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SOME ESTIMATIONS IN THE TOPOLOGY OF SIMPLY-CONNECTED ALGEBRAIC SURFACES

BY B. MOISHEZON

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The simplest nontrivial oriented topological surface is the 2-dimensional torus. It is well known that any compact Riemann surface is topologically equivalent to the 2-sphere with handles attached, that is, to a connected sum of 2-toruses. We can consider this decomposition as corresponding to the canonical decomposition of the (skew-symmetric) intersection form of 1-homologies on the given Riemann surface.

In the case of simply-connected algebraic surfaces the intersection form of 2-homologies plays a fundamental role because it defines completely the homotopy type of the corresponding 4-dimensional topological manifold (see [1], [2]).

Performing a σ-process on the given simply-connected algebraic surface V we obtain an algebraic surface V' which contains a 2-dimensional homology class with self-intersection equal -1 (which is an odd number). Then it is well known (see [3], [4]) that there exists a basis of $H_2(V', \mathbf{Z})$ such that the corresponding intersection matrix is diagonal. The corresponding "elementary blocks" $\|+1\|$ and $\|-1\|$ are the intersection matrices of the simplest nontrivial oriented simply-connected 4-manifolds:

P = complex projective plane with its usual orientation and Q = complex projective plane with orientation opposite to the usual. From the homotopy classification theorem [1], [2], it follows that V' is homotopy equivalent to a connected sum of P's and Q's. Of course, the "ideal situation" (analogous to the mentioned above topological decomposition of compact Riemann surfaces), which we could expect, is the existence of a homeomorphism of V' to this connected sum. However, there are some nondirect indications that V' is homeomorphic to a connected sum of P's and Q's if and only if V' is a rational algebraic surface. (This conjecture was formulated in [5].) The question is still open, but assuming the conjecture we can consider as a realistic aim only the problem of estimating how "far" topologically is the given nonrational simply-connected algebraic surface from an "ideal" topological model, that is, from a connected sum of P's and Q's.

In [6] Wall proved the following theorem: *If M_1, M_2 are simply-connected compact 4-manifolds, which are homotopically equivalent, then there exists*

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an integer $k \geqslant 0$ *such that* $M_1 \# k(S^2 \times S^2)$ *is diffeomorphic to* $M_2 \# k(S^2 \times S^2)$ (# is is the connected sum operation).

It follows almost immediately from this result that if M is a simply-connected compact 4-manifold, then there exists an integer $k \geqslant 0$ such that $M \# (k+1)P \# k \cdot Q$ is diffeomorphic to $lP \# mQ$ for some $l, m \geqslant 0$.

After the proof of his theorem Wall writes the following [**6**, p. 147]: "We remark that our results is a pure existence theorem; We have obtained, even in principle, no bound whatever on the integer k".

As it was remarked in [**5**], the operation $M \# P$ (resp. $M \# Q$) where M is an oriented 4-manifold could be considered as performing of certain blowing up of some point on M. We call this blowing up $\overline{\sigma}$-process (resp. σ-process). (The exact definition of σ-process and $\overline{\sigma}$-process is the following: In a small enough neighborhood N_x of a point $x \in M$ we can always take local coordinates giving N_x a complex structure. This complex structure will then have an orientation the same as that of M or opposite to that of M. Performing a classical σ-process using the local complex coordinates of N_x, we get an operation which in the first case we call "σ-process" and in the second case "$\overline{\sigma}$-process".)

We say that an oriented simply-connected 4-manifold W is completely decomposable (resp. almost completely decomposable) if W (resp. $W \# P$) is diffeomorphic to $lP \# mQ$ for some $l, m \geqslant 0$.

Let M be an oriented compact simply-connected 4-manifold. For $(k_1, k_2) \in \mathbf{Z} \times \mathbf{Z}$, $k_1 \geqslant 0$, $k_2 \geqslant 0$, let $M(k_1, k_2)$ be a 4-manifold obtained from M by k_1 $\overline{\sigma}$-processes and k_2 σ-processes. Denote by $\mathcal{W}(M) = \{(k_1, k_2) \in \mathbf{Z} \times \mathbf{Z} | k_1 \geqslant 0, k_2 \geqslant 0, M(k_1, k_2)$ is completely decomposable$\}$. It follows from the theorem of Wall that $\mathcal{W}(M) \neq \varnothing$.

