BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 4, July 1977

OBSTRUCTION THEORY IN 3-DIMENSIONAL TOPOLOGY: CLASSIFICATION THEOREMS

BY HARRIE HENDRIKS

Communicated by T. A. Chapman, January 24, 1977

We consider the classification up to homotopy of homotopy equivalences of compact 3-manifolds. Given two compact 3-manifolds (with base point and *CW*-decomposition) (M, m) and (N, n), in [3] we found an algebraic criterion for the existence of a degree 1 map $f: (M, \partial M) \longrightarrow (N, \partial N)$ extending a given map f^1 defined on the relative 1-skeleton $(M, \partial M)^1$. Here we consider the space $\mathcal{H}^1(M, m)$ of degree 1 homotopy equivalences $f: (M, m) \longrightarrow (M, m)$ such that $f|\partial M = \mathrm{Id}$ and f is homotopic rel $\partial M \cup \{m\}$ to a map coinciding with the identity on $(M, \partial M)^1$. If $\partial M = \emptyset$, it is equivalent to say that f induces the identity automorphism of $\pi_1(M, m)$. (If $\partial M \neq \emptyset$, we assume that $m \in \partial M$.) Important results are the following.

1. Following Waldhausen e.a. [6] a homotopy equivalence of $\mathbf{P}^2$-irreducible (closed) sufficiently large 3-manifolds is homotopic to a homeomorphism unique up to isotopy. Our result indicates that the exclusion of 2-sided projective planes is necessary. Indeed, suppose M is the connected sum of two nonsimply connected 3-manifolds, then we have

THEOREM ([2]). *If M contains 2-sided projective planes, M admits a self homotopy equivalence, in $\mathcal{H}^1(M, m)$, which is not homotopic to a homeomorphism* rel ∂M.

Recall that all elements of $\mathcal{H}^1(M, m)$ are *simple* homotopy equivalences (in the sense of Whitehead).

On the other hand, let S be an embedded 2-sphere in M with collar $S \times [0, 1]$. Then the *rotation along S* is the homeomorphism in $\mathcal{H}^1(M, m)$ defined by the identity outside $S \times [0, 1]$ and by a generator of $\pi_1 SO(3)$ within $S \times [0, 1]$.

THEOREM ([4]). *Let M be a 3-manifold which does not contain 2-sided projective planes, then every self homotopy equivalence in $\mathcal{H}^1(M, m)$ is homotopic* rel $\partial M \cup \{m\}$ *to a rotation along a sphere.*

2. Let $\mathcal{R}$ be the set of 2-spheres S in M such that we can express $M = M_1 \cup M_2$, where $M_1 \cap M_2 = S$, and where $M_1 \cup_S D^3$ is a connected sum of closed manifolds, each either with finite fundamental group whose 2-Sylow

AMS (MOS) subject classifications (1970). Primary 55D10, 55G37, 57A10.

subgroup is *cyclic* or homotopy equivalent to a S^2 or $\mathbf{P}^2$ fibration over S^1.

THEOREM ([4]). *The rotation along an embedded 2-sphere S is homotopic to the identity* rel ∂M *if and only if* $S \in \mathcal{R}$.

For nonseparating spheres S, this is contained in [5].

3. The composition in $\mathcal{H}^1(M, m)$ defines a multiplication in $\pi_0 \mathcal{H}^1(M, m)$.

THEOREM. $\pi_0 \mathcal{H}^1(M, m)$ *is a group of exponent* 2.

More completely, let $\Lambda = \{\lambda \in \pi_1(M, m); \lambda^2 = e, \lambda$ reverses the orientation$\}$.[1] By [1] there exists for each $\lambda \in \Lambda$ an immersion $\sigma_\lambda: (S^2, *) \longrightarrow (M, m)$ such that $\sigma_\lambda(S^2)$ is a (2-sided) projective plane carrying the loop λ. Let $W(\Lambda)$ denote the $\mathbf{Z}_2$-module with generators $\Lambda \times \Lambda$ and relations $\langle\lambda, \mu\rangle = \langle\mu, \lambda\rangle = \langle\xi\lambda\xi^{-1}, \xi\mu\xi^{-1}\rangle = \langle\lambda, \mu\lambda\mu\rangle$ for every $\lambda, \mu \in \Lambda$ and $\xi \in \pi_1 M$. Let R denote the $\pi_1 M$ submodule of $\pi_2 M$ generated by $\mathcal{R}$.

MAIN THEOREM ([4]). *Suppose* $\pi_2 M \neq 0$. *There is an exact sequence of* $\mathbf{Z}_2$ *modules*:

$$0 \longrightarrow \mathbf{Z}_2 \otimes_\pi (\pi_2 M)/R \xrightarrow{r} \pi_0 \mathcal{H}^1(M, m) \longrightarrow W(\Lambda) \oplus \mathbf{Z}_2 \otimes_\pi \mathbf{Z}[\Lambda] \longrightarrow 0,$$

where $\otimes_\pi$ *denotes the tensor product over* $\mathbf{Z}[\pi_1 M]$.

If $\sigma: (S^2, *) \longrightarrow (M, m)$ is an embedding or an immersion with image a 2-sided projective plane, $r(1 \otimes \sigma)$ is represented by the *rotation along* $\sigma(S^2)$. Let $\lambda, \mu \in \Lambda$, and suppose $f: M \longrightarrow M$ is a map different from the identity only in a 3-ball where it differs by $\sigma_\lambda \circ Hopf \in \pi_3 M$, where *Hopf* denotes the Hopf fibration $S^3 \longrightarrow S^2$ (resp. by the Whitehead product $[\sigma_\lambda, \sigma_\mu]$). Then the arc component of f is mapped to $1 \otimes \lambda$ (resp. $\langle\lambda, \mu\rangle$).

BIBLIOGRAPHY

1. D. B. A. Epstein, *Projective planes in 3-manifolds*, Proc. London Math. Soc. (3) **11** (1961), 469–484. MR **27** # 2968.

2. H. Hendriks, *Une obstruction pour scinder une équivalence d'homotopie en dimension* 3, Ann. Sci. École. Norm. Sup. (4) **9** (1976), 437–467.

3. ———, *Obstruction theory in 3-dimensional topology: an extension theorem*, Report 7617, Nijmegen (1976); J. London Math. Soc. (to appear).

4. ———, *Applications de la théorie d'obstruction en dimension* 3, available as Publ. math. d'Orday N^o 133–7535 (1975); also, Mémoire Soc. Math. France (to appear).

5. L. Pontryagin, *A classification of mappings of the three-dimensional complex into the two-dimensional sphere*, Réc. Math. [Mat. Sbornik] N. S. **9** (**51**) (1941), 331-363. MR **3**, 60.

6. G. P. Scott, *On sufficiently large 3-manifolds*, Quart. J. Math. Oxford Ser. (2) **23** (1972), 159–172; Correction, ibid (2) **24** (1973), 527–529. MR **52** # 4295.

MATHEMATISCH INSTITUUT, KATHOLIEKE UNIVERSITEIT, NIJMEGEN, NEDERLAND

---

[1] $\pi_1(M, m)$ acts on Λ by conjugation.