appear to some that years of herculean effort have yielded limited progress, but after all the antagonist is a formidable foe.

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The Selberg trace formula for PSL (2, **R**), Volume I, by Dennis A. Hejhal, Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 1976, iv + 516 pp., $ 15.20.

For the last twenty-five years or so the Selberg trace formula has had, in the general mathematical community, an aura of mystery which is only slowly dissipating. This circumstance makes it necessary for us to look a little at the history and nature of this formula in order to understand properly the position of this new book.

First of all, the Selberg trace formula has precedents some of which are very old indeed. The underlying technical ideas have been in common currency amongst applied mathematicians since the turn of the century; these arose in the study of Laplace's equation and we would now associate them with groups like $O(3, \mathbf{R})$. Furthermore, various versions are to be found in earlier investigations concerning automorphic forms. These were mostly number-theoretical and hinged around the class-number formulae discovered by Kronecker and studied further by Fricke, Mordell, Hecke and Eichler. But also from the differential-geometric point of view both J. Delsarte and H. Huber came very close to an explicit trace formula (for $PSL(2, \mathbf{R})$).

Yet, nevertheless, Selberg's discovery of this formula in the early 1950's was a revolutionary event and its impact is far from spent. This lies in the nature of the formula. Although I have continually referred to it as a *formula* it is much more a *method*; a method, that is, for probing more deeply into the nature of discontinuous groups and their function theory. In broad terms, the Selberg trace formula arises when one learns to think functional-analytically about automorphic functions and forms. This has been the *pons asinorum*; it forces one to shed preferences for complex-analytic functions and prejudices against 'soft analysis'. Once this has been done a new land, full of promise, opens up.

There are two approaches to the trace formula; that due to Selberg which uses differential and integral operators–and in fact the differential operators can be eliminated–and that due to Gelfand and his collaborators which uses representation theory. The latter is now almost indispensable for general, especially number-theoretic questions, whereas for the study of Fuchsian groups the former is more flexible. It is this that is used in this book and we shall first look at it a little more closely.

The basic idea is the following. Let S be a 'good' topological space and m a measure on S. Let A be a commutative family of compact integral operators on $L^2(S, m)$ and we suppose that the adjoint of any operator in A is also in A. Then, from spectral theory, we know that A can be 'diagonalised' and under our assumptions there exists a countable orthonormal basis $\{v_n;\ n \in \mathbf{N}\}$ of

$L^2(S, m)$ so that if $K(x,y)$ is the kernel of an operator in A then there is a function $n \mapsto c(K, n)$ so that

$$K(x,y) = \sum_n c(K, n) v_n(x) \overline{v_n(y)}.$$

At the moment this exists only in an L^2-sense but if we put a few more conditions on K it converges absolutely and uniformly. Thus if we set $x = y$ and integrate

$$\int_S K(x, x)\, dm(x) = \sum c(K, n).$$

This is the basic form of the trace formula and the argument here shows why it is so-called. Note that in this formula there is no reference to the v_n about which we cannot expect to know much in general.

The problem now is to construct a useful algebra A and to compute $c(K, n)$. As an example let $S = \mathbf{R}/\mathbf{Z}$ and let m be the Lebesgue measure. Let A be made up of all K of the form

$$K(x,y) = \sum_{m \in \mathbf{Z}} k(x - y + m)$$

where k is a continuous function of compact support on $\mathbf{R}$. Then $\{v_n\}$ can be taken to be $\{\exp(2\pi i x)\colon m \in \mathbf{Z}\}$ and the formula above becomes the Poisson summation formula. Here the structure of $\mathbf{R}$ as an additive group has come to our aid and this happens much more generally.

Suppose that we are given a group G (often a Lie group or a p-adic Lie group) acting transitively on a space X (and satisfying some further conditions) and that S is of the form $\Gamma \backslash X$ (quotient space with Γ acting on the left) where Γ is a discrete subgroup acting discontinuously on X so that S is compact. Then we can often form such an algebra A (e.g. from spherical functions) in a natural fashion. In such cases the two sides of the above formula can be brought in to more elegant forms; the left-hand side depends on the conjugacy classes of Γ, the right-hand side on the 'spectrum' of A, or by abuse of language, of Γ. The trace formula then puts these two into a sort of duality. To see how all this works in the case $G = PSL(2, \mathbf{R})$, $X =$ the upper half-plane, the reader can hardly do better than to read the first 30 pages of the book under review. This case is especially important as every Riemann surface of genus $\geqslant 2$ can be realized as such an S. One can, from this point of view, unify much of the classical function theory of compact Riemann surfaces and derive the Riemann-Roch theorem. This is because complex-analytic functions are solutions of differential equations, but the Selberg trace formula, which is not restricted to such functions, is much more general.

