locally flat modulo an $(n - 3)$-cell K such that K is flat relative to Σ and Σ can be pierced along K by a flat $(n - 2)$-cell, then Σ is flat.*

It is possible to produce examples [D7], [D12] illustrating the necessity of

the requirement that K be flat relative to Σ; the necessity of flatness in the other requirement is obvious.

In somewhat related fashion, we define a horizontal section X_t ($t \in E^1$) of a set X in E^n as the intersection of X with the $(n-1)$-plane $E^{n-1} \times t$. Generalizing the 3-dimensional Slicing Theorem of Eaton [E1] and Hosay [H4], we have shown:

3C.18. THEOREM [D1]. *If Σ is an $(n-1)$-sphere in E^n ($n \geqslant 6$) such that Σ_1 and Σ_{-1} are points and each horizontal section Σ_t ($t \in (-1, 1)$) is an $(n-2)$-sphere that is locally flat both as a subset of Σ and as a subset of $E^{n-1} \times t$, then Σ is flat.*

3C.19. *Question.* Is 3C.18 valid if one only requires of Σ_t ($t \in (-1, 1)$) that it be locally flat as a subset of $E^{n-1} \times t$?

Addendum. Daverman and Pax [D16] have produced an example to answer this question in the negative.

Since it has not been discussed elsewhere, we shall describe the validity of high dimensional analogues to the results of Burgess derived in [B25].

3C.20. THEOREM. *An $(n-1)$-sphere Σ in S^n ($n \geqslant 5$) is flat if each component U of $S^n - \Sigma$ is an open n-cell (homeomorphic to E^n) and Σ is locally peripherally collared from U in the following strong sense: each point s in Σ has arbitrarily small closed neighborhoods N in Σ such that Bd N is a simply connected $(n-2)$-manifold and Bd N is collared from U.*

The fundamental construction for the proof builds a neighborhood W of Σ in Cl U such that the complement in W of a collar C on some Bd N consists of two components, V_1 and V_2, with V_1 being a small neighborhood in Cl U of p. To establish that U is 1-LC at p, it suffices to consider a loop f: $\partial B^2 \to V_1 \cap U$ that extends to a map F: $B^2 \to V_1$. Since U is an open cell, there exists an $(n-1)$-sphere S in W separating $f(\partial B^2)$ and Σ, and the map F can be cut off on S, to define a map G: $B^2 \to S \cup (F(B^2) - \Sigma)$, after which the map G can be cut off on C, because of the simple connectedness of the collar, to define a map g of B^2 into the small set $(C \cup V_1) - \Sigma$.

The wild but cellular n-cells in S^n [D7], [D12] having boundaries locally flat modulo Cantor sets are locally peripherally collared in a weaker sense, not satisfying simple connectedness of the collar, but where each point p of the wild boundary sphere Σ is included in arbitrarily small closed neighborhoods N of p in Σ, N being an $(n-1)$-manifold, such that Σ is pierced along ∂N by a locally flat (in S^n) copy of $\partial N \times [-1, 1]$.

Finally, two homogeneity questions originally raised for 2-spheres in E^3 remain unresolved in any dimension $n \geqslant 3$.

3C.21. *Question.* Is an $(n-1)$-sphere Σ in E^n flat if it is homogeneous in the sense that for any two points p and q of Σ there exists a self-homeomorphism H of E^n such that $H(\Sigma) = \Sigma$ and $H(p) = q$?

3C.22. *Question.* Is an $(n-1)$-sphere Σ in E^n flat if it is strongly homogeneous (i.e., each self-homeomorphism h of Σ extends to a self-homeomorphism H of E^n)?

In high dimensions even the most outrageously powerful homogeneity property is not known to imply flatness.

3.C.23. *Question.* Is an $(n-1)$-sphere Σ in E^n flat if each isotopy of Σ extends to an isotopy of E^n?

Addendum. Gene C. Garza has pointed out that Theorem 3C.4 can be applied to answer this question in the affirmative.

4. Examples of wild $(n-1)$-spheres in E^n.

A. The primitive method for generating wild codimension one embeddings is to suspend lower dimensional examples: start with an $(n-2)$-sphere Σ' in E^{n-1} such that $E^{n-1} - \Sigma'$ fails to be 1-ULC and determine an $(n-1)$-sphere Σ in E^n as the geometric suspension of $\Sigma' \subset E^{n-1} = E^{n-1} \times 0 \subset E^n$ from the two points $(0, \ldots, 0, \pm 1)$–that is, Σ coincides with the union of all line segments in E^n joining $(0, \ldots, 0, \pm 1)$ to a point of Σ'. This sphere Σ cannot be flat because it also fails to satisfy the 1-LCC Property (in E^n).

B. Another construction technique has evolved from the classical 3-dimensional theory. In his initial display of wild 2-spheres in E^3, J. W. Alexander [A2] observed that Antoine's famous wild Cantor set [A6] can be viewed as a subset of a 2-sphere S in E^3, where S must then be wild because its complement cannot be simply connected. Later Blankenship showed, while constructing wild Cantor sets in E^n $(n > 3)$ [B11], that Alexander's observation extends significantly: for any Cantor set C in E^n there is an n-cell B in E^n such that Bd B contains C and is locally flat modulo C. This is formally stated by Osborne as well [O1]. Any time B is flat, the classical Klee trick [K5] indicates that each arc or Cantor set in Bd B is flat (tame); consequently, if C above is wild, so is Bd B.

C. Wild spheres can be easily detected in the product of the real line and certain decomposition spaces E^{n-1}/G of E^{n-1} for which $E^{n-1}/G \times E^1$ is known to be homeomorphic to E^n. A method generated by Brown [B16], this takes its most immediate application when G has a noncellular arc A as its only nondegenerate element, in which case Andrews and Curtis [A5] demonstrated the equivalence of $E^{n-1}/A \times E^1$ and E^n; one then can form a sphere Σ that contains a wild arc corresponding to the product of A and an interval in the E^1-factor.

D. An example is produced in [D7], from a construction more relentless than elegant, of a wild $(n-1)$-sphere in E^n $(n > 3)$ that is locally flat modulo a Cantor set and the Cantor set is tame relative to E^n. The example primarily serves to illustrate the sharpness of Kirby's result [K1] that a sphere locally flat modulo a twice tame Cantor set is flat (Theorem 3A.7), and because of its E^n flatness properties it also aids in reproducing more complex examples (for illustrations, see [D12, §13]). In §9 we shall set forth a more graceful alternative construction.

E. An example is produced in [D8] of a wild $(n-1)$-sphere Σ in E^n $(n > 3)$ such that each 2-cell in Σ is wildly embedded in E^n. Such examples, together with some earlier and simpler variants given in [D3], carry a form of wildness quantitatively more complicated than that of spheres previously described in the literature. Further evidence for this claim will be exhibited in §5A.