An important geometrical problem is to define minimal elements of $\mathcal{W}(M)$ (in any natural sense). A certain step for solving this problem could be the construction of some elements of $\mathcal{W}(M)$ in explicit form, say in terms of the 2-dimensional Betti number and of the signature of M. We can prove that such a construction is possible when M admits a complex structure. The main result is

THEOREM A. *Let M be a compact simply-connected 4-manifold which admits a complex structure. Take an orientation on M corresponding to certain complex structure on it. Let* $K(X)$, $L(X)$ *be cubic polynomial defined as follows*:

$$K(X) = \widetilde{K}(9(5X+4)) - X, \qquad L(X) = \widetilde{L}(9(5X+4)),$$

where $\widetilde{K}(t) = t(t^2 - 6t + 11)/3$, $\widetilde{L}(t) = (t-1)(2t^2 - 4t + 3)/3$

$$(K(X) = 30375X^3 + 68850X^2 + 52004X + 13092,$$

$$L(X) = 60750X^3 + 141750X^2 + 110265X + 28595).$$

Denote by b_+ *(resp.* b_-*) the number of positive (resp. negative) squares in the inter-*

section form of M and let $k_1' = K(b_+)$, $k_2' = \max(0, L(b_+) - b_-)$. *Then the pair* $(k_1', k_2') \in \mathcal{W}(M)$.

REMARKS ABOUT THE PROOF OF THEOREM A. From the Kodaira classification of compact complex surfaces [7] it follows that if M is a simply-connected compact complex surface, then there exists a nonsingular projective-algebraic complex surface V such that V is diffeomorphic to M and one of the following three possibilities holds:

(a) V is rational;

(b) V is elliptic;

(c) V is of general type.

In case (a) our theorem is evident. In case (b) we can prove a much stronger result.

THEOREM B. *Any simply-connected elliptic surface V is almost completely decomposable.* (*That is*, $(1, 0) \in \mathcal{W}(V)$.)

For the case (c) we first prove the following comparison theorem for topology of projective algebraic surface of given degree n and nonsingular hypersurface of degree n in $\mathbf{C}P^3$:

THEOREM C. *Let* V_n *be a projective algebraic surface of degree n embedded in* $\mathbf{C}P^N, N \geqslant 5$, *such that* V_n *is not contained in a proper projective subspace of* $\mathbf{C}P^N$. *Suppose that* V_n *is nonsingular or has as singularities only rational double-points. Let* h: $\widetilde{V}_n \longrightarrow V_n$ *be a minimal desingularization of* V_n (*that is,* $\widetilde{V}_n$ *has no exceptional curve of first kind s such that h(s) is a point on* V_n). *Suppose* $\pi_1(\widetilde{V}_n) = 0$. *Denote by* Y_n *the diffeomorphic type of a nonsingular hypersurface of degree n in* $\mathbf{C}P^3$.

Then

(i) $b_+(\widetilde{V}_n) < b_+(Y_n)$, $b_-(\widetilde{V}_n) < b_-(Y_n)$;

(ii) $\widetilde{V}_n \# [b_+(Y_n) - b_+(\widetilde{V}_n) + 1] P \# [b_-(Y_n) - b_-(\widetilde{V}_n)] Q$ *is diffeomorphic to* $Y_n \# P$.

In [5] it was proved that $Y_n \# P$ is completely decomposable. Thus we need only some estimation of a possible minimal degree for projective embeddings of V in terms of $b_+(V)$, $b_-(V)$. We obtain such an estimation from Bombieri's results on pluricanonical embeddings of algebraic surfaces of general type [8].

REMARK TO THEOREM B. Note that Theorem B together with results of [5], [9], [10] show that all big explicit classes of simply-connected algebraic surfaces considered until now have the property that their elements are almost completely decomposable 4-manifolds. That is, the "theoretical" Thereom A gives much weaker results than our "empirical" knowledge.

The interesting question is, how far we can move with such "empirical achievements" in more general classes of simply-connected algebraic surfaces.

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DEPARTMENT OF MATHEMATICS, TEL-AVIV UNIVERSITY, RAMAT-AVIV, ISRAEL