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UNICITY OF BEST L_2 APPROXIMATION BY SECOND-ORDER SPLINES WITH VARIABLE KNOTS

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1. Introduction and results. Let $S_N \subset C[0, 1]$ denote the class of all piecewise linear functions with at most $N + 1$ linear segments. In this article we announce some interesting and somewhat surprising approximation properties of S_N in the space $L_2[0, 1]$. Three main theorems will be stated in this section and the main idea of our proof of the first two theorems will be sketched in §2.

Theorem 1 describes a fairly large class of strictly convex functions which have, for each positive integer N, unique best $L_2[0, 1]$ approximants from the nonlinear (spline) manifold S_N. Theorem 2 states that any sufficiently smooth strictly convex function eventually, i.e. for all large N, has a unique best $L_2[0, 1]$ approximant from this manifold. This behavior will be called "eventual uniqueness". Theorem 3 indicates the sharpness of these two results.

We emphasize that S_N is not only a nonlinear manifold, but also a nonclosed subset of $L_2[0, 1]$. Hence, arguments regarding existence, uniqueness, and characterization of best approximants are nontrivial. Since it has been shown in [1] that, for every positive integer N, any continuous function has at least one best $L_2[0, 1]$ approximant from S_N, we are only concerned with uniqueness and eventual uniqueness of best approximants in this paper.

THEOREM 1. *Let $f \in C^2[0, 1]$ with $f'' > 0$ on $[0, 1]$. Suppose that $\log f''$ is concave in $(0, 1)$. Then for every positive integer N, f has a unique best $L_2[0, 1]$ approximant from S_N.*

THEOREM 2. *Let $f \in C^5[0, 1]$ with $f'' > 0$ on $[0, 1]$. Then there exists a positive integer N_0 such that for any integer $N > N_0$, f has a unique best $L_2[0, 1]$ approximant from S_N.*

THEOREM 3. *Let N be any positive integer. There exists a function $f \in C^\infty[0, 1]$ with $f'' > 0$ on $[0, 1]$, such that f has more than one best $L_2[0, 1]$ approximant from S_N.*

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We mention that strictly convex $C^2[0, 1]$ functions always have unique best $L_\infty[0, 1]$ approximants from S_N (cf. [2, p. 188]). Thus Theorem 3 shows that there is a fundamental difference in the two norms regarding approximation from S_N, and in addition suggests that Theorem 1 is in some sense sharp. This seems to be quite interesting since the class of functions considered in Theorem 1 is not commonly associated with approximation theoretic questions.

2. Main ideas in the proof of unicity. Theorems 1 and 2 are proved in the following manner. Let $\Sigma^N \subset \mathbf{R}^N$ be the open simplex $\{\mathbf{t} = (t_1, \ldots, t_N): 0 \equiv t_0 < t_1 < \cdots < t_N < t_{N+1} \equiv 1\}$. For $\mathbf{t} \in \Sigma^N$ and each i, $i = 0, 1, \ldots, N$, let l_i be a linear polynomial defined on $[t_i, t_{i+1}]$. Further, let s be defined on $[0, 1]$ as follows: $s(t) = l_i(t)$ if $t_i \leqslant t < t_{i+1}$, $i = 0, \ldots, N$, and $s(1) = l_N(1)$. Hence, $s \in S_N$ if and only if $(l_1(t_1) - l_0(t_1), \ldots, l_N(t_N) - l_{N-1}(t_N)) = \theta$, the zero vector in $\mathbf{R}^N$. Let $s(t) \equiv s(t; \mathbf{t}^*)$, $\mathbf{t}^* = (t_1^*, \ldots, t_N^*) \in \Sigma^N$, be a best $L_2[0, 1]$ approximant from S_N. A variational argument shows that, for each $i = 0, \ldots, N$, l_i, the restriction of s to $[t_i^*, t_{i+1}^*]$, is a best L_2 approximant to f on the interval $[t_i^*, t_{i+1}^*]$ from the space of all linear polynomials on $[t_i^*, t_{i+1}^*]$. This suggests the consideration of the map F: $\Sigma^N \longrightarrow \mathbf{R}^N$, defined by $F(\mathbf{t}) = (l_1(t_1) - l_0(t_1), \ldots, l_N(t_N) - l_{N-1}(t_N))$, where, for each $i = 0, \ldots, N$, l_i denotes the unique best L_2 linear polynomial approximant of f on $[t_i, t_{i+1}]$. Therefore, a necessary condition for $\mathbf{t} \in \Sigma^N$ to be a knot sequence for a best $L_2[0, 1]$ approximant to f from S_N is that $F(\mathbf{t}) = \theta$. It can be shown that the topological degree of the map F with respect to Σ^N and the point θ is equal to 1 if $f'' > 0$. Under the hypotheses of Theorem 1, an additional argument shows that the topological degree actually counts the number of solutions. Hence, there can be only one set of knots for the best approximation to f from S_N and unicity easily follows. Similarly, if f satisfies the hypotheses of Theorem 2, one can conclude, using a similar but more complicated argument, that, for all large N, $F(\mathbf{t}) = \theta$ has a unique solution in the interior of Σ^N.

Complete proofs of the above three theorems and further results will appear elsewhere.

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