F. A novel variation to the suspension technique described under (A) is made possible by R. D. Edwards' recent solution to the Double Suspen-

sion Problem [E6], in which he gives a PL embedding of Zeeman's Dunce Hat (a contractible but noncollapsible 2-complex) in S^4 and a regular neighborhood N of its image such that $S^4 - N$ is nonsimply connected, ∂N is nonsimply connected, but the double suspension $\Sigma^2 \partial N$ is S^5. Curtis and Zeeman [C17] have shown how this is tied to wildness, by regarding $\Sigma^2 \partial N$ as the natural subset of the double suspension $\Sigma^2 S^4 = S^6$. One then perceives $\Sigma^2 \partial N$ to be wild because the component $S^6 - \Sigma^2 N$ of $S^6 - \Sigma^2 \partial N$ fails to be simply connected. This is especially surprising since $\Sigma^2 \partial N$ can be determined as the underlying point set of a subcomplex (but for which the triangulation cannot be PL–recall Theorem 3A.9) of a PL triangulation of S^6. Hence, $\Sigma^2 \partial N$ is a wild sphere having a manifold mapping cylinder neighborhood, thus accounting for what might appear to be extraneous hypotheses in 3C.6 and 3C.7. On the other hand, $S^5 = \Sigma^2 \partial N$ has an unusual triangulation induced from the suspension structure, unusual because it includes a wildly embedded 5-simplex, namely, any simplex containing a subarc of the suspension circle.

§9 presents further examples of wildly embedded spheres in E^n that possess geometric properties contrary to what is possible in E^3.

5. Properties of spheres in E^n. In 1912 Brouwer [B13] extended the planar Jordan Curve Theorem, deriving the fundamental separation property of codimension one spheres:

5A.1. THEOREM. *Each $(n - 1)$-sphere Σ in E^n $(n > 1)$ separates E^n into two components, each having Σ as its boundary.*

Later Wilder [W2] discovered a comparable local property:

5A.2. THEOREM. *If Σ is an $(n - 1)$-sphere in E^n $(n > 1)$, then each component U of $E^n - \Sigma$ is 0-ULC.*

5A.3. COROLLARY. *If Σ is an $(n - 1)$-sphere in E^n $(n > 1)$ and U is a component of $E^n - \Sigma$, then each point p of Σ is accessible from U (that is, there exists an arc A containing p such that $A - \{p\} \subset U$).*

Bing has shown that for any $(n - 1)$-sphere Σ in E^n, the closure of Int Σ (= bounded component of $E^n - \Sigma$) is a compact absolute retract.

5A.4. THEOREM [B8]. *For any $(n - 1)$-sphere Σ in E^n there exists a retraction of E^n to the closure of Int Σ.*

The wildness of a sphere Σ is partially revealed by the structure and location of sets F in Σ such that very small loops in $E^n - \Sigma$ can be contracted in small subsets of $(E^n - \Sigma) \cup F$. Although the set W consisting of all points in Σ at which Σ is wildly embedded always serves as one possible choice for F, generally F can be a proper subset of W (scattered exceptions occur when $n = 3$). Bing [B9] proved that each 2-sphere Σ in E^3 contains a 0-dimensional F_σ set F such that both $F \cup$ Int Σ and $F \cup$ Ext Σ are 1-ULC. An extension of sorts has been obtained by myself [D2] for $n \geqslant 5$ and recently by Ancel and McMillan [A4] for $n = 4$.

5A.5. THEOREM. *Each $(n - 1)$-sphere Σ in E^n $(n \geqslant 4)$ contains a 1-dimensional F_σ set F such that both $F \cup$ Int Σ and $F \cup$ Ext Σ are 1-ULC.*

5A.6. COROLLARY. *Each $(n - 1)$-sphere Σ in E^n $(n \geqslant 4)$ contains a 1-dimensional G_δ set G such that for each $\varepsilon > 0$ there exists an ε-map f of $\mathrm{Cl}(\mathrm{Int}\ \Sigma)$ (or $\mathrm{Cl}(\mathrm{Ext}\ \Sigma)$) into $G \cup \mathrm{Int}\ \Sigma$ (or $G \cup \mathrm{Ext}\ \Sigma$) that reduces to the identity outside the ε-neighborhood of Σ.*

5A.7. *Question.* Can "1-dimensional" in the statements of 5A.5 and 5A.6 be improved to "0-dimensional"?

Implications of Theorem 5A.5 include generalizations of Bing's result [B6] that 2-spheres in E^3 contain many flat arcs, extended first by Seebeck [S3] for $n \geqslant 5$ and shortly thereafter by Sher [S5] for $n = 4$, each of whom showed that every k-cell $(k > 1)$ in E^n, for the appropriate values of n, contains many flat arcs.

5A.8. THEOREM [B6], [S3], [S5]. *Let Σ denote an $(n - 1)$-sphere in E^n, A an arc in Σ, and $\varepsilon > 0$. Then there exists an ε-embedding ψ of A in Σ such that $\psi(A)$ is flat relative to E^n. Moreover, if A is flatly embedded in Σ, ψ can be obtained as the restriction to A of an ε-homeomorphism θ of Σ to itself that leaves points fixed outside the ε-neighborhood of A.*

Examples cited in §4E indicate that Bing's theorem about flat arcs on 2-spheres in E^3 cannot be transformed to a theorem establishing the existence of flat $(n - 2)$-cells on each $(n - 1)$-sphere in E^n $(n > 3)$, since the examples do not even include a single flat 2-cell. The degree to which 2-cells are flat is a pivotal property, however, for if most 2-cells in the $(n - 1)$-sphere Σ are flat, then most k-cells in Σ are flat $(k \leqslant n - 2)$.

5A.9. THEOREM [B18], [D6]. *Let Σ be an $(n - 1)$-sphere in E^n $(n \geqslant 6)$. The following statements are equivalent:*

(a) *There exist (curvilinear) PL triangulations T of Σ with arbitrarily small mesh for which the 2-skeleta $T^{(2)}$ are tame relative to E^n.*

(b) *There exists a 0-dimensional F_σ set F in Σ, where F is the union of a countable collection of Cantor sets that are tame in Σ, such that $F \cup W$ is 1-ULC for each component W of $E^n - \Sigma$.*

(c) *For each k-dimensional $(k \leqslant n - 3)$ polyhedron P in Σ and $\varepsilon > 0$, there exists an ε-homeomorphism θ of Σ to itself such that $\theta(P)$ is tame relative to E^n.*

(d) *For any k-cell K in Σ $(k \leqslant n - 2)$ and $\varepsilon > 0$, there exists an ε-homeomorphism θ of Σ onto itself such that $\theta(K)$ is flat relative to E^n.*

REMARK. Condition (d), unlisted in earlier versions of 5A.9, depends on the Locally Flat Approximation Theorem (5B.1) and requires its consequence that each $(n - 2)$-cell in a flat $(n - 1)$-sphere in E^n is flat in E^n (5B.5). A somewhat more complex variation to Theorem 5A.9 holds for $n = 5$ [D13].

