

STOKES PHENOMENA

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1. Introduction. Let x be a complex variable, and set $\mathcal{D}(R) = \{x; |x| > R\}$. Consider two differential equations

$$(E_1) \quad dy/dx = A_1(x)y$$

and

$$(E_2) \quad du/dx = A_2(x)u,$$

where y and u are n -dimensional vectors, and A_1 and A_2 are n -by- n matrices whose components are holomorphic in $\mathcal{D}(R_1)$ and $\mathcal{D}(R_2)$ respectively and meromorphic at $x = \infty$. Note that if A_1 and A_2 are replaced by their Laurent expansions at $x = \infty$, then (E_1) and (E_2) become two differential equations defined in a vector space over the field of quotients of the ring of formal power series in x^{-1} . The two equations (E_1) and (E_2) are said to be *formally equivalent* if there exists an n -by- n matrix $T(x)$ such that (i) $T(x) = x^q \sum_{h=0}^{\infty} T_h x^{-h}$ is a formal series (with an integer q and constant matrices T_h), (ii) $\det T(x) \neq 0$ as a formal series, and (iii) the transformation $u = T(x)y$ reduces (E_1) to (E_2) formally. The two equations (E_1) and (E_2) are said to be *meromorphically equivalent* if there exists an n -by- n matrix $T(x)$ such that (i) the components of $T(x)$ are holomorphic in $\mathcal{D}(R_3)$ for some R_3 and meromorphic at $x = \infty$, (ii) $\det T(x) \neq 0$ in $\mathcal{D}(R_3)$, and (iii) the transformation $u = T(x)y$ reduces (E_1) to (E_2) in $\mathcal{D}(R_3)$.

Let \mathfrak{A} be an equivalence class of such differential equations with respect to the formal equivalence. The main concern in this report is to characterize the equivalence classes with respect to the meromorphic equivalence within \mathfrak{A} .

2. Asymptotic properties of an analytic fibre bundle. For a sector S , set $S(R) = \{x \in S; |x| > R\}$. An n -by- n matrix $F(x)$ is said to have an asymptotic expansion $A_{sp}(F)$ as x tends to ∞ in S , if (i) the components of $F(x)$ are holomorphic in $S(R)$ for some R , (ii) $A_{sp}(F) = x^q \sum_{h=0}^{\infty} G_h x^{-h}$ is a formal series, and (iii) for every positive integer N there exists a positive number K_N such that $|F(x) - x^q \sum_{h=0}^N G_h x^{-h}| \leq K_N |x|^{q-N-1}$ in $S(R)$. For an n -by- n matrix $F(x)$, the notation $F(x) \in A(S(R))$ means that (i) $F(x)$ is holomorphic

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in $S(R)$, (ii) $\det F(x) \neq 0$ in $S(R)$, and (iii) F and F^{-1} have asymptotic expansions $A_{sp}(F)$ and $A_{sp}(F^{-1})$ respectively as x tends to ∞ in S .

Throughout this section we fix a finite number of sectors S_1, \dots, S_N . Set $S_j = \{x; a_j < \arg x < b_j\}$. We assume that $0 < b_j - a_j < \pi$ and that $\{S_j\}$ covers the entire x -plane. Set $S_{jk} = S_j \cap S_k$, where (j, k) is an ordered pair. Let $\{F_{jk}(x)\}$ be a finite set of n -by- n matrices such that (i) $F_{jk}(x) \in A(S_{jk}(R))$ for some R whenever $S_{jk} \neq \emptyset$, and (ii) $F_{jh}(x)F_{hk}(x) = F_{jk}(x)$ whenever $S_j \cap S_h \cap S_k \neq \emptyset$. Such a system $\{F_{jk}(x)\}$ is called an asymptotic system at $x = \infty$. An asymptotic system $\{F_{jk}(x)\}$ is said to be trivial at $x = \infty$, if there exist n -by- n matrices $P_1(x), \dots, P_N(x)$ such that (i) $P_j(x) \in A(S_j(R))$ for some R , and (ii) $F_{jk}(x) = P_j(x)^{-1}P_k(x)$ whenever $S_{jk} \neq \emptyset$. The set $\{P_j(x)\}$ is called a cross-section of $\{F_{jk}(x)\}$ at $x = \infty$. An asymptotic system $\{F_{jk}(x)\}$ is said to be formally trivial if there exist n -by- n matrices $Q_1(x), \dots, Q_N(x)$ such that (i) $Q_j(x) = x^q \sum_{h=0}^{\infty} Q_{jh} x^{-h}$ is a formal series, (ii) $\det Q_j(x) \neq 0$ as a formal series, and (iii) $A_{sp}(F_{jk}) = Q_j(x)^{-1}Q_k(x)$ whenever $S_{jk} \neq \emptyset$.

THEOREM 1. *An asymptotic system $\{F_{jk}(x)\}$ is trivial at $x = \infty$ if and only if it is formally trivial.*

To prove this theorem, the following lemma is essential.

