Since further books on fuzzy set theory are unavoidable, we may at least ask them to show a greater sensitivity to the relevant diverse sources of literature, and provide a comparative analysis which shows when and where the language of fuzzy set theory helps, and where it only adds fuzziness to the theory without in any way smoothing the original problem.

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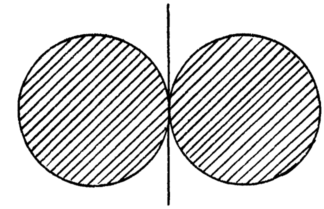
MICHAEL A. ARBIB

Etude géométrique des espaces vectoriels, une introduction, by Jacques Bair and René Fourneau, Lecture Notes in Mathematics, no. 489, Springer-Verlag, Berlin, Heidelberg, New York, 1975, vii + 184 pp., $8.20.

In the axiomatic study of linear topological spaces over the real field, which flourished forty to twenty years ago, it soon became clear that the absolutely minimal requirement for a topology in a linear space L is that each line in L carry a copy of much of the structure of R. This finds expression in two aspects of segments, first, that each open segment in R is a neighborhood of all its points–an aspect that can usefully be generalized to linear spaces over all topological fields, and, second, that the two endpoints of each segment are accessible from the interior of the segment–an aspect which generalizes to linear spaces over ordered fields. These two attitudes lead to analogues of interior of a set in L and of derived set of a set in L.

The first attitude leads to a definition: x is called a core point of a subset A of L if for each line l through x the subset $l \cap A$ contains an open interval (in l) which contains x. Two topologies in L are suggested: For T, the neighborhoods of x are all the subsets of L which have x as a core point; for T_n, the neighborhoods of x are all the *convex* subsets of L which have x as a core point. T is not badly related to the linear operations in L; translation by an element of L or dilation by a real number is a continuous function for T, but addition in L is not a continuous function of both variables, even if L is only two-dimensional.



(The picture shows a set, two tangent circular discs and an interval of their common tangent, which has zero as a core point, which cannot contain any $U + V$ for which U and V have 0 as a core point.)

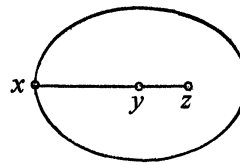
The topology T_n, on the other hand, makes L into a locally convex linear topological space, and T_n is the finest locally convex topology which L can carry. It is reported that thirty years ago in lectures at the University of Chicago, M. H. Stone called this the natural topology of L; since then it has frequently been called the convex core topology, a terminology probably due to MacShane or his student Klee in the late 1940's.

Starting from the end-point attitude leads to an analog of derived set, originally used by Nikodym and his students for convex subsets of L. Say that x is linearly accessible from A if there is a point a such that the half-open segment $[a, x[$ is contained in A. The book at hand uses ${}^{a}A$ for the set of all points linearly accessible from A; the early studies used $\operatorname{lin} A$, defined as $A \cup {}^{a}A$, the natural closure associated with ${}^{a}(\ )$. When A is a convex set with at least two points, ${}^{a}A \supseteq A$. Klee showed that L is finite-dimensional if and only if for each convex subset A of L, $\operatorname{lin}(\operatorname{lin} A) = \operatorname{lin} A$.

In early 1949 in a referee's report on one of Klee's early notes I sent him what I called "the standard example" of a convex set K for which each linear function bounded below on K is identically zero: L has a countably-infinite Hamel basis (e_i) and K is the set of all x whose last coordinate, when represented in terms of the e_i, is positive. (I do not now remember why I thought then that everybody knew that example; I came to it myself while thinking about ordered linear spaces after years of drudgery with lexicographic ordering in products of general ordered systems.) Klee pounced on the example, generalized it, called a convex set K ubiquitous in L if $\operatorname{lin} K = L$, and studied ubiquitous convex sets very thoroughly. (In the example above $K \cup -K = L$, $K \cap -K = \{0\}$, and both K and $-K$ are ubiquitous convex cones in L, both quite without core points.)

In that same period (1948-1953) Nikodym was working with the operation lin and with two other line-determined operations; Nikodym called a convex set K in L linear-bounded (linear-closed) if for each line l in L the intersection $A \cap l$ is contained in a segment $[x, y]$ of l ($A \cap l$ is a closed subset of l). He showed that if $\operatorname{lin}^{\alpha}$ is defined for all ordinals α by transfinite repetition of lin, then for each infinite-dimensional L and each $\alpha < \Omega$, the first uncountable ordinal, there is a convex A in L such that $\operatorname{lin}^{\alpha+1} A = \operatorname{lin}^{\alpha} A \supset \operatorname{lin}^{\beta} A \supset \operatorname{lin}^{\gamma} A$ for all $\gamma < \beta < \alpha$, while if the dimension of L is uncountable, there is even a convex set A with $\operatorname{lin}^{\alpha} A$ a strictly increasing function of $\alpha \leqslant \Omega$. However, for each convex set A in each linear space L $\operatorname{lin}^{\Omega+1} A = \operatorname{lin}^{\Omega} A$. Nikodym also showed by example that the vector sum of two linear-closed or linear-bounded sets need not be linear-closed or linear-bounded. Also, Nikodym and Berg showed how to split every convex set into disjoint convex subsets (faces) each of which, relative to its smallest containing flat set, contains only core points. Later Klee picked this up and defined an order in K: For x and y in K say that $x \leqslant y$ if $x = y$ or there is a z in K such that y is in the open segment $]x, z[$. Then Nikodym's faces are the equivalence classes determined by the equivalence relation $x \leqslant y$ and $y \leqslant x$.

