BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 4, July 1977

A TRUNCATION PROCESS FOR REDUCTIVE GROUPS

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Communicated by J. A. Wolf, January 26, 1977

Let G be a reductive group defined over $\mathbf{Q}$. Index the parabolic subgroups defined over $\mathbf{Q}$, which are standard with respect to a minimal ${}^{(0)}P$, by a partially ordered set $\mathfrak{I}$. Let 0 and 1 denote the least and greatest elements of $\mathfrak{I}$ respectively, so that ${}^{(1)}P$ is G itself. Given $u \in \mathfrak{I}$, we let ${}^{(u)}N$ be the unipotent radical of ${}^{(u)}P$, ${}^{(u)}M$ a fixed Levi component, and ${}^{(u)}A$ the split component of the center of ${}^{(u)}M$. Following [1, p. 328], we define a map ${}^{(u)}H$ from ${}^{(u)}M(\mathbf{A})$ to ${}^{(u)}\mathfrak{a} = \operatorname{Hom}(X({}^{(u)}M)_{\mathbf{Q}}, \mathbf{R})$ by

$$e^{\langle \chi, {}^{(u)}H(m)\rangle} = |\chi(m)|, \qquad \chi \in X({}^{(u)}M)_{\mathbf{Q}},\ m \in {}^{(u)}M(\mathbf{A}).$$

If K is a maximal compact subgroup of $G(\mathbf{A})$, defined as in [1, p. 328], we extend the definition of ${}^{(u)}H$ to $G(\mathbf{A})$ by setting

$${}^{(u)}H(nmk) = {}^{(u)}H(m), \qquad n \in {}^{(u)}N(\mathbf{A}),\ m \in {}^{(u)}M(\mathbf{A}),\ k \in K.$$

Identify ${}^{(0)}\mathfrak{a}$ with its dual space via a fixed positive definite form $\langle\ ,\ \rangle$ on ${}^{(0)}\mathfrak{a}$ which is invariant under the restricted Weyl group Ω. This embeds any ${}^{(u)}\mathfrak{a}$ into ${}^{(0)}\mathfrak{a}$ and allows us to regard ${}^{(u)}\Phi$, the simple roots of $({}^{(u)}P, {}^{(u)}A)$, as vectors in ${}^{(0)}\mathfrak{a}$. If $v \leqslant u$, ${}^{(v)}P \cap {}^{(u)}M$ is a parabolic subgroup of ${}^{(u)}M$, which we denote by ${}^{(v)}_{(u)}P$ and we use this notation for all the various objects associated with ${}^{(v)}_{(u)}P$. For example, ${}^{(v)}_{(u)}\mathfrak{a}$ is the orthogonal complement of ${}^{(u)}\mathfrak{a}$ in ${}^{(v)}\mathfrak{a}$ and ${}^{(v)}_{(u)}\Phi$ is the set of elements $\alpha \in {}^{(v)}\Phi$ which vanish on ${}^{(u)}\mathfrak{a}$.

Let R be the regular representation of $G(\mathbf{A})$ on $L^2(ZG(\mathbf{Q})\backslash G(\mathbf{A}))$, where we write Z for ${}^{(1)}A(\mathbf{R})^0$, the identity component of ${}^{(1)}A(\mathbf{R})$. Let f be a fixed K-conjugation invariant function in $C_c^\infty(Z\backslash G(\mathbf{A}))$. Then $R(f)$ is an integral operator whose kernel is

$$K(x, y) = \sum_{\gamma \in G(\mathbf{Q})} f(x^{-1}\gamma y).$$

If $u < 1$ and $\lambda \in {}^{(u)}\mathfrak{a} \otimes \mathbf{C}$, let $\rho(\lambda)$ be the representation of $G(\mathbf{A})$ obtained by inducing the representation

$$(n, a, m) \longrightarrow {}_{(u)}R_{\text{disc}}(m) \cdot e^{\langle \lambda, {}^{(u)}H(m)\rangle}$$

from ${}^{(u)}P(\mathbf{A})$ to $G(\mathbf{A})$. Here ${}_{(u)}R_{\text{disc}}$ is the subrepresentation of the representation

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AMS (MOS) subject classifications (1970). Primary 22E55; Secondary 32N10.

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of ${}^{(u)}M(\mathbf{A})$ on $L^2({}^{(u)}A(\mathbf{R})^0 \cdot {}^{(u)}M(\mathbf{Q})\backslash {}^{(u)}M(\mathbf{A}))$ which decomposes discretely. We can arrange that $\rho(\lambda)$ acts on a fixed Hilbert space ${}^{(u)}H$ of functions on ${}^{(u)}N(\mathbf{A}) \cdot {}^{(u)}A(\mathbf{R})^0 \cdot {}^{(u)}M(\mathbf{Q})\backslash G(\mathbf{A})$. If $u = 1$, we take ${}^{(1)}H$ to be the *orthogonal complement* of the cusp forms in the subspace of $L^2(ZG(\mathbf{Q})\backslash G(\mathbf{A}))$ which decomposes discretely.

