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WHITEHEAD GROUPS OF FINITE GROUPS[1]

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In 1966, Milnor surveyed in this Bulletin [23] the concept of Whitehead torsion, focusing on the definition, topological significance and computation of Whitehead groups and their relationship to algebraic K-theory and the congruence subgroup problem. As Milnor showed in that survey [23, Appendix 1], an affirmative solution to the congruence subgroup problem for algebraic number fields would imply that for any finite abelian group G, $SK_1(\mathbf{Z}G) = 0$; i.e. that the Whitehead group of a finite abelian group G is torsion-free. At that time the status of the congruence subgroup problem was uncertain [23, pp. 360, 416]; it was subsequently shown to have a negative solution by Bass, Milnor and Serre [7]. Nevertheless, until 1972 all finite abelian groups for which computations could be made had trivial SK_1 (cf. [5, p. 624]) and the question of whether these groups could be nontrivial remained open [6].

An intensive study of Milnor's K_2-functor on discrete valuation rings [10] and the application of Mayer-Vietoris sequences in algebraic K-theory led to the first examples of finite abelian groups with nontrivial SK_1 and have provided an algorithm for the computation of such SK_1's in general. In addition, the first steps towards the computation of $SK_1(\mathbf{Z}G)$ for nonabelian finite groups have been taken by several authors.

It is my purpose to survey these techniques and computations, beginning where Milnor left off in 1966. I will rely heavily on his article for background material; all unexplained notations and terminology should be sought there.

If G is a finite group, its order is denoted $|G|$ and its abelianization, G^{ab}. A finite field with q elements is denoted $\mathbf{F}_q$. The units of a ring A are denoted A^* or $U(A)$.

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1. Whitehead groups. Let R be an associative ring with 1, and suppose $n \geqslant 1$. Let $GL_n(R)$ be the group of all invertible $n \times n$ matrices with entries in R. We can embed $GL_n(R)$ in $GL_{n+1}(R)$ by sending an $n \times n$ matrix A to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(R)$. This yields homomorphisms $GL_1(R) \to GL_2(R) \to \dots$; their direct limit is denoted $GL(R)$.

An $n \times n$ matrix is called *elementary* if it differs from the identity by a single off-diagonal entry. The subgroup generated by all elementary matrices is denoted $E(R)$, and J. H. C. Whitehead proved that $E(R)$ is precisely the

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commutator subgroup of $GL(R)$. We define $K_1(R) = GL(R)/E(R) = GL(R)^{ab}$.

If the ring R is commutative, similar considerations apply to the matrices of determinant 1, and lead to the definition $SK_1(R) = SL(R)/E(R)$. In this case there is a direct sum decomposition

$$K_1(R) \approx U(R) \oplus SK_1(R),$$

where $U(R)$ denotes the group of units of R, since the inclusion $U(R) = GL_1(R) \subset GL(R)$ splits the determinant homomorphism $GL(R) \to U(R)$ whose kernel is $SL(R)$.

If I is a (two-sided) ideal in R, we let $GL(R, I) = \ker(GL(R) \to GL(R/I))$ and $E(R, I)$ be the normal subgroup of $E(R)$ generated by matrices with all off-diagonal entries in I. The *relative group* $K_1(R, I) = GL(R, I)/E(R, I)$ fits into an exact sequence

(1.1) $$K_1(R, I) \to K_1(R) \to K_1(R/I);$$

when R is commutative, similar definitions and remarks apply to $SK_1(R, I)$ as well.

The following theorem will be crucial to our calculations.

(1.2) THEOREM [7, COROLLARY 4.3]. *Let $\mathfrak{O}$ be the ring of integers in an algebraic number field F and let $\mathfrak{a}$ be an ideal of $\mathfrak{O}$. Then $SK_1(\mathfrak{O}, \mathfrak{a})$ is canonically isomorphic to a cyclic subgroup of order l of the roots of unity, $\mu(F)$, of F, where $l = 1$ if $\mathfrak{a} = \mathfrak{O}$, or $\mathfrak{a} = (0)$, or $\mathfrak{O}$ is not totally imaginary; and*

$$\operatorname{ord}_p(l) = \min_{\mathfrak{p}|p} \left[\frac{\operatorname{ord}_{\mathfrak{p}}(\mathfrak{a})}{\operatorname{ord}_{\mathfrak{p}}(p)} - \frac{1}{p-1} \right]_{[0,s]},$$

where $s = \operatorname{ord}_p(\mu(F))$, otherwise.

Here $[x]_{[0,s]}$, $x \in \mathbf{R}$, $s \in \mathbf{Z}$, denotes the nearest integer is the interval $[0, s]$ to the largest integer $\leqslant x$.