5A.10. COROLLARY [D12], [D13]. *If the $(n - 1)$-sphere Σ in E^n $(n \geqslant 5)$ is the suspension of an $(n - 2)$-sphere in E^{n-1}, then each arc in Σ is flat relative to E^n and the conditions of 5A.9 all hold (even for the case $n = 5$).*

5B. *The Locally Flat Approximation Theorem and its consequences.* Surprisingly, the absence, until now, of a high dimensional analogue to Bing's

Polyhedral Approximation Theorem [B3], the progenitor of results concerning embeddings of surfaces in 3-manifolds, has not obstructed developments concerning embeddings of codimension one manifolds. Very recently Ancel and Cannon [A3] have obtained a satisfactory generalization.

5B.1. THEOREM [A3]. *Let M be an $(n-1)$-manifold in an n-manifold N $(n > 5)$ and let $\varepsilon\colon M \to (0, \infty)$ be continuous. Then there exists a locally flat embedding f of M in* Int N *such that $\rho(x, f(x)) < \varepsilon(x)$ for each $x \in M$.*

5B.2. COROLLARY. *If M is an $(n-1)$-manifold embedded as a 2-sided subset of a PL n-manifold N $(n > 6)$, then M admits a PL structure.*

PROOF. Because any sufficiently close approximation to M in N is also 2-sided, some locally flat embedding of M in N is bicollared. According to the Kirby-Siebenmann Product Structure Theorem [K4],[S7], M admits a PL structure consistent with the PL structure on the bicollar inherited from N.

An immediate consequence is the 1-ULCC characterization of one-sided flatness.

5B.3. COROLLARY [A3], [C13], [S4]. *If Σ is an $(n-1)$-sphere in E^n $(n > 5)$ such that* Int Σ *is 1-ULC, then Σ is collared from* Int Σ *(that is,* Cl(Int Σ) *is an n-cell).*

As part of a review of results similar to Corollary 5B.3, it should be emphasized, first of all, that both methods for proving the two sided version (3B.1) made inflexible use of the 1-ULC property in *each* complementary domain. Price and Seebeck [P1] discovered a stronger result, showing that any $(n-1)$-sphere Σ in E^n $(n > 5)$ such that Int Σ is 1-ULC and Σ can be approximated by locally flat spheres is collared from Int Σ, which combines with 5B.1 to produce Corollary 5B.3. Until learning of Ancel and Cannon's approach, I believed that resolution of 5B.1 would reverse the procedure for the 3-dimensional case by first reembedding the crumpled n-cube $C = \Sigma \cup$ Int Σ in S^n so the closure of its complement is an n-cell, obviating the need for the Price-Seebeck result. (*Addendum.* Very recently J. W. Cannon succeeded in reversing this procedure.) Both Bryant, Edwards and Seebeck [B19] and Štan′ko [S11] claimed Theorem 5B.1 but obtained only partial solutions, because the arguments could not account for potential embeddings giving negative answers to Question 5A.7 (in fact, Štan′ko's argument will not account for embeddings like those of [D12, Example 13.3]). Černavskiĭ [C13] has announced the following stronger version of 5B.1, and Seebeck [S4] independently but, it appears, subsequently deduced the same fact.

5B.4. THEOREM [C13], [S4]. *Suppose U is an open n-manifold and M is an open $(n-1)$-manifold. Suppose $X = U \cup M$ is a locally compact metric space such that $U \cap M = \varnothing$, U is dense in X, and U is k-LC at each point of M $(k = 0, 1, \ldots, n; n > 5)$. Then X is an n-manifold with boundary M.*

Another consequence of 5B.1 completes the theory of flatness for cells and spheres in hyperplanes begun by Klee [K5]. For codimension 3 cells or spheres, the analogue is due to Bryant-Seebeck [B23], and for codimension 1 cells, to Černavskiĭ [C11].

5B.5. THEOREM [A3], [B19]. *Suppose Σ is an $(n-2)$-sphere or $(n-2)$-cell in $E^{n-1} \times 0 \subset E^n$ $(n \geqslant 5)$. Then Σ is flat in E^n.*

Furthermore, instead of the expected reversal of the 3-dimensional approach, the Locally Flat Approximation Theorem functions as the primary constituent in reembedding crumpled cubes, yielding the high dimensional analogue to work of Hosay [H3] and Lininger [L1].

5B.6. THEOREM [D13]. *For each crumpled n-cube C in S^n $(n \geqslant 5)$ and $\varepsilon > 0$, there is an ε-homeomorphism h of C in S^n such that the closure of $S^n - h(C)$ is an n-cell.*

5B.7. COROLLARY. *Let C be a crumpled n-cube $(n \geqslant 5)$. There exists an upper semicontinuous decomposition G of B^n into points and flat arcs such that each arc g of G intersects ∂B^n in an endpoint of g and $B^n / G \approx C$.*

PROOF. Think of C as a closed n-cell-complement in S^n so that there is a collar θ: Bd $C \times I \to S^n - $ Int C for which θ(Bd $C \times 0) = $ Bd C. Then $C \cup \theta$(Bd $C \times [0, 1/2])$ is homeomorphic to B^n, and the decomposition G consists of points and the arcs corresponding to

$$\left\{ \theta\left(c \times [0, 1/2]\right) | c \in \text{Bd } C \right\}.$$

Bing reconstructed his proof of the Polyhedral Approximation Theorem for 2-spheres to obtain what he called the Side Approximation Theorem [B7], providing better controls on the approximations. Exploiting the crumpled n-cube results. we refine the Locally Flat Approximation Theorem to a high dimensional Side Approximation Theorem. Given an $(n-1)$-sphere Σ in E^n and a component W of $E^n - \Sigma$, we say that Σ can be *almost approximated from W* if for each $\varepsilon > 0$ there exists in ε-embedding h of Σ in E^n such that $h(\Sigma)$ is locally flat and each component of $h(\Sigma) - W$ has diameter less than ε. (Thus, there exist disjoint open ε-sets $A_1, \ldots, A_k$ in $h(\Sigma)$ such that $(h(\Sigma) - \cup A_i) \subset W$.)

5B.8. SIDE APPROXIMATION THEOREM [D14]. *Each $(n-1)$-sphere Σ in E^n $(n \geqslant 5)$ can be almost approximated from either component of $E^n - \Sigma$.*

PROOF. Consider, say, the unbounded component W of $E^n - \Sigma$. We only establish the theorem under the assumption that Cl(Int Σ) is an n-cell.

