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Ergodic theory on compact spaces, by Manfred Denker, Christian Grillenberger and Karl Sigmund, Lecture Notes in Math., vol. 527, Springer-Verlag, Berlin, Heidelberg, New York, iv + 360 pp., \$13.20.

Ergodic theory and topological dynamics, by James R. Brown, Academic Press, New York, San Francisco, London, x + 190 pp., \$19.50.

The term "ergodisch" was coined by Ludwig Boltzmann more than 100 years ago, with a rather intuitive meaning. The first real theorem in ergodic theory was Poincaré's recurrence theorem, proved around 1890. Ironically, the professed anti-Cantorian Poincaré had to wait for the development of rigorous set theoretical measure theory in order to see his theorem fully established. Weyl's equidistribution mod 1 was the second major event in ergodic theory (1916). But in a way all this was prenatal, and the real moment of birth of ergodic theory happened only in 1931, when G. D. Birkhoff and J. v. Neumann proved the individual and mean ergodic theorems, respectively. To be precise, what was born in these years, was measure theoretical ergodic theory, based on the notion of a dynamical system (Ω,B,m,T), where T: $\Omega \to \Omega$ is an m-preservering B-measurable transformation. This is the result of implementing the simple combinatorial framework (Ω,T) with a measure theoretical structure. One could implement it with a topological structure rather than a measure theoretical one choosing a topology (mostly compact metrizable) in Ω and requiring T to be continuous; the result would be what is called topological dynamics. Its origins are largely due to G. D. Birkhoff whose monograph on that subject dates from 1927. Choosing a differentiable structure instead, one arrives at what may be called differentiable dynamics, and is closest to the physical origins of the whole subject. Actually, the physicist would see Ω as the state space of a, say, mechanical system, and T as the result of an evolution over a unit of time. What he wants, e.g., to know is the mean sojourn time

$$\overline{1_E}(\omega) = \lim_{n\to\infty} \frac{1}{n}\left(1_E(\omega) + 1_E(T\omega) + \cdots + 1_E(T^{n-1}\omega)\right)$$

of a state $\omega \in \Omega$ in a set $E \subseteq \Omega$. G. D. Birkhoff's individual ergodic theorem says that

$$\bar{f}(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\omega)$$

exists and is finite m-a.e. if $f \in L^1_m$, for any measure theoretical dynamical system (Ω,B,m,T). The mean ergodic theorem states the same convergence in the sense of L^2_m-mean, for $f \in L^2_m$, and in the L^1_m-mean for any $f \in L^1_m$ in case $m(\Omega) < \infty$. This proves the basic existence statement needed for a rigorous answer to the old physical question whether, in the case of an (Ω,B,m,T) modelling an ideal gas in a box, $\overline{1_E}(\omega)$ is independent of ω and (hence) equals $m(E)/m(\Omega)$. This statement is called the classical ergodic hypothesis. It has been waiting for its full proof for about 100 years. J. G. Sinai sketched a proof in 1963, and the marvelous theory of the Sinai billiard has grown out of attempts to work out the details.

This being the situation with respect to the classical physicists' question, all aspects are changed as soon as one adopts a purely mathematical viewpoint. Here the basic aim is to classify all dynamical systems with respect to a natural isomorphy concept. Ergodic theory has been no exception to the rule that this task be tackled by means of invariants, and the spectral type invariant (Koopman, 1931, v. Neumann, 1932) and the entropy invariant (Kolmogoroff and Sinai, 1956/57) mark the big inventive steps of the mathematical development of ergodic theory. In this development, a wealth of subsidiary results has grown up, from the improvements of ergodic theorems (due to, among others, E. Hopf, Chacon, Ornstein, Dunford, Schwartz, Akcoglu, Rota, Choquet, Foiaş, McMillan, Breiman) to the generator theorems of Rokhlin and Krieger. Topological dynamics has gone its own, however closely related, way, with Furstenberg's structure theorem for distal flows as a solitary culmination point. The key word "Axiom A" characterizes the highlights of differentiable dynamics which grew under the hands of Anosov, Sinai, Smale, Bowen and Ruelle. There are various cross-relations between the three branches of ergodic theory. We mention the Kryloff-Bogoljuboff theory of invariant measures on compact spaces, the theory of topological entropy and its relations to measure theoretical entropy, and the theory of periodic and nonwandering points for Axiom A diffeomorphisms.