Let A be any commutative ring and G a finite group. The inclusion of A into the group ring AG splits the augmentation $AG \to A$ and shows that $K_1(A)$ occurs as a direct summand of $K_1(AG)$. Also, the composite

$$G \to U(AG) = GL_1(AG) \to GL(AG) \to K_1(AG)$$

induces a homomorphism $G^{ab} \to K_1(AG)$ which is injective because the composition

$$G^{ab} \to K_1(AG) \to K_1(AG^{ab}) \overset{\det}{\to} U(AG^{ab})$$

is just the usual inclusion. We define the *Whitehead group*, Wh(AG), to be the quotient of $K_1(AG)$ by $K_1(A) \oplus G^{ab}$. When $A = \mathbf{Z}$, we write Wh(G) for Wh$(\mathbf{Z}G)$. Bass has shown [4] that Wh(G) is a finitely generated abelian group of rank $r(G) - q(G)$, where $r(G)$ (resp. $q(G)$) denotes the number of irreducible real (resp. rational) representations of G.

Note that when G is finite abelian, we have the decomposition

(1.3) $$\mathrm{Wh}(AG) \approx (SK_1(AG)/SK_1(A)) \oplus (U(AG)/(U(A) \oplus G)).$$

When A is the ring of integers in an algebraic number field, $\mathrm{SK}_1(A) = 0$ by

Theorem 1.2 and we have $\mathrm{Wh}(AG) \approx SK_1(AG) \oplus U(AG)/(A^* \oplus G)$. In this case Higman [13] has shown that $A^* \oplus G$ contains all the units of finite order in AG, so the summand $U(AG)/(A^* \oplus G)$ is torsion-free. We shall see later that $SK_1(AG)$ is finite.

2. Group rings of finite abelian groups; some easy examples. Let F be a finite extension field of $\mathbf{Q}$, G a finite abelian group, and $\chi: G \to \mathbf{C}^*$ an irreducible character of G. Then $\chi(G)$ is a finite cyclic group of roots of unity of order m dividing $|G|$. Let $\bar{\chi}$ be a primitive mth root of unity. Then χ induces a surjective homomorphism, which we shall also denote χ, from the group algebra FG to $F(\bar{\chi})$. Since FG is semisimple (Maschke's theorem), χ must be split and $F(\bar{\chi})$ occurs as a simple component of FG. If $\mathfrak{O}$ is the ring of integers in F, the restriction to $\mathfrak{O}G$ of χ has image $\mathfrak{O}[\bar{\chi}]$.

Conversely, every simple component of FG affords an irreducible representation, and thus an irreducible character, of G. This means that every simple component of FG arises in the way described above.

It is possible, however, for distinct irreducible characters of G to give rise to the same simple component of FG. For example, if $F = \mathbf{Q}$, two characters χ_1, χ_2 of G will give rise to the same simple component of $\mathbf{Q}G$ if and only if their kernels are the same. The relation "$\chi_1 \sim \chi_2$ if the simple components of FG they give rise to are the same" is an equivalence relation on the set of irreducible characters of G; we shall denote by $S_F(G)$ (or simply S when F and G are understood) a set of representatives for the equivalence classes under this relation. When $F = \mathbf{Q}$, $S_{\mathbf{Q}}(G)$ may be taken to be the set of irreducible rational characters of G.

It follows from the preceding discussion that for any finite abelian group G, there is an isomorphism

$$\alpha: FG \xrightarrow{\approx} \prod_{\chi \in S_F(G)} F(\bar{\chi}) \tag{2.1}$$

where $\alpha = \prod_{\chi \in S_F(G)} \chi$.

Let us write $B = \prod_{\chi \in S_{\mathbf{Q}}(G)} \mathbf{Z}[\bar{\chi}]$. The restriction to $\mathbf{Z}G$ of α is an injective homomorphism $\mathbf{Z}G \to B$. The *conductor*, $\mathfrak{c}$, from B to $\mathbf{Z}G$ is the largest ideal of B contained in $\mathbf{Z}G$; equivalently, $\mathfrak{c} = \{x \in \mathbf{Z}G \mid xB \subset \mathbf{Z}G\}$. Moreover, $|G|B \subset \mathfrak{c}$ [5, Chapter IX, Corollary 1.2]. Similar remarks hold if we replace $\mathbf{Z}$ by the ring of integers in a number field F throughout. It is a theorem of Bass and Murthy [8, Lemma 10.5] that under the above hypotheses, the map $SK_1(\mathbf{Z}G, \mathfrak{c}) \to SK_1(B, \mathfrak{c})$ is an isomorphism.[2] Since $|G|\mathbf{Z}G \subset |G|B \subset \mathfrak{c}$, $\mathbf{Z}G/\mathfrak{c}$ is a finite ring and has trivial SK_1 [5, Chapter V, Corollary 9.2]. We thus deduce from (1.1) that $SK_1(\mathbf{Z}G)$ is a quotient of $SK_1(B, \mathfrak{c})$.