Selberg wrote his original paper [2] emphasising the nature of the method but without going into the detailed calculations of any particular case. Such case-studies require considerable analytical ingenuity and facility with special functions–precisely those virtues that were taught in older mathematics courses. Selberg did, however, discuss such cases in lectures at Princeton and Göttingen and there is extant in Göttingen a partial set of lecture notes that is very illuminating.

Despite what has been said above, several people have understood and made use of the Selberg trace formula. The antinomy arises as those who have done this have usually not given an exposition in general terms of the lines of thought that led to their understanding. Thus the often spectacular results have appeared like black magic and, to use a cliché, the gulf between the 'haves' and 'have-nots' has widened. The most notable exception to this has been the German school (Maass, Roelcke, Elstrodt) but their work has not become as widely known as it deserves.

This "brings us by a commodius vicus of recirculation back to " Professor Hejhal's book. It describes the author's own process of understanding and applying the ideas of Selberg. In other cases this might well be considered a weakness but here, I am sure, it will lead many people into this beautiful circle of ideas. There are a great number of informal notes, especially at the ends of chapters, which motivate and explain what is being done, and, sometimes, what is not being done. This informal and unpolished style is entirely in keeping with the declared intentions of the 'Springer Lecture Notes' series. To say it again, this is not the final treatise but a set of notes of considerable interest.

The book splits naturally into two parts, Chapter 2 being different in kind to the other four. In these other four the Selberg trace formula for Fuchsian groups of compact quotient (cocompact) is developed in its simplest form (in Chapter 1) and later extended to cover 'vector-valued' automorphic functions, automorphic forms to nonzero integral weight and modular correspondence (i.e. Hecke operators). These essentially involve working out Selberg's ideas in these cases and the material is not novel. On the other hand the formulae, which elsewhere, are usually left as exercises for the energetic reader, are proved in detail. This material could have been unified and generalized by a slightly more sophisticated use of spectral theory and Petersson's theory of unitary multipliers of arbitrary real weight and dimension. This would have had the advantage of yielding the full Riemann-Roch theorem (a point left open on p. 433), although then the calculations have to be approached rather differently. Much of this is hinted at in the notes.

The most substantial part of the book is Chapter 2. It deals with the theory of the Selberg zeta-function. This is a function whose analytic properties essentially contain the Selberg trace formula; it can also be regarded as more or less a determinant of the resolvent of the Laplace-Beltrami operator. The relation of this function to the Selberg trace formula is the same as that subsisting between the Riemann zeta-function and the 'explicit formulae' of prime number theory. Moreover the 'Riemann hypothesis' for the Selberg zeta-function is, with a small caveat, true. It also has an 'Euler product' expansion. Thus one is led to try to apply the methods of analytic number theory to study the Selberg zeta-function. This is precisely what Professor Hejhal does in Chapter 2, although only for the 'simplest' trace formula. The chief consequence drawn is an asymptotic description of the conjugacy classes of the group in question; this is analogous to the prime number theorem. This is a remarkable fact that, by using spectral theory and only the most rudimentary geometrical information one can deduce quite refined geometrical results that are automatically (but, in a sense, not universally) true. Not even the leading terms of the expansions have been proved by geometric (i.e. packing and covering) arguments.

In this chapter the author strives both towards the best possible results of this type and towards discovering their limitations by proving what are often called Ω-theorems. There is, as one would expect, still quite a large gap between these. The development has not been made as slick as possible and the results proved in one section are often improved later using better techniques. But it is quite possible, as the author remarks, that the expertise so gained in handling the Selberg zeta-function will be more significant in the long run than the specific results presented here.

In this connection it is worth pointing out that the techniques lay stress on the 'explicit formulae', especially where these are not absolutely convergent. Such questions are very delicate; the author has courageously undertaken this investigation and has found several interesting results. This part of the book form in itself a useful little compendium on analytic number theory containing much that is not readily accessible. This, and the excellent bibliography will prove useful to many.