The development of ergodic theory to its present wealth has, at every stage, been accompanied by monographs, the first being E. Hopf's Ergebnisbericht *Ergodentheorie* of 1937, and, most influential, P. R. Halmos' *Lectures on ergodic theory* of 1953.

After that date, the development of ergodic theory has become so multiversal that a comprehensive monograph seems to be an unmanageable task for any single author. The survey article of Katok, Sinai and Stepin (1975) in the Russian monograph series Matematiceskij Analiz lists 726 references. Accordingly, the mathematician who wants to inform himself about the current state of ergodic theory is being offered a bundle of small books, partly introductory, partly selective. Here is a list:

N. Friedman: *Introduction into ergodic theory*, 1970.
P. Walters: *Ergodic theory–introductory lectures*, 1975.
M. Smorodinski: *Ergodic theory, entropy*, 1971.
D. Ornstein: *Ergodic theory, randomness and dynamical systems*, 1974.
W. Parry: *Entropy and generators in ergodic theory*, 1969.
P. Shields: *The theory of Bernoulli shifts*, 1973.
J. Sinai: *Introduction into ergodic theory* (Russian), 1973.
K. Petersen: *Introductory ergodic theory*, 1971.
K. Sibirskii: *Introduction into topological dynamics* (Russian), 1970.

The two monographs which are under review here implement this list in two different ways.

Brown's book is a well-written introduction into the rudiments of ergodic theory and topological dynamics, suitable for graduate students and other future research workers with some background in measure theory and topology, provided they are willing to work themselves through a vast theory-oriented mass of exercises and to accept several deep theorems without proof. The material covered is a bit more than in P. Walters' *Introductory lectures* (Springer Lecture Notes, no. 458, 1975) as can be guessed from the slightly larger number of pages already. Chapter I starts with the basic notion of a (measure-theoretical) dynamical system. The maximal ergodic lemma is proved for positive contractions in L^1 (after A. Garsia, of course). Birkhoff's individual ergodic theorem is given with full proof, and so is Yosida's mean ergodic theorem for doubly stoachstic operators in L^p and Poincaré's recurrence theorem. The theorems of Dunford and Schwartz, Akcoglu and Chacon and Ornstein are formulated without proof. Sections on ergodicity and mixing, Lebesgue spectrum, products and factors, inverse limits and induced systems follow. Flows built under a function are treated in the exercises. Chapter II introduces the reader to topological dynamics: symbolic dynamics, minimal sets (a.p. points in the exercises only), unique ergodicity, equicontinuity, distality and expansiveness. Sums, products and inverse limits are studied. The Ellis semigroup is introduced. Furstenberg's structure theorem on distal systems is given without proof. Chapter III deals with group automorphisms and affine transformations, discrete and quasidiscrete spectrum both in the measure-theoretical and the topological setting, and the affine transformation induced on the group of all complex functions of constant modulus 1. In Chapter IV a swift introduction into entropy theory is given. It starts with measure theoretical entropy after Kolmogoroff and Sinai. The theorems of Goodwyn, Dinaburg and Goodman suggest the author's definition of topological entropy as the supremum of measure-theoretical entropies. The section on affine transformations deals with positivity and vanishing of entropy, and not with its evaluation in the case of torus automorphisms. The last section of this chapter gives the L^1-norm convergence for McMillan's theorem and mentions convergence a.e. Abramov's theorem on the entropy of induced systems is proved. The book culminates with Chapter V on Ornstein's isomorphy theorem (1969) for Bernoulli schemes, the greatest event in measure theoretical ergodic theory since the invention of entropy. It states that two Bernoulli shifts are isomorphic if and only if their entropy is equal. This theorem is proved in detail. Further related results of Ornstein are given without proof. Like Walters' notes, the book does not contain any proof of a generator theorem for measure theoretical dynamical systems. This lack is compensated thoroughly by the lecture notes of Denker, Grillenberger and Sigmund. These notes are a must for everyone who has already acquired some background in ergodic theory and topological dynamics (e.g. from the introductory books of Brown or Walters) and wants to penetrate more deeply into the subject. In particular, he will get a full-fledged training in topological entropy and in generator theorems. The style is clear and careful.

I can sum up my personal opinion as follows:

1. The all-comprehensive treatise on ergodic theory has still to be written; this will be possible only after a certain relaxation of the present state of tremendous activity in this field.

2. The ergodic beginner is at present best off with a combination of Brown or Walters plus Denker, Grillenberger and Sigmund. He cannot avoid to proceed to study further monographs, notably Ornstein's, and to consult the survey article of Katok, Sinai and Stepin.

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