Following the decomposition $B = \prod_{\chi \in S} \mathbf{Z}[\bar{\chi}]$, we may write $\mathfrak{c} = \prod_{\chi \in S} \mathfrak{c}_\chi$; since SK_1 commutes with finite products, we see that $SK_1(\mathbf{Z}G)$, for G finite abelian, is a quotient of $\prod_{\chi \in S} SK_1(\mathbf{Z}[\bar{\chi}], \mathfrak{c}_\chi)$, and each term of this product is known by Theorem 1.2. Thus $SK_1(\mathbf{Z}G)$ is a finite abelian group and is precisely the torsion part of $\mathrm{Wh}(G)$ when G is finite abelian. (A more refined analysis shows that any exponent for G is also an exponent for $SK_1(\mathbf{Z}G)$.)

[2]This property is special to the circumstances described here. In general if A is a subring of B and I is an ideal in both, $SK_1(A, I)$ is not necessarily isomorphic to $SK_1(B, I)$ [30].

Similar arguments can be used to show that $SK_1(AG)$ is finite when A is the ring of integers in an algebraic number field.

Here is an example. Suppose G is an elementary abelian 2-group of rank k (i.e. a product of k cyclic groups of order 2). All characters of G take values in $\{\pm 1\}$; hence $B \approx \mathbf{Z}^q$, where $q = 2^k - 1$. Thus $SK_1(B, \mathfrak{c}) = \prod_{\chi \in S} SK_1(\mathbf{Z}, \mathfrak{c}_\chi) = 0$ (since $\mathbf{Z}$ has a real embedding), and $SK_1(\mathbf{Z}G) = 0$.

3. Mayer-Vietoris sequences; K_2. Early workers who tried to exploit the exact sequence $\prod SK_1(\mathbf{Z}[\bar{\chi}], \mathfrak{c}_\chi) \to SK_1(\mathbf{Z}G) \to 0$ to compute Whitehead groups were frustrated by the absence of an extension of this sequence to the left. The algebraic K_2 functor introduced by Milnor [**24**] and the resulting Mayer-Vietoris sequences went far towards alleviating this difficulty. I will describe these techniques in some generality before specializing to our particular computational problems.

Suppose

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \alpha\downarrow & & \downarrow\beta \\ A' & \xrightarrow[f']{} & B' \end{array} \tag{3.1}$$

is a commutative diagram of rings and ring homomorphisms with the property that f induces an isomorphism between the ideals $I = \ker \alpha$ and $J = \ker \beta$. Such a diagram is a *pullback* in the category of rings. Let us assume, in addition:

(3.2) For some $i \geqslant 1$, $K_i(A, I) \to K_i(B, J)$ is onto and $K_{i-1}(A, I) \to K_{i-1}(B, J)$ is an isomorphism.

(In Milnor's original work [**5**, Chapter VII, §4] he took $i = 1$. We are assuming in this general discussion the existence of a "higher" algebraic K-theory as, for example, in [**27**].) Then an easy diagram chase shows that there exists a connecting homomorphism ∂ such that the following sequence is exact:

$$\begin{array}{l} K_i(A) \to K_i(B) \oplus K_i(A') \to K_i(B') \\ \xrightarrow{\partial} K_{i-1}(A) \to K_{i-1}(B) \oplus K_{i-1}(A') \to K_{i-1}(B'). \end{array} \tag{3.3}$$

Such a sequence is called a Mayer-Vietoris sequence. It exists for $i = 1$ so long as f' or β is onto. When $i = 2$, Milnor showed (3.2) is satisfied provided f' *and* β are onto. This leads to an easy proof that $SK_1(\mathbf{Z}G) = 0$ for a finite *cyclic* group G.

Let us assume, first of all, that G is cyclic of prime order p (we shall see in §5 that we can reduce to this case). Then

$$\begin{array}{ccc} \mathbf{Z}G & \to & \mathbf{Z}[\zeta] \\ \downarrow & & \downarrow \\ \mathbf{Z} & \to & \mathbf{F}_p \end{array}$$

is a pullback with all arrows surjective, where ζ is a primitive pth root of unity. Here a generator of G is mapped to $\zeta \in \mathbf{Z}[\zeta]$ and to $1 \in \mathbf{Z}$, and the other maps are reduction mod $1 - \zeta$ and p. From (3.3) with $i = 2$ we deduce the exact sequence

$$K_2(\mathbf{F}_p) \to SK_1(\mathbf{Z}G) \to SK_1(\mathbf{Z}) \oplus SK_1(\mathbf{Z}[\zeta])$$

whose last term is trivial by Theorem 1.2. Since K_2 of a finite field is also trivial [**29**], we conclude that $SK_1(\mathbf{Z}G) = 0$.