To end this review I shall state two problems, both of which are quite old now, but which are more 'structural' than those considered in this book. They appear to me to be the most significant problems about the type of Selberg zeta-function considered in this book; however if one widens the class of Fuchsian groups under consideration one is led to many difficult problems, some of which are to appear in the promised "Volume 2". These problems will, I hope, pose a challenge to the readers of this book.

At this late stage it is necessary to be more specific. Let G be a Fuchsian group acting on the upper half-plane $\mathbf{H}$ so that $G \setminus \mathbf{H}$ is compact and suppose that G has no elements of finite order. Let $\hat{G}$ be the group of characters of G (1-dimensional unitary representations); this is then naturally isomorphic to the Jacobian variety of $G \setminus \mathbf{H}$. If $\chi \in \hat{G}$ let $Z_G(s, \chi)$ be the Selberg zeta-function attached to G and χ. If χ is trivial we write $Z_G(s)$ for $Z_G(s, \chi)$; this is the function studied in this book. The following problem is due to Ray and Singer [**1**] (where there are also more details about $Z_G(s, \chi)$):

(a) *Express* $Z_G(1, \chi)$ *in terms of theta functions.*

There is a fair amount of circumstantial evidence that this is possible and such a formula would represent a far-reaching generalisation of the second Kronecker limit formula. More generally, and more vaguely, if χ is an arbitary unitary representation of G does $Z_G(1, \chi)$ have an interesting interpretation in terms of the moduli spaces of vector bundles introduced by Weil, Narashiman and Mumford?

The second question is attributed to Gelfand:

(b) *Does* $Z_G(s)$ *determine* G?

It is known that there are at most finitely many G with the same $Z_G(s)$ (McKean) but it is not, as far as I know, even known if two G of arithmetic type need have distinct zeta-functions. Both of these problems touch on the relation of Selberg's theory to Teichmüller theory (d'après Ahlfors, Bers, Rauch, . . .). Some results in this direction have been obtained by John D. Fay but the general picture remains very obscure.

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Gaussian processes, function theory, and the inverse spectral problem, by H. Dym and H. P. McKean, Probability and Mathematical Statistics, vol. 31, Academic Press, New York, San Francisco, London, 1976, xi + 333 pp., $35.00.

A stationary Gaussian process is a continuous map $t \to \xi_t$ from the real line into the real L^2 space of a probability measure, P, with the following properties:

(i) $\int \xi_t \, dP = 0$ for all t;

(ii) $\int \xi_s \xi_t \, dP$ depends only on the difference $t - s$ (and so can be written as $Q(t-s)$, where Q is a continuous positive definite function on the line, known as the covariance function of the process);

(iii) every function in the linear span of the functions ξ_t is normally distributed.

By Bochner's theorem, the covariance function Q admits a representation

$$Q(t) = \int e^{itx} \, d\Delta(x),$$

where Δ is a positive measure on the line, symmetric with respect to the origin. This leads to what is called the spectral representation of the process: the map sending ξ_t to the function e^{itx} on the line extends to an isometry sending the span, in complex $L^2(P)$, of the functions ξ_t onto the space $Z = L^2(\Delta)$.

The Gaussian condition (that is, condition (iii)) enables one to give geometric interpretations to various probabilistic aspects of the process. The simplest instance is the statement that, in the L^2 span of the functions ξ_t, orthogonality is equivalent to stochastic independence. Because of the spectral representation, one can go a step further, translating probabilistic questions about the process into questions in analysis. The questions in analysis that arise usually involve the theory of Hardy spaces in the upper half-plane and the theory of entire functions of exponential type. It is to them that the book under review is devoted.

The process is called deterministic if its past determines its future. This means, in probabilistic terms, that every function ξ_t is measurable with respect to the σ-algebra generated by the family $\{\xi_s: s \leq 0\}$. Because the process is Gaussian, the latter reduces to the condition that every ξ_t belong to the span, in $L^2(P)$, of the family $\{\xi_s: s \leq 0\}$. Application of the spectral representation now shows that the process is deterministic if and only if Z is spanned by the functions e^{isx}, $s \leq 0$. A criterion is provided by a theorem