LEMMA 1. *An asymptotic system $\{F_{jk}(x)\}$ is trivial at $x = \infty$ if $A_{sp}(F_{jk}) = I$ (the identity matrix) whenever $S_{jk} \neq \emptyset$. (Cf. Y. Sibuya [3, Theorem 6.4.1, p. 172].)*

THEOREM 2. *Assume that an asymptotic system $\{F_{jk}(x)\}$ is trivial at $x = \infty$. Let $\{P_j(x)\}$ and $\{\tilde{P}_j(x)\}$ be two cross-sections of $\{F_{jk}(x)\}$ at $x = \infty$. Then there exists an n -by- n matrix $T(x)$ such that (i) the components of $T(x)$ are holomorphic in $\mathcal{D}(R)$ for some R and meromorphic at $x = \infty$, and (ii) $\tilde{P}_j(x) = T(x)P_j(x)$ for every j .*

3. Stokes structures. Let \mathfrak{U} be an equivalence class of differential equations with respect to the formal equivalence (cf. §1). Let us fix an equation

$$(E_0) \quad dy/dx = A_0(x)y$$

which belongs to \mathfrak{U} . Let

$$(E) \quad du/dx = A(x)u$$

be an arbitrary equation in \mathfrak{U} .

THEOREM 3. *There exist a finite number of sectors S_1, \dots, S_N such that (i) $\{S_j\}$ satisfies the two conditions given in §2, and (ii) for every equation (E) in \mathfrak{U} there exist n -by- n matrices $P_1(x), \dots, P_N(x)$ which satisfy the following conditions: (1) $P_j(x) \in A(S_j(R))$ for some R , and (2) the transformation $u = P_j(x)y$ reduces (E_0) to (E) in $S_j(R)$.*

Throughout this section, we fix sectors S_1, \dots, S_N and fundamental matrix solutions $\Phi_1(x), \dots, \Phi_N(x)$ of (E_0) which are given in $S_j(R)$ for some R (i.e. $d\Phi_j(x)/dx = A_0(x)\Phi_j(x)$ and $\det \Phi_j(x) \neq 0$ in $S_j(R)$) respectively.

A finite set $\{C_{jk}\}$ of n -by- n constant matrices is called a *Stokes system associated with* \mathfrak{A} if (i) $C_{jk} \in \text{GL}(n, \mathbf{C})$ whenever $S_{jk} \neq \emptyset$, (ii) $C_{jh}C_{hk} = C_{jk}$ whenever $S_j \cap S_h \cap S_k \neq \emptyset$, (iii) $\Phi_j(x)C_{jk}\Phi_k(x)^{-1} \in A(S_{jk}(R))$ for some R whenever $S_{jk} \neq \emptyset$, and (iv) the asymptotic system $\{\Phi_j(x)C_{jk}\Phi_k(x)^{-1}\}$ is trivial at $x = \infty$. Two Stokes systems $\{C_{jk}\}$ and $\{\tilde{C}_{jk}\}$ are said to be *equivalent* if there exist n -by- n constant matrices C_1, \dots, C_N such that (i) $C_j \in \text{GL}(n, \mathbf{C})$, (ii) $\Phi_j(x)C_j\Phi_j(x)^{-1} \in A(S_j(R))$, and (iii) $C_{jk}C_k = C_j\tilde{C}_{jk}$ whenever $S_{jk} \neq \emptyset$. An equivalence class \mathcal{C} of Stokes systems is called a *Stokes structure associated with* \mathfrak{A} .

Let $\{P_j(x)\}$ be the matrices given in Theorem 3. Then $P_j(x)\Phi_j(x)$ is a fundamental matrix solution of (E) in $S_j(R)$. Set $C_{jk} = (P_j\Phi_j)^{-1}(P_k\Phi_k)$ whenever $S_{jk} \neq \emptyset$. Then $\{C_{jk}\}$ is a Stokes system. Let \mathcal{C} be the Stokes structure to which $\{C_{jk}\}$ belongs. Then \mathcal{C} is independent of the choice of $\{P_j(x)\}$. This Stokes structure \mathcal{C} is called *the Stokes structure determined by* (E) and denoted by $\mathcal{C}(E)$.

THEOREM 4. *For every Stokes structure \mathcal{C} associated with \mathfrak{A} , there exists an equation (E) in \mathfrak{A} such that $\mathcal{C} = \mathcal{C}(E)$.*

THEOREM 5. *Two equations (E_1) and (E_2) in \mathfrak{A} are meromorphically equivalent if and only if $\mathcal{C}(E_1) = \mathcal{C}(E_2)$.*

REMARK. The definition of Stokes structures depends on the choice of $S_j, (E_0), \Phi_j$. Another choice produces another isomorphic definition. Certain particular choices were given, for example, by G. D. Birkhoff [2] and Balsler, Jurkat and Lutz [1].

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