A second circumstance in which a Mayer-Vietoris sequence exists for $i = 2$ is the conductor situation described above which was first studied by Bass and Murthy. More generally, we have:

(3.4) THEOREM [**9**]. *Let $B = \prod_{i=1}^n B_i$ be a direct product of (not necessarily commutative) rings and suppose $A \subset B$ is a subring such that each projection of A into a direct factor B_i of B is surjective. Let I be any two-sided ideal of B contained in A. Then* (3.2) *holds for the square*

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ A/I & \to & B/I \end{array}$$

and $i = 2$.

4. An algorithm for finite abelian groups. Let me now describe how Theorem 3.4 can be used to calculate, in principle, $SK_1(\mathbf{Z}G)$ for any finite abelian group G. We return to the notation introduced in §2; in particular, $B = \prod_{\chi \in S} \mathbf{Z}[\chi]$ is the integral closure in $\mathbf{Q}G$ of $\mathbf{Z}G$, where $S = S_{\mathbf{Q}}(G)$. We choose an ideal I contained in the conductor from B to $\mathbf{Z}G$ and containing a rational integer ($|G|B = I$ is one such choice). We obtain from (3.3) the exact sequence

(4.1) $$K_2(B) \oplus K_2(\mathbf{Z}G/I) \to K_2(B/I) \to SK_1(\mathbf{Z}G) \to 0.$$

This involves noting that K_1 can be replaced by SK_1 when the rings are commutative, and applying Theorem 1.2 and the fact that $\mathbf{Z}G/I$ is finite to conclude that

$$SK_1(B) = \prod SK_1(\mathbf{Z}[\chi]) = 0 = SK_1(\mathbf{Z}G/I).$$

Our first task is to analyze the map $K_2(B) \to K_2(B/I)$. This may be done componentwise by considering the exact sequences

(4.2) $$K_2(\mathbf{Z}[\chi]) \to K_2(\mathbf{Z}[\chi]/I_\chi) \to SK_1(\mathbf{Z}[\chi], I_\chi) \to 0.$$

To simplify our task, let us assume henceforth that G is a p-group for some prime p (we will shortly show how we may always reduce to this case). The group $K_2(\mathbf{Z}[\chi]/I_\chi)$ has been computed by Dennis and Stein [**10**, Theorem 5.1]; combining their calculation with Theorem 1.2 we conclude from (4.2) that $K_2(\mathbf{Z}[\chi]) \to K_2(\mathbf{Z}[\chi]/I)$ is the 0-map unless $\chi = \pm 1$, in which case the map is onto. Thus we may rewrite (4.1) as follows:

(4.3) $$K_2(\mathbf{Z}G/I) \xrightarrow{\varphi} \prod_{\chi \in S^*} K_2(\mathbf{Z}[\chi]/I_\chi) \to SK_1(\mathbf{Z}G) \to 0,$$

where S^* is the subset of $S = S_{\mathbf{Q}}(G)$ different from ± 1 (of course, $\chi = -1$ can occur only when $p = 2$) and φ is the product map induced by the characters in S^*. Since all the K_2's which occur in (4.3) are generated by

Steinberg symbols [**28**, Theorem 2.13], explicit calculation is often possible. Here is a simple example which indicates the technique.

Let ζ be a primitive cube root of unity and let χ be the character of a cyclic group of order 3 which maps a fixed generator to ζ. Let G be the product of 2 cyclic groups of order 3 with generators σ, τ, repsectively (G is an elementary abelian 3-group of rank 2).

Then $\varphi = (\chi_1, \chi_2, \chi_3, \chi_4)$, where $\chi_1 = \chi \times 1$, $\chi_2 = 1 \times \chi$, $\chi_3 = \chi \times \chi$ and $\chi_4 = \chi^{-1} \times \chi$ (these are the elements of S^*), and we may take $I_\chi = (\zeta - 1)^3$. By [**10**, Theorem 3.8(f)] we know that $K_2(\mathbf{Z}[\zeta]/(\zeta - 1)^3)$ is generated by $s = \{\zeta, 1 + (\zeta - 1)^2\}$. Also, since each χ_i maps the augmentation ideal, J, of $\mathbf{Z}G$ to $(\zeta - 1)\mathbf{Z}[\zeta]$, it follows that $\chi_i(J^3) = 0$, and we may replace (4.3) by the exact sequence

$$K_2(\mathbf{Z}G/(I, J^3)) \xrightarrow{\varphi} K_2(\mathbf{Z}[\zeta]/(\zeta - 1)^3)^4 \rightarrow SK_1(\mathbf{Z}G) \rightarrow 0.$$

Define elements $s_i \in K_2(\mathbf{Z}G/(I, J^3))$ by

$$s_1 = \{\sigma, 1 + (\sigma\tau - 1)(\sigma\tau^2 - 1)\}, \quad s_2 = \{\tau, 1 + (\sigma\tau - 1)(\sigma^2\tau - 1)\},$$

$$s_3 = \{\tau, 1 + (\sigma - 1)(\sigma^2\tau^2 - 1)\}, \quad s_4 = \{\sigma^{-1}, 1 + (\tau - 1)(\sigma\tau^2 - 1)\}.$$

Then $\chi_i(s_j) = \delta_{ij}s$ (Kronecker delta), proving φ is surjective and $SK_1(\mathbf{Z}G) = 0$.