Those aficionados of general topology who care to recall the usual connections between a monotone closure and an associated neighborhood system (as described in my paper *Convergence, closure, and neighborhoods*, Duke Math. J. **11** (1944), 181–199) can see that the set-operation ${}^{a}(\,)$ is determined by the neighborhood system $\mathfrak{N}^{a}$ for which U is a neighborhood of x if and only if for each $y \neq x$ there is a point u in $U \cap]x,y[$. For lin the neighborhoods of x are subject to the additional condition that $x \in U$. ${}^{a}(\,)$ and lin are not idempotent so they are not definable in the usual way from any family of open sets. ${}^{a}(\,)$ and lin are also definable by convergence, but, since they are not additive, directed systems are not adequate for them; some very-wide-spreading oriented systems must be used.



$x \leqslant y$, not $y \leqslant x$.

The book at hand emphasizes generalizations of these themes and applies them to not-necessarily-convex sets. ${}^{l}A$ is the smallest flat subset which contains A, and ${}^{i}A$ is the relative core of A; that is, the only lines through x which are considered are those in ${}^{l}A$. Of course, ${}^{i}A$ and ${}^{a}A$ are empty for most subsets of L, so certain special classes of sets, between starlike and convex, get extra attention.

Part I discusses finite iteration of these operations. Examples are given of (a) an A in R for which $\mathrm{lin}^{\alpha} A$ is strictly increasing for each $\alpha < \omega$, the first infinite ordinal, and (b) a subset B_{ω} of R^{ω} which has $\mathrm{lin}^{\alpha} A$ strictly increasing for all $\alpha \leqslant \omega$.

Part II studies transfinite iteration of these operations. It is shown that for each subset A of L iteration of either ${}^{i}(\,)$ or ${}^{a}(\,)$ becomes constant eventually. It seems to me that the use of a larger limit ordinal β than ω in a construction similar to (b) above would give a subset B_{β} of R^{β} with $\mathrm{lin}^{\alpha} B_{\beta}$ increasing for all $\alpha \leqslant \beta$. This would show that convexity was a necessary part of Nikodym's result that $\mathrm{lin}^{\Omega+1} A = \mathrm{lin}^{\Omega} A$.

Part III is devoted to applications, almost exclusively to convex problems: (i) Decompositions of L into finitely many convex subsets. (ii) Ordered linear spaces over R, with special emphasis on totally ordered linear spaces and their appearance in separation and extension problems. (iii) A few words on optimization problems with a proof that any maximizing points of a nonconstant convex function on a convex set A must lie in ${}^{m}A$, the margin of A, defined as ${}^{a}A \setminus {}^{i}A$. (iv) A return to 'faces' of convex sets to present a proof of

H. S. Bear that the Gleason parts of the spectrum of a function algebra are determined by mapping into an appropriate convex set K and showing that the Gleason parts are just the inverse images of the sets of the Nikodym decomposition of K.

Part IV looks at the effects of choosing an ordered field other than R. Part V returns to linear spaces over R to compare these algebraic-geometric operations in L with more usual topologies for L. The book ends with some account of the natural topology in L.

The authors have surveyed and digested the literature of this topic quite thoroughly. This set of lecture notes gives interested mathematicians a very full account of the kind of topological structure forced on a linear space by its scalar field.

M. M. DAY

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Model theoretic algebra: *Selected topics*, by Greg Cherlin, Lecture Notes in Mathematics, vol. 521, Springer-Verlag, Berlin and New York, 1976, iv + 232 pp., $9.50.

In an address to the International Congress of Mathematicians at Cambridge, Massachusetts in 1950, Abraham Robinson pointed out that "contemporary symbolic logic can produce useful tools–though by no means omnipotent ones–for the development of actual mathematics, more particularly for the development of algebra and, it would appear, of algebraic geometry." A similar observation was made by Alfred Tarski in an address to the same Congress in which he defined some of the basic notions of that branch of logic which is now called model theory–that is the study of the properties of mathematical structures expressible in formal mathematical languages.

That the expectations of these two giants of model theory were more than fully realized in the succeeding decades is indicated by the scope of the volume under review, which is an exposition of selected results in the model theory of such diverse algebraic systems as groups, rings, modules, fields, division rings, ordered fields and valued fields. Not all of the results presented are applications of model theory to algebra in the strict sense that they are theorems expressed in conventional algebraic terms and proved by model-theoretic methods; but many of the others are applications in the broader sense that they show how–in the words of Robinson in a later paper **[9]**–"certain basic facts and notions of Algebra, for example the notion of an algebraically closed field, can be placed and generalized within the framework of Model Theory."

The book under review, which consists of lecture notes of a course given by the author at M.I.T. in 1974 and again at the University of Heidelberg in 1975, constitutes an expeditious and extensive introduction to the burgeoning field of "model theoretic algebra." The author is a knowledgeable and informative guide, who provides a broad view of the subject, never losing sight of the forest