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EXISTENCE AND APPLICATIONS OF REMOTE POINTS

BY ERIC K. VAN DOUWEN

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All spaces are completely regular, X^* is $\beta X - X$.

A point p of X^* will be called a *remote point of X* if $p \notin \mathrm{Cl}_{\beta X} D$ for every nowhere dense $D \subseteq X$. Fine and Gillman, [*FG*] showed that **Q**, the rationals, and **R**, the reals, have remote points if *CH* holds; their proof shows that X has remote points if X is separable and not pseudo-compact. We prove the existence of remote points without assuming additional set theoretic axioms, under slightly stronger conditions on X.

Recall that a *π-base* (or *pseudo-base*) for a space X is a family $\mathcal{B}$ of nonempty open sets such that every nonempty open set of X includes a member of $\mathcal{B}$. The *π-weight* of a space is the smallest cardinality for a π-base.

THEOREM A. *If X is a nonpseudocompact space with countable π-weight, then X has $2^{\mathfrak{c}}$ remote points.*

I originally proved this only for $X = \mathbf{Q}$, improving a technique from [$\mathrm{v}D_1$]. I am indebted to Mary Ellen Rudin for showing me how to make my ideas work for $X = \mathbf{R}$. The above theorem is a further improvement.

For applications we need a "pointed" version of extremal disconnectedness.

DEFINITION. *If $p \in X$,* then X is called *extremally disconnected at p* if for all disjoint open $U, V \subseteq X$, $p \notin \overline{U} \cap \overline{V}$.

One can show that βX is extremally disconnected at every remote point of X. Without much effort one deduces the following theorem. (X is *nowhere locally compact* if no point has a compact neighborhood.)

THEOREM B. *Let X be a nonpseudocompact space with countable π-weight.*

(a) *βX is extremally disconnected at some point of X^*.*

(b) *If X is nowhere locally compact, X^* is extremally disconnected at some point.*

Frolík, [*F*], proved *that* X^* is not homogeneous if X is not pseudocompact. Theorem B can be used to show *why* X^* is not homogeneous, for suitable X.

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THEOREM C. *Let X be a nowhere locally compact nonpseudocompact separable first countable space. Then X^** is *not homogeneous because X^* is extremally disconnected at some but not at all points.*

This applies if e.g. $X = \mathbf{Q}$, or $X = \{\text{irrationals}\}$, or $X = \{\text{Sorgenfrey line}\}$. As another application we show that certain spaces cannot be factored as a product of spaces without isolated points. The key observation is that $X \times Y$ is not extremally disconnected at any point if X and Y are separable spaces without isolated points.

THEOREM D. *Y is not the product of two spaces without isolated points in each of the following two cases*:

(a) *$Y = \beta X$ for some nonpseudocompact X with countable π-weight*;

(b) *$Y = X^*$ for some nowhere locally compact nonpseudocompact X with countable π-weight.*

(Actually one does not need the condition on the π-weight in (a), see [vD_2], but I do not know if it can be avoided in (b).)

COROLLARY. $\mathbf{Q}^*$ *and* $(\mathbf{Q}^*)^\kappa$ *are not homeomorphic, for $\kappa \geqslant 2$.*

I do not know if $(\mathbf{Q}^*)^2$ and $(\mathbf{Q}^*)^3$ are nonhomeomorphic.

As yet another application we mention the following curiosities.

EXAMPLE. *There is an extremally disconnected space which has a connected compactification.*

Indeed, if X is any connected nowhere locally compact separable metrizable space, like $\mathbf{R}^\omega$, then the subspace E of all points at which the connected space βX is extremally disconnected turns out to be dense in βX, but then E is extremally disconnected.

For the other application, recall that a space is called *ω-bounded* if every countable subset has compact closure.

THEOREM E. $\mathbf{R}^*$ *is the union of three pairwise disjoint dense ω-bounded subspaces.*

If one calls a point p of X^* a *far point of X* if $p \notin \mathrm{Cl}_{\beta X} D$ for every closed discrete subset D of X, [vD_1], then the three subspaces are the remote points of $\mathbf{R}$, the far points of $\mathbf{R}$ which are not remote and the points of $\mathbf{R}^*$ which are not even far. Under CH there is a family of 2^c such subspaces, [W].

ADDED IN PROOF: If κ and λ are cardinals with $\kappa > \lambda \geqslant 1$, then $(Q^*)^\kappa$ and $(Q^*)^\lambda$ are not homeomorphic. The nontrivial proof will appear elsewhere.

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INSTITUTE FOR MEDICINE AND MATHEMATICS, MATHEMATICS BUILDING, OHIO UNIVERSITY, ATHENS, OHIO 45701