Of course, the above result was known before the advent of the functor K_2, and our wish is to produce examples where $SK_1(\mathbf{Z}G)$ is *not* trivial. One method, based on [**2**], and suitable for machine computation, works as follows. Choose a collection of Steinberg symbols s_j which generate $K_2(\mathbf{Z}G/I)$. Form the matrix with rows indexed by these symbols and columns indexed by S^* whose ijth entry is the Steinberg symbol $\chi_i(s_j) \in K_2(\mathbf{Z}[\chi_i]/I_{\chi_i})$. This K_2 is a cyclic group which may be identified with a subgroup of the roots of unity in $\mathbf{Z}[\chi_i]$ by interpreting $\chi_i(s_j)$ as a norm residue symbol [**10**, §4]. Since explicit formulas for the evaluation of norm residue symbols are known [**3**], [**14**], we obtain a relation matrix describing the image of φ as a subgroup of $\amalg_{\chi \in S^*} K_2(\mathbf{Z}[\chi]/I_\chi)$ and the order of $SK_1(\mathbf{Z}G) = \operatorname{coker}(\varphi)$ may be computed. This method has been used by Roy G. Fuller to obtain the following results by machine calculation:

G	$SK_1(\mathbf{Z}G)$
$\mathbf{Z}/p^2 \times \mathbf{Z}/p^2$ ($p = 3, 5, 7$)	$(\mathbf{Z}/p)^{p-1}$
$\mathbf{Z}/p^2 \times \mathbf{Z}/p \times \mathbf{Z}/p$ ($p = 3, 5, 7$)	$(\mathbf{Z}/p)^{p(p-1)}$
$\mathbf{Z}/27 \times \mathbf{Z}/9$	$(\mathbf{Z}/3)^4$
$\mathbf{Z}/27 \times \mathbf{Z}/3 \times \mathbf{Z}/3$	$(\mathbf{Z}/3)^9$
$\mathbf{Z}/9 \times \mathbf{Z}/9 \times \mathbf{Z}/3$	$(\mathbf{Z}/3)^{15} \times (\mathbf{Z}/9)^2$

General computations of $SK_1(\mathbf{Z}G)$ depend, of course, on finding a method for computing the image of φ. The only general result so far has been for elementary abelian p-groups. In that case $\amalg_{\chi \in S^*} K_2(\mathbf{Z}[\chi]/I_\chi)$ is an $\mathbf{F}_p$-vector space of dimension $(p^k - 1)/(p - 1)$, where k is the rank of G. The dimension of $V = \text{image}(\varphi)$ as a subspace of this vector space may be computed by interpreting V as a certain vector subspace of the polynomial functions from the character group of G to $\mathbf{F}_p$. The result is:

(4.4) THEOREM [2]. *Let p be an odd prime and G an elementary abelian p-group of rank k. Then $SK_1(\mathbf{Z}G)$ is an elementary abelian p-group of rank*

$$\frac{p^k - 1}{p - 1} - \binom{p + k - 1}{p}.$$

In particular, $SK_1(\mathbf{Z}G) \neq 0$ for $k \geqslant 3$.

Some other results obtained by hand computation are given below [Dennis and Stein, unpublished].

G	$SK_1(\mathbf{Z}G)$
$\mathbf{Z}/4 \times \mathbf{Z}/4$	$\mathbf{Z}/2$
$\mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}/4$	$\mathbf{Z}/2$
$(\mathbf{Z}/2)^3 \times \mathbf{Z}/4$	$(\mathbf{Z}/2)^3 \times \mathbf{Z}/4$

5. Reduction to p-groups. The method outlined above for calculating $SK_1(\mathbf{Z}G)$ when G is a finite abelian p-group applies more generally to the computation of $SK_1(\mathfrak{O}G)$, where $\mathfrak{O}$ is the ring of integers in any finite Galois extension of $\mathbf{Q}$ in which the prime p does not ramify. We obtain an exact sequence

$$(5.1) \qquad K_2(\mathfrak{O}G/\mathfrak{J}) \xrightarrow{\psi} \prod_{\chi \in S_0} K_2(\mathfrak{O}[\chi]/\mathfrak{J}_\chi) \to SK_1(\mathfrak{O}G) \to 0$$

analogous to (4.3), where S_0 is a certain collection of irreducible characters of G [2]. The norm N: $\mathfrak{O} \to \mathbf{Z}$ induces, by extension of scalars, compatible homomorphism $\mathfrak{O}G/\mathfrak{J} \to \mathbf{Z}G/I$ and $\mathfrak{O}[\chi]/\mathfrak{J}_\chi \to \mathbf{Z}[\chi]/I_\chi$. These, in turn, induce a map of (5.1) to (4.3), which is an isomorphism $SK_1(\mathfrak{O}G) \to SK_1(\mathbf{Z}G)$ for $p \neq 2$. (When $p = 2$, S_0 may be larger than S^*.) The precise result is as follows.