Fix $\varepsilon > 0$. There exists a very small (its smallness will be prescribed below) neighborhood U of Σ in $N_\varepsilon(\Sigma)$, and there exists a compact PL n-manifold Q such that $\Sigma \subset$ Int $Q \subset Q \subset U$. Determine a triangulation T of Q for which each simplex has diameter less than $\varepsilon/3$, and let P denote the 2-skeleton of T. The neighborhood U can be chosen so that there exists an $(\varepsilon/3)$-homeomorphism h of E^n onto itself such that $h|W - Q = 1$ and $h(W)$ contains the 1-skeleton of T and that there exists another $\varepsilon/3$-homeomorphism g of E^n such that $g|$Int $\Sigma - Q = 1$ and $g($Int $\Sigma)$ contains the $(n-3)$-skeleton of T' dual to P. Thicken the open 2-simplexes of $h(P)$ to obtain an open set O such that the diameter of each component of O is less than ε and $O \cup W \supset h(P)$, and set $W' = O \cup W$. Stretch $g($Int $\Sigma)$ across the join structure of T' by an $(\varepsilon/3)$-homeomorphism f of E^n such that

$f|E^n - Q = 1$ and

$$fg(\text{Int } \Sigma) \cup h(W') \supset Q.$$

Thus,

$$fg(\text{Int } \Sigma) \cup h(W') \supset (\text{Int } \Sigma - Q) \cup Q \cup (\text{Ext } \Sigma - Q) = E^n.$$

Define H as $h^{-1}fg$, observe that H is an ε-homeomorphism, and observe that $fg(\Sigma) \subset h(W')$, implying that $H(\Sigma) \subset W'$. Since Σ is collared, $H(\Sigma)$ is approximated arbitrarily closely by locally flat spheres in $W' = O \cup W$. Such approximations satisfy the definition for this choice of ε.

The proof of the general case appeals to Corollary 5B.7.

This result stands as one of the rare instances in which engulfing theory contributes substantive information about 4-space. The preceding proof for the simplified situation where Σ bounds a cell can be readily adapted to the general situation, provided $n = 4$, since all that need be engulfed from either side is a 1-complex.

5B.9. THEOREM. *A 3-sphere* Σ *in* E^4 *can be almost approximated from either component of* $E^4 - \Sigma$ *if and only if* Σ *can be approximated by locally flat spheres.*

Spheres that almost approximate $\Sigma \subset E^n$ from a complementary domain W intersect Σ in sets having small components. However, unlike what Bing obtains for 2-spheres in E^3, one cannot obtain, in general, approximations whose intersection with Σ is covered by the interiors of finitely many pairwise disjoint small open $(n - 1)$-cells either in Σ or in the approximation; the former would imply that all spheres satisfy the equivalent conditions of Theorem 5A.9. For amplification on this subject, see [D14].

As justification for the emphasis placed on codimension one spheres, we show how the Locally Flat Approximation Theorem transforms local portions of a codimension one manifold embedded in an n-manifold into the canonical sphere-in-E^n situation.

5B.10. THEOREM. *Let* M *denote an* $(n - 1)$*-manifold topologically embedded in the interior of an* n*-manifold* N *($n \geqslant 5$) and* $p \in \text{Int } M$. *Then there exist a neighborhood* U_p *of* p *in* N *and an embedding* h *of* U_p *in* E^n *such that* $h(M \cap U_p)$ *lies in an* $(n - 1)$*-sphere.*

PROOF. Looking at a coordinate chart, we can simplify the setting so that $p \in \text{Int } M \subset E^n \subset S^n$. Fix an $(n - 1)$-cell B such that $p \in \text{Int } B \subset B \subset M$, and choose a neighborhood U of p for which $M \cap \text{Cl } U \subset \text{Int } B$. According to 5B.1, we can assume that M is locally flat modulo $M \cap \text{Cl } U$. Let S^{n-1} denote the standard sphere in S^n. Since Bd B is flat (again by 5B.1), we can assume that Bd $B \subset S^{n-1}$ standardly so that, moreover, a collar on Bd B in B also lies in S^{n-1}, where the components of $S^{n-1} - \text{Bd } B$ are denoted as C_+ and C_- and where the collar on Bd B is contained in C_+.

Using the universal cover of $S^n - \text{Bd } B$, as in [C12] or [K1], we can construct an embedding h of a neighborhood U' of Int B into $S^n - \text{Cl } C_-$ that is the inclusion near Bd B. Then $h(M \cap U')$ is contained in the $(n - 1)$-sphere $C_- \cup h(B)$, as required.

5B.11. *Question.* Does each $(n - 1)$-sphere (n-cell) in an n-manifold have a neighborhood that can be embedded in E^n?

Addendum. A negative answer seems likely. D. R. McMillian, Jr. [**M2**], has constructed an arc in an n-manifold ($n \geqslant 4$) having no neighborhood that can be embedded in E^n.

Price and Seebeck, in their attack on the 1-ULCC characterization of flatness, established a useful property of codimension one embeddings:

5B.12. THEOREM [**P1**]. *Let Σ denote an $(n - 1)$-sphere in E^n ($n > 5$) and ε a positive number. Then there exists $\delta > 0$ such that for any two locally flat δ-embeddings h_0 and h_1 of Σ in E^n, there exists an ε-isotopy θ_t of E^n onto itself satisfying $\theta_0 = 1$, $\theta_t | E^n - N_\varepsilon(\Sigma) = 1$, and $\theta_1 h_0 = h_1$.*

Combining this with the Locally Flat Approximation Theorem produces the following corollary.

5B.13. COROLLARY. *Let Σ denote an $(n - 1)$-sphere in E^n ($n \geqslant 5$) and ε a positive number. There exist a locally flat ε-embedding e of Σ in E^n and a pseudo-isotopy P_t of E^n onto itself (a level-preserving map $P: E^n \times I \to E^n \times I$ for which $P_t: E^n \to E^n$ is a homeomorphism whenever $t < 1$) such that $P_0 = 1$, $\rho(P_t, 1) < \varepsilon$, $P_t | E^n - N_\varepsilon(\Sigma) = 1$, $P_1 e = 1_\Sigma$, and $x \in E^n - \Sigma$ implies $P_1^{-1}(x)$ is a singleton.*

In contrast with Corollary 5B.13 and, for one direction, like Corollary 3C.5, the Cellular Approximation Theorem ([**S6**], also [**C6**]) gives another flatness characterization involving cellular maps for which the singularities do not touch the $(n - 1)$-spheres.

