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Character theory of finite groups, by I. Martin Isaacs, Pure and Applied Math., Academic Press, New York, San Francisco, and London, 1976, xii + 303 pp., $29.50.

Several approaches to the representation theory of finite groups have been taken in recent works on the subject. Because representation theory has become part of applicable mathematics, there have been books for prospective users in physics, chemistry, and combinatorics. Another point of view was advanced by E. Noether, who observed that the representation theory of finite groups was equivalent to the study of modules over group algebras, and there are books treating group representations as part of the general theory of modules over algebras. A third theme, historically the oldest, is the application of representations and characters to finite group theory, represented in the recent books of Dornhoff [3], Feit [4], Gorenstein [5], Huppert [6], Serre [7], and the book which is the subject of this review, by I. M. Isaacs.

A prospective reader may ask why Isaacs emphasizes character theory rather than representations and modules. The characters, as trace functions of the representations, often yield the most efficient proofs of theorems in group theory. The proofs are usually based on simple, but interesting and often ingenious, computations. In comparison, arguments based on representations and modules, although sometimes conceptually simple, tend to require more machinery.

The first book treating the applications of character theory to group theory is the second edition [1] of Burnside's famous book on group theory, published in 1911. That book contained proofs of the following two results, which have been the inspiration for efforts to widen the scope of the methods used in their proofs up to the present time.

THEOREM A (FROBENIUS). *Let H be a subgroup of a finite group G such that, for all $g \in G - H$, $g^{-1}Hg \cap H = \{1\}$. Then the set of elements N not belonging to any conjugate $g^{-1}Hg$ of H, together with* 1, *form a normal subgroup such that $G = HN$, and $H \cap N = \{1\}$.*

THEOREM B (BURNSIDE). *Let G be a finite group of order p^aq^b, where p and q are primes. Then either G is cyclic of prime order, or G contains a proper normal subgroup. In any case, G is a solvable group, that is, there exist subgroups $G \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_s = \{1\}$, with each G_i normal in G_{i-1}, and each factor group G_{i-1}/G_i abelian.*

Both theorems assert the existence of proper normal subgroups of a finite group. This phenomenon is related to character theory in the following way. Let T: $G \to GL(V)$ be a representation of G by linear transformations on a finite dimensional vector space over the complex field, and let χ: $G \to \mathbf{C}$ be the character of T, given by $\chi(g) = \text{Trace}(T(g))$. Then the kernel N of the homomorphism T, which is a normal subgroup of G, satisfies the condition $N = \{g \in G: \chi(g) = \chi(1)\}$. Conversely, if θ is the character of an arbitrary representation, the set $\{g \in G: \theta(g) = \theta(1)\}$ is always a normal subgroup of G. In this way characters can be used to prove the existence of normal subgroups.

Burnside called attention in [1] to his conjecture that all finite groups of odd order are solvable. This problem, and the related problem of classifying finite simple groups, were taken up again by R. Brauer and a number of talented students, collaborators, and others who became drawn to the subject. The result has been a period of extraordinary growth in the development of finite group theory, highlighted by such major achievements as the solution of Burnside's problem on the solvability of groups of odd order by W. Feit and J. Thompson in 1963, and extensive progress towards the classification of nonabelian finite simple groups. (A group is simple if it has no proper normal subgroups.)

Theorems on the use of characters to prove the existence of normal subgroups are a recurrent theme in Isaacs' book. Perhaps unexpectedly, results of this kind are also used in the classification of finite simple groups, by showing that groups cannot be simple if they have this or that kind of subgroup etc., thereby closing the net on the simple groups that can exist. An example of this approach is to find what are the possible Sylow 2-subgroups in a nonabelian simple group (the Feit-Thompson theorem implies that a nonabelian simple group has even order, and hence the 2-Sylow groups are nontrivial). Dihedral 2-Sylow groups can occur in finite simple groups, while generalized quaternion groups, and the quaternion group of order 8, cannot. The latter result, due to Brauer and Suzuki, is proved for generalized quaternion groups in Isaacs' book, and provides an illustration of the technique of exceptional characters, which also led to methods required for the proof of the Feit-Thompson theorem.

A fundamental role in modern character theory is played by a theorem of R. Brauer, which describes how characters, as complex valued functions on the group, can be identified by looking at their restrictions to subgroups of a fairly simple type. More precisely, the *elementary subgroups* of a finite group G are defined as direct products of cyclic groups with subgroups of prime power order. A *generalized character* of G is an integral linear combination of the irreducible characters.

THEOREM C (R. BRAUER). *A complex valued class function on a finite group G is a generalized character if and only if its restriction to each elementary subgroup E is a generalized character of E.*

Here is an example of how the theorem is applied. It is wished to prove that a given irreducible character χ vanishes on a set S of elements with a certain property. Define a class function $\tilde{\chi}$ which agrees with χ on $G - S$, and vanishes on S. Then $\tilde{\chi}$ is a class function, and in favorable cases it can be shown that $\tilde{\chi}$ is a generalized character using Theorem C. It is then shown that $\tilde{\chi} = \chi$ using the orthogonality relations for irreducible characters.

An irreducible representation $T: G \to GL(V)$ of a finite group can always be realized in an algebraic number field K (i.e. a finite extension of the rational field Q). This means that for a suitable basis of the vector space, the matrices of the transformations corresponding to the elements of the group, with respect to that basis, will have entries in K. In particular, the character values $\{\chi(g), g \in G\}$, where χ is the character of the representation, generate a number field $Q(\chi)$ contained in K, $Q \subset Q(\chi) \subset K$. It frequently happens, however, that the irreducible representation cannot be realized in the smaller field generated by the character values. A measure of this discrepancy is given by what is called the *Schur index*. There are intriguing connections between algebraic number theory and the subgroup-structure of finite groups, and more recently with division algebras over the rational field, which have a common focus on the Schur index. Isaacs, in collaboration with Goldschmidt, has contributed to this subject himself, and includes some of his own work in this area, along with a number of related results.

Isaacs' book concludes with some topics which are either more specialized, or lead outside his intended scope. In particular, an introduction is given to Brauer's theory of modular characters. In this theory, the irreducible complex characters, whose values all lie in the ring of algebraic integers R in some number field K, are classified into subsets, called p-blocks, according to their behaviour with respect to some prime ideal P in R, dividing a fixed rational prime p. This theory, which achieves a finer analysis of the characters, has been used to prove results not originally accessible by the method of ordinary character theory, such as the result of Brauer and Suzuki that the quaternion group of order 8 cannot be the 2-Sylow group of a finite simple group. Brauer's theory involves the study of representations of groups in fields of characteristic $p > 0$, where the representations are not necessarily completely reducible, and the study of indecomposable modules as well as irreducible ones, becomes of crucial importance. The point of view required for the analysis of indecomposable modules is presented more fully in [2], [3] and [7].

Isaacs' book includes extensive lists of carefully thoughtout problems, which extend the scope of the book considerably. The book is a pleasure to read. There is no question but that it will become, and deserves to be, a widely used textbook and reference, as well as a place where curious mathematicians from other fields can find a clear and authoritative introduction to a fascinating part of group theory.

References

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Mathematical logic, by J. Donald Monk, Springer-Verlag, New York, x + 531 pp., $19.80.

"On the banks of the Rhine a beautiful castle had been standing for centuries. In the cellar of the castle an intricate network of webbing had been constructed by the industrious spiders who lived there. One day a great wind