(5.2) THEOREM. *Let p be a prime and G a finite abelian p-group. Let $K_1 \subset K_2$ be a finite Galois extension of number fields with rings of integers $\mathfrak{O}_1$, $\mathfrak{O}_2$, respectively, in which p is unramified. Then $SK_1(\mathfrak{O}_2G) \approx SK_1(\mathfrak{O}_1G)$ in case p is odd, or, when $p = 2$, both $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are totally imaginary or both have real embeddings.*

I now want to indicate how Theorem 5.2 can be used to reduce the computation of $SK_1(\mathbf{Z}G)$ from general finite abelian groups to the case of p-groups. For any finite abelian group G, let us write G_p for its Sylow p-subgroup. Thus $G = H \times G_p$, where H has order prime to p. The integral closure, C, of $\mathbf{Z}H$ in $\mathbf{Q}H$ is, by the discussion in §2, the direct product of $q(H) = q(G/G_p)$ factors of the form $\mathbf{Z}[\zeta]$, where ζ is a root of unity of order prime to p. In particular, each factor $\mathbf{Z}[\zeta]$ satisfies the hypothesis of Theorem 5.2, and it follows that if p is odd, $SK_1(C[G_p]) \approx SK_1(\mathbf{Z}G_p)^{q(G/G_p)}$. In particular, $SK_1(C[G_p])$ is a finite abelian p-group (cf. §2).

On the other hand, we have the homomorphisms

$$\mathbf{Z}G = \mathbf{Z}[H \times G_p] \approx \mathbf{Z}[H][G_p] \subset C[G_p]$$

and the induced map $SK_1(\mathbf{Z}G) \to SK_1(C[G_p])$ kills all torsion other than p-torsion, thus inducing a homomorphism

$$SK_1(\mathbf{Z}G)_p \xrightarrow{\alpha} SK_1(C[G_p]).$$

(5.3) THEOREM [2]. *The homomorphism α is an isomorphism for all p.*

Combining Theorems 5.2 and 5.3, we have

(5.4) THEOREM. *Let G be a finite abelian group with Sylow subgroups $\{G_p\}$. Then*

$$SK_1(\mathbf{Z}G) \approx \prod_{\substack{p \text{ odd} \\ p \mid |G|}} SK_1(\mathbf{Z}G_p)^{q(G/G_p)} \times SK_1(\mathbf{Z}G_2) \times SK_1(\mathbf{Z}[\zeta_3]G_2)^{q(G/G_2)-1},$$

where ζ_3 is a primitive cube root of unity.

6. Induction theorems for finite groups. Let G be a finite abelian group. Since $\mathbf{Q}G$ is semisimple, it follows from the stability theorems for K_1 [5, Chapter V, Theorem 4.2 ff.] that the determinant map from $K_1(\mathbf{Q}G)$ to $U(\mathbf{Q}G)$ is an isomorphism, and, therefore, that $\ker(K_1(\mathbf{Z}G) \to K_1(\mathbf{Q}G)) = SK_1(\mathbf{Z}G)$. We may thus generalize the definition of SK_1 to nonabelian finite groups by setting $SK_1(\mathbf{Z}G) = \ker(K_1(\mathbf{Z}G) \to K_1(\mathbf{Q}G))$. (There is an alternative method of defining SK_1 using reduced norms [4, §1]; we will not need this definition here.) More generally, if A is the ring of integers in an algebraic number field F, we set $SK_1(AG) = \ker(K_1(AG) \to K_1(FG))$.

We have already seen in §§1 and 2 that when G is abelian, the torsion subgroup of $\mathrm{Wh}(AG)$ is precisely $SK_1(AG)$. Our next task is to prove Wall's result that this remains true if G is finite, but not necessarily abelian (or, equivalently, to show that $\mathrm{tor}(K_1(AG)) = \mathrm{tor}(A^*) \oplus G^{\mathrm{ab}} \oplus SK_1(AG)$ where A is the ring of integers in an algebraic number field and G is finite). To do so will require a brief sketch of the use of induction techniques in algebraic K-theory, as developed by Swan, Lam and Dress.

Our starting point is the calculus of induction and restriction for group representations. Let H be a subgroup of a finite group G. Any $\mathbf{Z}H$-module M can be made into an induced $\mathbf{Z}G$-module $i_*(M) = M \otimes_{\mathbf{Z}H} \mathbf{Z}G$. Conversely, by restricting scalars from $\mathbf{Z}G$ to $\mathbf{Z}H$, any $\mathbf{Z}G$-module N can be made into a $\mathbf{Z}H$-module denoted $i^*(N)$. The maps i_*, i^* are, in fact, functors between the appropriate categories of modules, and are related by the Frobenius reciprocity theorem, which, for our purposes, can be formulated as follows.