5B.14. THEOREM. *Let Σ denote an $(n - 1)$-sphere in E^n ($n \geqslant 5$). Suppose f is a cellular map of E^n onto itself such that $f(E^n - \Sigma) = E^n - f(\Sigma)$ and $f | \Sigma$ is 1-1. Then Σ is flat if and only if $f(\Sigma)$ is flat.*

Boyd [**B12**] has proved that for any 2-sphere Σ in E^3 there exists a monotone map m (that is, each set $m^{-1}(x)$ is compact and connected) of E^3 onto itself such that $m(E^3 - \Sigma) = E^3 - m(\Sigma)$, $m | \Sigma$ is 1-1, and $m(\Sigma)$ is flat. A slight modification of the suspension of such a map m associated with a wild 2-sphere indicates that "cellular" cannot be reduced to "monotone" in the statement of 5B.14.

5B.15. *Question.* If Σ is an $(n - 1)$-sphere in E^n, is there a monotone map m of E^n to itself such that $m | \Sigma$ is 1-1, $m(\Sigma)$ is flat, and $E^n - m(\Sigma) = m(E^n - \Sigma)$?

6. Flat subsets of spheres. As mentioned previously, each $(n - 1)$-sphere in E^n contains many flat arcs but need not contain a single flat k-cell ($k \geqslant 2$). To what extent then are subsets of spheres flat?

We say that a compactum X that lies in some $(n - 1)$-sphere Σ in E^n is *flat* if there exists a flat $(n - 1)$-sphere in E^n containing X, and we say that *the pair (Σ, X) can be flattened* if there exists a flat embedding f of Σ in E^n for which $f | X = 1_X$. A nonobvious consistency occurs with this terminology, for if X ($\subset \Sigma$) denotes a compact manifold, X is flat in the sense given above if and only if X is a locally flat submanifold in E^n.

While writing specifically about the 3-dimensional situation, Burgess and Cannon [B27, §7] direct attention toward two questions:

What properties characterize the flat subsets of $(n - 1)$-spheres?

What information about the embedding of a sphere can be deduced from information about its flat subsets?

For the second question no completely satisfactory answer has been found to date, but for the first a pleasing answer in homotopy-theoretic terms can be given, as a precise analogue to that for the 3-dimensional case discovered by Cannon [C3]. This answer, which also has been perceived by Ancel and Cannon, is based first on a reduction permitted by the Locally Flat Approximation Theorem and, second, on an observation about the applicability of Štan'ko reembedding techniques [S10], [S11].

Cannon's work [C3] reveals the homotopy-theoretic invariant.

6.1. THEOREM. *Suppose X is a compact subset of an $(n - 1)$-sphere Σ in E^n. Then $E^n - \Sigma$ is 1-ULC in $E^n - X$ if and only if $E^n - X$ is 1-ALG (that is, for each $\varepsilon > 0$ there exists $\delta > 0$ such that any loop in $E^n - X$ that is null-homologous in a δ-subset of $E^n - X$ is null-homotopic in an ε-subset of $E^n - X$).*

A more familiar variation holds under restriction on the dimension of X.

6.2. COROLLARY. *Suppose X is a compact subset of an $(n - 1)$-sphere Σ in E^n and $\dim X \leqslant n - 3$. Then $E^n - \Sigma$ is 1-ULC in $E^n - X$ if and only if $E^n - X$ is 1-ULC.*

6.3. COROLLARY [C4]. *Suppose X is a compact subset of $(n - 1)$-spheres Σ_1 and Σ_2 in E^n. Then $E^n - \Sigma_1$ is 1-ULC in $E^n - X$ if and only if $E^n - \Sigma_2$ is 1-ULC in $E^n - X$.*

The main result captures the strong conclusion that if $X \subset \Sigma$ is flat in E^n, then the pair (Σ, X) can be flattened by embeddings close to the inclusion.

6.4. THEOREM. *Suppose X is a compact subset of an $(n - 1)$-sphere Σ in E^n $(n > 5)$. Then the following statements are equivalent.*

(A) *X is flat.*

(B) *$E^n - X$ is 1-ALG.*

(C) *The pair (Σ, X) can be flattened.*

(D) *For each $\varepsilon > 0$ there exists an ε-embedding e of Σ in E^n such that $e|X = 1_X$ and $e(\Sigma)$ is flat.*

REMARKS ABOUT A PROOF. The crucial implication (B) $\Rightarrow$ (C) rests upon an observation about the arguments of Bryant, Edwards and Seebeck [B19] or Štan'ko [S11]. First, however, according to the Locally Flat Approximation Theorem, the setting can be simplified to one in which Σ is locally flat modulo X, without ruining the hypothesis that $E^n - \Sigma$ is 1-ULC in $E^n - X$ (6.3). For the closure C of either component of $E^n - \Sigma$, the wildness of Σ as measured from C has the most elementary possible form (in the language developed in §8, C is said to be of Type 1). This permits the Štan'ko immersion techniques employed in [B19] and [S11] to be applied for reembedding C in E^n via an embedding h such that $E^n - h(C)$ is 1-ULC. The sequence of immersions e_i essential to the construction of h can be obtained

so that $e_i | X = 1_X$, simply because the support of each e_i can be confined to small neighborhoods of 2-cells in $E^n - X$ bounding given loops in $E^n - e_i(C)$; consequently, the desired h can be determined subject to the additional requirement that $h | X = 1_X$. Inverting the process leads to an embedding h^* of $C^* = \text{Cl}(E^n - h(C))$ in E^n such that $h^* | X = 1_X$ and $E^n - h^*(C^*)$ is 1-ULC. As a result, $h^* h(\Sigma)$ is flat, for each of its complementary domains is 1-ULC. With epsilonic controls added, such an argument yields that (B) implies (D).

6.5. COROLLARY. *Suppose X is a compact $(n - 3)$-dimensional subset of some $(n - 1)$-sphere in E^n $(n > 3)$ such that $E^n - X$ is 1-ULC. Then X is flat.*

As a cautionary word concerning countable unions, recall that there exist *wild* spheres Σ that are locally flat modulo flat subsets X (for example, the suspension of a certain Fox-Artin example is locally flat modulo a flat $(n - 3)$-cell); such spheres Σ can be expressed as the countable union of flat subcompacta, by merely rewriting $\Sigma - X$ as the countable union of compact sets.

Contrary to the potential absence of flat k-cells $(k > 2)$ in an $(n - 1)$-sphere in E^n, Σ must contain flat $(n - 2)$-dimensional compacta.

6.6. THEOREM. *Let Σ be an $(n - 1)$-sphere in E^n $(n > 5)$. For each $\varepsilon > 0$ there exists a flat compact set X_ε in Σ such that the components of $\Sigma - X_\varepsilon$ form a null sequence of sets, each of diameter less than ε.*

The proof consists of a relatively uncomplicated construction of an open subset $\Sigma - X$ of Σ such that the components of $\Sigma - X$ form a null sequence, as desired, and $E^n - \Sigma$ is 1-ULC in $E^n - X$.