THEOREM 1. *There exist orthonormal bases* ${}^{(u)}\mathfrak{B}$ *of* ${}^{(u)}H$, $u \in \mathfrak{J}$, *such that*

$$K_E(x, y) = \sum_{u\in\mathfrak{J}} \int_{i\{^u_1\}\mathfrak{a}} \sum_{\phi,\phi'\in{}^{(u)}\mathfrak{B}} (\rho(\lambda, f)\phi', \phi)E(\phi, \lambda, x)\overline{E(\phi', \lambda, y)}d|\lambda|$$

converges uniformly for x and y in compact subsets of $ZG(\mathbf{Q})\backslash G(\mathbf{A})$. (*Here* $E(\phi, \cdot\, , \cdot\,)$ *is the Eisenstein series associated with* ϕ *as in* [3, *Appendix* II].) *Moreover,* $R_{\mathrm{cusp}}(f)$, *the restriction of the operator* $R(f)$ *to the space of cusp forms, is of trace class, and if the Haar measures* $d|\lambda|$ *on* $i\{^u_1\}\mathfrak{a}$ *are suitably normalized,*

$$\operatorname{tr} R_{\mathrm{cusp}}(f) = \int_{ZG(\mathbf{Q})\backslash G(\mathbf{A})} (K(x, x) - K_E(x, x))\, dx. \quad \square$$

For any $u \in \mathfrak{J}$, let ${}^{(u)}\hat{\Phi}$ be the basis of $\{^u_1\}\mathfrak{a}$ which is dual to ${}^{(u)}\Phi$. We write $|u|$ for the number of elements in ${}^{(u)}\Phi$ or ${}^{(u)}\hat{\Phi}$. Let ${}^{(u)}\hat{\chi}$ be the characteristic function of $\{H \in {}^{(u)}\mathfrak{a} : \langle\mu, H\rangle > 0, \mu \in {}^{(u)}\hat{\Phi}\}$. Fix a point $T \in {}^{(0)}\mathfrak{a}$ such that $\langle\alpha, T\rangle$ is suitably large for each $\alpha \in {}^{(0)}\Phi$. Motivated by the results of [2, §9], we define

$$(\Lambda\phi)(x) = \sum_{u\in\mathfrak{J}} (-1)^{|u|} \sum_{\delta\in{}^{(u)}P(\mathbf{Q})\backslash G(\mathbf{Q})} \int_{{}^{(u)}N(\mathbf{Q})\backslash{}^{(u)}N(\mathbf{A})} \phi(n\delta x)\, dn \cdot {}^{(u)}\hat{\chi}({}^{(u)}H(\delta x) - T),$$

for any continuous function ϕ on $ZG(\mathbf{Q})\backslash G(\mathbf{A})$. Let $\tilde{k}^T(x)$ and $\tilde{k}^T_E(x)$ be the functions obtained by applying Λ to each variable in $K(x, y)$ and $K_E(x, y)$ separately, and then setting $x = y$. If ϕ is a cusp form, $\Lambda\phi = \phi$. From this it follows that

$$\tilde{k}^T(x) - \tilde{k}^T_E(x) = K(x, x) - K_E(x, x).$$

THEOREM 2. *The functions* $\tilde{k}^T(x)$ *and* $\tilde{k}^T_E(x)$ *are both integrable over* $ZG(\mathbf{Q})\backslash G(\mathbf{A})$, *and the integral of* $\tilde{k}^T_E(x)$ *equals*

$$\sum_{u\in\mathfrak{J}} \int_{i\{^u_1\}\mathfrak{a}} \sum_{\phi,\phi'\in{}^{(u)}\mathfrak{B}} (\rho(\lambda, f)\phi', \phi) \int_{ZG(\mathbf{Q})\backslash G(\mathbf{A})} \Lambda E(\phi, \lambda, x) \cdot \overline{\Lambda E(\phi', \lambda, x)}\, dx\, d|\lambda|. \quad \square$$

It should eventually be possible to calculate the integrals in Theorem 2 by extending the methods of [2, §9]. On the other hand, $\tilde{k}^T(x)$ is not a natural truncation of $K(x, x)$. This defect is remedied by the following

THEOREM 3. *The function*

$$k^T(x) = \sum_{u \in \mathfrak{J}} (-1)^{|u|} \sum_{\delta \in {}^{(u)}P(\mathbf{Q})\backslash G(\mathbf{Q})} \int_{{}^{(u)}N(\mathbf{A})} \sum_{\mu \in {}^{(u)}M(\mathbf{Q})} f(x^{-1}\delta^{-1}\mu n \delta x)\, dn$$

$$\cdot {}^{(u)}\hat{\chi}({}^{(u)}H(\delta x) - T)$$

is integrable over $ZG(\mathbf{G})\backslash G(\mathbf{A})$. *For sufficiently large* T, *the integrals over* $ZG(\mathbf{Q})\backslash G(\mathbf{A})$ *of* $k^T(x)$ *and* $\tilde{k}^T(x)$ *are equal.* □

The proofs will appear elsewhere.

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