Let $G_{\mathbf{Z}}(G)$ (resp. $G_{\mathbf{Z}}(H)$) be the Grothendieck group on the category of all finitely generated $\mathbf{Z}G$ (resp. $\mathbf{Z}H$)-modules which are $\mathbf{Z}$-projective. Tensor product over $\mathbf{Z}$ induces ring structures on $G_{\mathbf{Z}}(G)$ and $G_{\mathbf{Z}}(H)$, and the Frobenius reciprocity law in this setting states that for $x \in G_{\mathbf{Z}}(G)$, $y \in G_{\mathbf{Z}}(H)$,

$$i_*(i^*(x)y) = xi_*(y).$$

As usual, similar definitions and results apply when $\mathbf{Z}$ is replaced by a ring of algebraic integers A.

Now suppose $\mathfrak{D}$ is some collection of subgroups of G, and let $G_A(G)_{\mathfrak{D}}$ be the ideal in $G_A(G)$ generated by $i_*(G_A(H))$ for all $H \in \mathfrak{D}$.

(6.1) THEOREM [ARTIN]. *Let $\mathcal{C}$ be the set of all cyclic subgroups of the finite group G. Then*

$$|G| G_{\mathbf{Q}}(G) \subset G_{\mathbf{Q}}(G)_{\mathcal{C}}.$$

We shall often be interested in the collection of hyperelementary subgroups of G. A finite group H is said to be *p-hyperelementary* (for some prime p) if it contains a cyclic normal subgroup of index a power of p (equivalently: if H is the semidirect product $N \rtimes P$ with N normal cyclic and P a p-group). Any *dihedral group* is 2-hyperelementary. All p-groups are p-hyperelementary.

(6.2) THEOREM [WITT, BERMAN]. *Let $\mathcal{H}$ be the collection of p-hyperelementary subgroups of the finite group G for all primes dividing $|G|$. Then $G_F(G) = G_F(G)_{\mathcal{H}}$ for any algebraic number field F.*

These theorems were extended by Swan [31], [32].

(6.3) THEOREM. *Let R be a Dedekind domain with field of fractions K and let G be a finite group. Suppose $\mathfrak{D}$ is a collection of subgroups of G. If $nG_K(G) \subset G_K(G)_{\mathfrak{D}}$, then*

(i) *$nG_{R/\mathfrak{p}}(G) \subset G_{R/\mathfrak{p}}(G)_{\mathfrak{D}}$ for all maximal ideals $\mathfrak{p}$ of R; and*

(ii) *$n^2 G_R(G) \subset G_R(G)_{\mathfrak{D}}$.*

Swan's work, in turn, was formalized and extended by Lam [21], who showed that whenever theorems such as (6.3) hold for $G_R(G)$, they are valid as well for $F(G)$, where F is any functor from finite groups and their monomorphisms to abelian groups which can be given the structure of a "Frobenius module" over G_R. Examples of such functors are $K_0(RG)$, $K_1(RG)$, $SK_1(RG)$, $\mathrm{Wh}(RG)$.

7. Some computations for nonabelian finite groups. We have seen above that $SK_1(\mathbf{Z}G) = 0$ when G is cyclic. Hence $SK_1(\mathbf{Z}G)_{\mathcal{C}} = 0$ for any finite group G, where $\mathcal{C}$ is the set of cyclic subgroups of G. Using the work of Lam together with Theorems 6.1 and 6.3, we see that $|G|^2 SK_1(\mathbf{Z}G) = 0$; i.e. that $SK_1(\mathbf{Z}G)$ is a torsion group. Since it is also finitely generated (cf. [5, p. 553, (v)]), we conclude that $SK_1(\mathbf{Z}G)$ is finite.

Another corollary of the work of Swan and Lam is:

(7.1) COROLLARY. *If $SK_1(AH) = 0$ for every hyperelementary subgroup H of a finite group G, then $SK_1(AG) = 0$ as well.*

By restricting attention to p-hyperelementary subgroups for a fixed prime p, we can obtain information about the p-torsion in $SK_1(\mathbf{Z}G)$. For example:

(7.2) THEOREM [SWAN-LAM]. *If G has a cyclic normal Sylow p-subgroup, $SK_1(\mathbf{Z}G)$ has no p-torsion.*

Let me now indicate, in outline, Wall's argument for proving $SK_1(AG) = \mathrm{tor}(\mathrm{Wh}(AG))$. Here A is the ring of integers in an algebraic number field F, and we write $K_1'(AG)$ for the image of $K_1(AG)$ in $K_1(FG)$. We have an exact commutative diagram (cf. §1):