7. Spheres that are locally flat modulo flat subsets. The literature of 3-space topology extensively presents results of the general type: *If Σ is an $(n - 1)$-sphere in E^n that is locally flat modulo a subset X such that . . . , then Σ is flat.* Eventually the 3-dimensional information of this sort was collected and refined by Cannon into what he called taming set theory; in his terminology a subset X of some sphere in E^n is a *taming set* (for $(n - 1)$-spheres in E^n) if whenever S is an $(n - 1)$-sphere in E^n containing X and embedded locally flatly modulo X, then S itself is flat. Building upon work of Hosay and Loveland, Cannon [C1] characterized the compact taming sets in E^3 as those compacta that lie on flat spheres and that have no degenerate component.

In comparison, the characterization in higher codimensions seems to constrict taming set theory dramatically.

7.1. THEOREM [D7]. *A compact proper subset X of an $(n - 1)$-sphere in E^n $(n > 4)$ is a taming set if and only if X is countable.*

After developing 7.1, I dismissed in print, possibly prematurely, higher dimensional taming set theory as essentially being without substance. Now, following a suggestion of J. W. Cannon, I would like to mention a closely related, less flexible concept that may be more valuable.

Let X denote a compact subset of some $(n - 1)$-sphere S in E^n. The disadvantages of high dimensional taming set theory manifested in Theorem 7.1 stem from the definitional disregard of the embedding of X in S. To

counter this, we specify the embedding, calling X an (S, X)-*taming set* if for each embedding e of S in E^n such that $e|S - X$ is locally flat and $e|X$ is the inclusion, $e(S)$ is flat. The Locally Flat Approximation Theorem implies that each (S, X)-taming set X in E^n $(n \geqslant 5)$ is flat. As an immediate corollary to Theorem 3C.12 and Corollary 3C.13, we obtain a homotopy-theoretic means for detecting pairwise taming sets.

7.2. THEOREM. *If X is a compact subset of an $(n - 1)$-sphere S in E^n $(n \geqslant 5)$ such that $S - X$ is 1-ULC and $E^n - S$ is 1-ULC in $E^n - X$, then X is an (S, X)-taming set.*

What accounts for the richness of 3-dimensional taming set theory is that the compact subsets X of a 2-sphere S for which $S - X$ is 1-ULC are characterized intrinsically, ignoring properties of the embedding, by the existence of a positive lower bound to the diameters of the components of X.

Theorem 7.2 falls short of providing a complete representation of the (S, X)-taming sets, because countable unions preserve pairwise taming sets.

7.3. THEOREM. *Suppose X is a compact subset of an $(n - 1)$-sphere S in E^n $(n \geqslant 5)$ and suppose X can be expressed as a countable union of compacta $\{X_i\}$ such that each X_i is an (S, X_i)-taming set. Then X is an (S, X)-taming set.*

The crux of the matter is that, if X_i is an (S, X_i)-taming set, then Cl $W - X_i$ is 1-ULC, where W denotes either component of $E^n - S$ (see [C1, §5]). From this it is quite easy to prove that Cl $W - \bigcup X_i$ is 1-ULC [C1], [C4], [D1].

7.4. *Question.* Does the combination of 7.2 and 7.3 characterize (S, X)-taming sets? That is, if X is an (S, X)-taming set, can X be expressed as a countable union of compacta $\{X_i\}$ for which each $S - X_i$ is 1-ULC?

The affirmative answer to 7.4 for (pairwise) taming sets in E^3 given by Cannon [C1] leads to the improvement characterizing (absolute) taming sets in that case. Despite the unsettled nature of 7.4, the narrower concept of pairwise taming set nevertheless appears to be the reasonable high-dimensional analogue to that of taming set.

It may also be possible to find a narrower high dimensional analogue to that of *-taming set introduced and applied so effectively by Cannon ([C2] and its sequels); however, it remains unclear whether that would be effectively beneficial and consequently we do not pursue the idea here.

8. Sewings of crumpled n-cubes. A *crumpled n-cube* C is a space homeomorphic to the closure of a complementary domain of an $(n - 1)$-sphere Σ embedded in S^n. The set of points corresponding to Σ is called the *boundary of C*, written Bd C, and $C - $ Bd C is called the *interior of C*, written Int C. A *sewing* of two crumpled n-cubes C and D is a homeomorphism h between their boundaries; associated with any such sewing h is the *sewing space*, denoted as $C \cup_h D$, namely, the quotient space obtained from the disjoint union of C and D under the identification of each point $p \in$ Bd C with its image $h(p) \in$ Bd D.

The topic of sewings arises as a dual to that of reembedding crumpled n-cubes–instead of simplifying the embedding of an $(n - 1)$-sphere by

straightening the structure on one side, it contributes to the production of pathology whenever two crumpled cubes are pasted together so as to identify some pair of wild points from each. Interest in the sewing process usually centers on the problem of whether the sewing space is an n-manifold. In our restricted setting ($n \geqslant 5$), the Generalized Poincaré Conjecture [S8] implies:

8.1. THEOREM. *If $C \cup_h D$ is an n-manifold, then $C \cup_h D \approx S^n$.*

Accordingly, the question about sewings that is basic to the study of embeddings asks:

8.2. *Question.* Under what conditions does a sewing of two crumpled n-cubes yield S^n?

In case $n = 3$, Eaton [E3] has determined an effective answer to this question, which will be discussed later in this chapter. Eaton [E4] also devised the original example for $n \geqslant 4$ of a sewing of crumpled n-cubes (precisely, of a crumpled n-cube to itself) that does not yield S^n.

8A. *Closed n-cell-complements.* While examining the pathology of a codimension one embedded sphere, one occasionally finds it a convenient simplification to assume that the wildness is confined to one side by supposing that the embedded sphere is collared from the other side. Adopting terminology sporadically employed in the literature, we say that a crumpled n-cube C is a *closed n-cell-complement* if there exists an embedding h of C in S^n such that $S^n - h(\text{Int } C)$ is an n-cell, and we translate Theorem 5B.6 into this language.

8A.1. THEOREM [D13]. *Each crumpled n-cube C ($n \geqslant 5$) is a closed n-cell-complement.*

8A.2. COROLLARY. *Let h be a sewing of crumpled n-cubes C and D ($n \geqslant 5$). Then there exists an upper semicontinuous decomposition G of S^n into points and flat arcs such that $S^n/G = C \cup_h D$.*

8.B. *Sewings that yield S^n and the types of crumpled cubes.* The basic result is the analogue to one direction of Eaton's 3-dimensional Mismatch Theorem [E3], which shows that a sewing of two crumpled 3-cubes yields S^3 iff it satisfies the Mismatch Property. Following Eaton, we say that a sewing h of crumpled n-cubes C_0 and C_1 satisfies the Mismatch Property if $\text{Bd } C_e$ contains a set F_e such that $F_e \cup \text{Int } C_e$ is 1-ULC ($e = 0, 1$) and $h(F_0) \cap F_1 = \varnothing$.

8B.1. MISMATCH THEOREM [D12, THEOREM 5.1]. *If a sewing h of crumpled n-cubes C_0 and C_1 ($n \geqslant 5$) satisfies the Mismatch Property, then $C_0 \cup_h C_1 \approx S^n$.*

Unlike the 3-dimensional situation, satisfaction of the Mismatch Property is not a necessary condition for a sewing to yield S^n, but for an important class of sewings it is necessary. To expose which class, we have found it beneficial, as a crude measure of the one-sided wildness along the boundary sphere, to distinguish certain types of crumpled n-cubes C by enumerating properties that allow the complexity of the wildness to expand, as follows:

Type 1. There exists a 0-dimensional F_σ set F in $\text{Bd } C$ such that F is a

countable union of Cantor sets that are tame relative to Bd C and $F \cup \text{Int } C$ is 1-ULC.

Type 2. There exists a 0-dimensional F_σ set F in Bd C such that $F \cup \text{Int } C$ is 1-ULC, but C is not of Type 1.

Type 3. There exists a 1-dimensional F_σ set F in Bd C such that $F \cup \text{Int } C$ is 1-ULC, but C is not of Type 2.

Theorem 5A.5 reveals that every crumpled n-cube can be typified in one of these three ways. It was discovered, after this scheme was originally introduced, that the crumpled cubes C of Type 2 should be split further into two subtypes, (Type 2A) those for which $C \cup_{\text{Id}} C = S^n$ and (Type 2B) those for which $C \cup_{\text{Id}} C \neq S^n$, due to the formation of examples from each subtype in [**D12**, §13]. The theory of demension (embedding dimension) developed by Stan'ko [**S9**] and extended by Edwards [**E5**] gives another valuable perspective for distinguishing these types, under the additional assumption that Bd C is collared from $E^n - C$: the particular set F called for in the definition of a Type 1 crumpled cube C has demension 0 relative to Bd C and also, it turns out, relative to E^n; in a Type 2A crumpled n-cube C, the set F has demension $n - 3$ relative to Bd C and demension 0 relative to E^n; in crumpled cubes C of Type 2B or Type 3, the set F has demension $n - 3$ relative to Bd C and can have demension $n - 2$ relative to E^n. The failure of demension theory to separate these last two types constitutes the intrinsic difficulty encountered when searching for a Type 3 crumpled n-cube (Question 5A.7).

8B.2. Theorem [D12]. *Let h denote a sewing of Type* 1 *crumpled n-cubes C_0 and C_1 ($n \geqslant 5$). Then $C_0 \cup_h C_1 \approx S^n$ if and only if h satisfies the Mismatch Property.*

8B.3. Example [D12, Example 13.2]. There exists a sewing s of crumpled n-cubes D_1 and D_2 of Types 1 and 2, respectively, such that $D_1 \cup_s D_2 \approx S^n$ but s does not satisfy the Mismatch Property.

If one of the crumpled n-cubes involved is of Type 1, a sewing can be approximated by another that satisfies the Mismatch Property.

8B.4. Theorem [D12]. *Let C and D denote crumpled n-cubes ($n \geqslant 5$), with C of Type* 1 *and D of arbitrary type, and let h: Bd $C \to$ Bd D be a sewing. Then for each $\varepsilon > 0$ there exists a sewing g: Bd $C \to$ Bd D such that $\rho(g, h) < \varepsilon$ and $C \cup_g D \approx S^n$.*

8B.5. *Question.* If C and D are arbitrary crumpled n-cubes, is there a sewing g of C and D such that $C \cup_g D \approx S^n$?

Up to topological equivalence, the involutions of S^n leaving topologically embedded $(n - 1)$-spheres pointwise fixed stand in bijective correspondence with the crumpled n-cubes C for which the identity sewing Id: Bd $C \to$ Bd C yields S^n. Consequently, the following criterion can be used to describe peculiar involutions on S^n; in addition, it will be applied later as a fresh method for constructing wild embeddings.

8B.6. Theorem [D12]. *A crumpled n-cube C ($n \geqslant 5$) is of Type* 1 *or of Type* 2A *if and only if $C \cup_{\text{Id}} C \approx S^n$.*

Since crumpled n-cubes of Type 1 form such a tractable class, some

geometric properties by which they can be identified are collected from [D5], [D6], [D12] and are listed in the following theorem. See also Theorem 5A.9.

8B.7. THEOREM. *Let C be a crumpled n-cube in E^n ($n \geqslant 5$). Then C is of Type 1 if any one of the following conditions hold*:

(i) *C is homeomorphic to the suspension of a compact space X*;

(ii) *Bd C has curvilinear PL triangulations T of arbitrarily small mesh such that $C - T^{(2)}$ is 1-ULC*;

(iii) *Bd C is locally flat modulo a k-dimensional polyhedron K, $k \leqslant n - 3$, such that K is tame in Bd C*;

(iv) *Bd C is locally flat modulo a k-dimensional polyhedron K, $k \leqslant n - 2$, such that K is tame in E^n*.

8C. *Universal crumpled n-cubes*. A crumpled n-cube C is *universal* if, for each sewing h of C to another (variable) crumpled n-cube D, $C \cup_h D = S^n$. Obviously 8A.1 implies that the n-cell is a universal crumpled n-cube. At present there exists no characterization of universality, but there do exist several conditions implying it.

8C.1. THEOREM [D12]. *Under any one of the following conditions, the crumpled n-cube C is universal ($n \geqslant 5$)*:

(i) *Bd C contains a countable set Q such that $Q \cup$ Int C is 1-ULC*;

(ii) *For each compact 1-dimensional set X in Bd C, $C - X$ is 1-ULC*;

(iii) *C is the suspension of a compact space*.

8C.2. COROLLARY. *Let Z denote Blankenship's wild Cantor set [B11] in S^n ($n \geqslant 5$), and let C denote a crumpled n-cube such that Bd C contains Z as a tame subset and Bd C is locally flat modulo Z. Then C is universal*.

8C.3. COROLLARY. *If h is a sewing of crumpled $(n - 1)$-cubes C and D, then the suspension of $C \cup_h D$ is homeomorphic to S^n*.

See [D15] for the case $n = 4$.

All the universal crumpled n-cubes extant are of Type 1, but apparently the only formal restriction concerning type is the one resulting from 8B.6. A similar lack of information prevails concerning the weaker notion of self-universality (a crumpled n-cube C for which every sewing of C to itself yield S^n). Further details can be found in [D12, §12].