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 \to & SK_1(A) & \to & K_1(A) & \oplus G^{ab} & \to & K_1'(A) \oplus G^{ab} & \to & 0 \\
 & \downarrow & & \downarrow & & & \downarrow & & \\
0 \to & SK_1(AG) & \to & K_1(AG) & & \to & K_1'(AG) & \to & 0 \\
 & & & \downarrow & & & \downarrow & & \\
 & & & \mathrm{Wh}(AG) & & \to & \mathrm{Wh}'(AG) & & \\
 & & & \downarrow & & & \downarrow & & \\
 & & & 0 & & & 0 & &
\end{array}$$

Note that $SK_1(A) = 0$ by Theorem 1.2. Wall shows that $\mathrm{Wh}'(AG)$ is torsion-free, hence that $\mathrm{tor}(A^*)$, G^{ab} and $SK_1(AG)$ generate $\mathrm{tor}(K_1(AG))$. Since the diagram remains exact when we restrict to torsion subgroups, it follows that $\mathrm{tor}(A^*) \oplus G^{ab}$ projects isomorphically to $K_1'(AG)$, which proves that $SK_1(AG)$ is a direct summand of $\mathrm{tor}(K_1(AG))$.

To prove that $\mathrm{Wh}'(AG)$ is torsion-free, Wall invokes the induction techniques described above, showing that $\mathrm{Wh}'(AG)$ is p-torsion free if $\mathrm{Wh}'(AH)$ is p-torsion free for every p-hyperelementary subgroup H of G. Next he proves that if K is a normal l-subgroup of H for some prime $l \neq p$, $\mathrm{Wh}'(AH)$ is p-torsion free if $\mathrm{Wh}'(A[H/K])$ is. This allows him to reduce to the case when H is a p-group, for which direct arguments are possible.

Theorem 7.2 has been used to prove that $SK_1(\mathbf{Z}D_{2p}) = 0$, where D_{2p} is the dihedral group of order $2p$, p an odd prime. Let I be the ideal of $\mathbf{Z}D_{2p}$ generated by all $g - 1$, where g lies in the Sylow p-subgroup of D_{2p}. Then $SK_1(\mathbf{Z}D_{2p}/I) = 0$, and it follows that $SK_1(\mathbf{Z}D_{2p})$ is a quotient of $SK_1(\mathbf{Z}D_{2p}, I)$. Direct computation shows that this relative group is a p-group. Since Theorem 7.2 implies that $SK_1(\mathbf{Z}D_{2p})$ has no p-torsion, it must be trivial. These results are due to Lam [21] for $p = 3$ and to Keating [18] and Obayashi [25].

Similar arguments have been used by Keating to show $SK_1(\mathbf{Z}G) = 0$ for any metacyclic group G containing a normal subgroup H of prime index s relatively prime to $|H|$. He has also noted [19] that these methods prove the triviality of $SK_1(\mathbf{Z}G)$ if the normal subgroup H has order p, $|G/H|$ does not divide $p - 1$ and G/H embeds in the automorphism group of H.

Along slightly different lines, Keating [17] and Obayashi [26] have proved $SK_1(\mathbf{Z}D) = 0$ when D is a dihedral 2-group. Keating produces an order $\mathfrak{O}$ in $\mathbf{Q}D$ and an ideal $I \subset \mathfrak{O}$ such that the usual map $SK_1(\mathfrak{O}, I) \to SK_1(\mathfrak{O})$ factors as

$$SK_1(\mathfrak{O}, I) \xrightarrow{\alpha} SK_1(\mathbf{Z}D) \xrightarrow{\beta} SK_1(\mathfrak{O}),$$

with α onto and β injective. Explicit computation then shows that $\beta\alpha$ is the 0-map. The same technique gives a similar result for semidihedral 2-groups [Keating, unpublished].

Finally, Magurn [22] has generalized the results of Keating and Obayashi to show $SK_1(\mathbf{Z}D) = 0$ for *all* dihedral groups D. His method uses Mayer-Vietoris sequences reminiscent of §3 to proceed inductively from the cases $|D| = 2^r$ and $|D| = 2p$. Since these dihedral groups are hyperelementary, Magurn is able to apply his result in conjunction with Corollary 7.1 to prove:

(7.3) THEOREM. *Let* $\mathfrak{S}_n$ *be the nth symmetric group.* $SK_1(\mathbf{Z}\mathfrak{S}_n) = 0$ *for* $n = 4$, 5, 6. *More generally,* $SK_1(\mathbf{Z}G) = 0$ *for any permutation group* G *of degree* $\leqslant 6$.

Similarly, $SK_1(\mathbf{Z}G) = 0$ when G is the binary tetrahedral or icosahedral group. The same is true for the binary octahedral group, *provided* that $SK_1(\mathbf{Z}H) = 0$ for the generalized quaternion group H of order 16. Whether this is, in fact, true, remains an open question.

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