8C.4. *Question*. Is every universal crumpled n-cube of Type 1?

8C.5. *Question*. If C is a universal crumpled n-cube, does every sewing h: Bd $C \to$ Bd D satisfy the Mismatch Property?

9. More wild $(n - 1)$-spheres in E^n. An effective method for constructing wild examples trades on a simple technique called *inflating* introduced in [D12], for which the primary constituent is a crumpled $(n - 1)$-cube C in E^{n-1} (presumption: Int C is not 1-ULC) and the secondary one is an arbitrary continuous function f of C in $[0, 1]$ such that Bd $C = f^{-1}(0)$. Whenever $C \cup_{\mathrm{Id}} C \approx S^n$, the graphs of f and $-f$, considered as subsets of $C \times E^1 \subset E^n$, combine to produce an $(n - 1)$-sphere S, which cannot be flat since the vertical retraction of Int S to Int $C \times 0$ reveals that Int S fails to be 1-ULC. Clearly S is locally flat modulo Bd $C \times 0$; furthermore, in case Bd C

is locally flat (as a subset of E^{n-1}) modulo $X \subset \mathrm{Bd}\ C$, then S is locally flat modulo $X \times 0$.

The crumpled n-cubes obtained by inflating satisfy a weak universality property.

9.1. THEOREM [D12]. *If the crumpled n-cube C is obtained by inflating* $(n \geqslant 5)$, *then C is self-universal.*

Inflating affords an elementary means for constructing a wild $(n-1)$-sphere locally flat modulo a tame (in E^n) Cantor set (recall Example 4D).

9.2. EXAMPLE [A1], [D7], [D12]. *For $n \geqslant 3$ there exists a wild $(n-1)$-sphere in E^n that is locally flat modulo a Cantor set tame relative to E^n.*

Construction. We consider only $n \geqslant 6$. Determine a crumpled $(n-1)$-cube C in E^{n-1} such that $\mathrm{Bd}\ C$ is locally flat modulo a Cantor set X that is tame relative to $\mathrm{Bd}\ C$ (C not an $(n-1)$-cell). Since C is of type 1, C inflates to a wild $(n-1)$-sphere Σ in E^n, and Σ is locally flat modulo the tame (in E^n) Cantor set $X \times 0 \subset E^{n-1} \times 0$.

This inflating technique is particularly useful for generating wild codimension one spheres in E^n ($n > 3$) possessing geometric properties known to imply flatness in case $n = 3$. We illustrate with three examples.

First of all, in contrast with the 3-dimensional result of Jensen and Loveland [J1] it is immediately obvious that there can be no general vertical taming theorem in high dimensions.

9.3. EXAMPLE. *There is a wild $(n-1)$-sphere in E^n intersecting every straight line parallel to the x_n-axis in at most two points ($n \geqslant 4$).*

Second, in contrast with a 3-dimensional result of J. Cobb [C14] (see also [B27, Theorem 9.3.2]), there is no general flattening theorem associated with the concept of visibility. A closed subset X of an $(n-1)$-sphere S in E^n is *visible from the point* $p \in \mathrm{Int}\ S$ if each ray emanating from p and meeting X intersects S in precisely one point.

9.4. EXAMPLE. *There exists a wild $(n-1)$-sphere S in E^n ($n \geqslant 4$) such that S is locally flat modulo $X \subset S$ and X is visible from some point of $\mathrm{Int}\ S$.*

Construction. Choose a crumpled $(n-1)$-cube C in E^{n-1} such that (i) $C \cup_{\mathrm{Id}} C \approx S^{n-1}$, (ii) $\mathrm{Bd}\ C$ is locally flat modulo a tame (in E^{n-1}) Cantor set X, and (iii) C contains a flat (in E^{n-1}) $(n-1)$-cell B having X in its boundary, and then rearrange the embedding of C so that B is a round ball in E^{n-1} with center point p. The wild sphere S defined by

$$(C \times 0) \cup \left\{ \langle c, \rho(c, E^{n-1} - C) \rangle \in C \times E^1 \subset E^n | c \in C \right\}$$

is locally flat modulo $X \times 0$, and $X \times 0$ is visible from the point $\langle p, \frac{1}{2}\rho(p, E^{n-1} - C) \rangle$.

Third, there is no general high dimensional flattening theorem comparable to the work of L. A. Weill [W1]. Weakening Bing's condition that there exist arbitrarily close reembeddings of a given codimension one sphere in its interior, Weill has shown that a necessary and sufficient condition, in case $n = 3$, for an $(n-1)$-sphere S in E^n to bound an n-cell is (Weill's

Condition): *for each compact subset F of* Int S *there exists an $(n - 1)$-sphere S' in* Int S *such that $F \subset$* Int S' *and each $x \in S'$ can be joined to some point of S by an arc α in $\{x\} \cup$ Ext S', with* diam $\alpha < \rho(S, F)$. Certainly Weill's Condition is necessary in all dimension for flatness.

9.5. EXAMPLE. *There exists a wild $(n - 1)$-sphere S in E^n that does not bound an n-cell but satisfies Weill's Condition.*

Construction. Choose a crumpled $(n - 1)$-cube C in E^{n-1} such that (i) $C \cup_{\mathrm{Id}} C \approx S^{n-1}$, (ii) Int C is an open $(n - 1)$-cell, and (iii) Bd C is collared from $E^{n-1} - C$, and use (ii) to obtain a map $f: C \to [0, 1]$ for which Bd $C = f^{-1}(0)$ and $f^{-1}(t)$ is a bicollared $(n - 2)$-sphere whenever $t \in (0, 1)$. Inflate C to a wild $(n - 1)$-sphere S.

To see that S satisfies Weill's Condition, fix a compact set F in Int S and determine $\delta > 0$ so that $\delta < \rho(S, F)$ and the vertical projection of F into Int $C \times 0$ is contained in $f^{-1}((\delta, 1]) \times 0$. Let B denote the $(n - 1)$-cell in Int C bounded by $f^{-1}(\delta)$. For the sphere

$$S' = \{\langle x, \pm (f(x) - \delta)\rangle \mid x \in B\}$$

the required arcs readily can be discerned as vertical ones of diameter δ. It can also be shown that the wild sphere S fulfills the appropriate analogue of Weill's Condition for Ext S.

Weill [W1] also derives a consequence of his flatness criterion for which the high dimensional analogue remains unresolved.

9.6. *Question.* Must an $(n - 1)$-sphere S in E^n be flat if for each $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that the metric δ-envelope of S, defined as $\{x \in E^n \mid \rho(x, S) = \delta\}$, is the union of two $(n - 1)$-spheres?

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