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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 4, July 1977

Mathematics of organization, by Mircea Malita and Corneliu Zidăroiu, Abacus Press, Tunbridge Wells, Kent, England, 1974, 383 pp., $30.00.

The Nobel Prize in Economics for 1975 was awarded to Leonid V. Kantorovich and Tjalling C. Koopmans for their contributions to the theory of optimum allocations of resources. This event emphasized the fact that the mathematics of operations research has been developed in parallel with economic theory. Books on operations research, such as the one under review, emphasize optimization problems, especially linear programming, game theory and control theory. These topics have been developed in the past thirty years and a sketch of this development may help to put in perspective the mathematics, presented in this book in a rather terse style.

In 1928 John von Neumann [**16**] gave a mathematical formulation of games of strategy and proved the celebrated minimax theorem justifying his definition of the value of a noncooperative game. This work was not pursued further until the economist Oskar Morgenstern, having been forced to leave Vienna, came to Princeton University and, during the classical tea in Fine Hall, talked with von Neumann about games and economics. This conversation led to the collaboration between Morgenstern and von Neumann which resulted in the publication in 1944 [**17**] of their famous book *The theory of games and economic behavior*. A fascinating account of this collaboration may be found in [**11**].

In 1939 the Russian mathematician Leonid Kantorovich published a paper in which he discussed several problems which we now call linear programming problems. An example of the type of problem that he solved is the following. A plywood producer has eight peeling machines and uses five different kinds of material. The productivity of each machine for each kind of material is given. How should the work time for the machines be distributed to maximize the output per day? He discussed a number of such problems and showed that each led to the problem of maximizing certain linear functions under linear inequality constraints. He presented an iterative method for solving these problems based on quantities he called resolving multipliers and which are now called dual variables in linear programming. These variables have important economic interpretions and Kantorovich made use of this fact. See [7] for a more detailed discussion of the work of Kantorovich. This work was not to become known to the Western world until 20 years later when Koopmans learned of it and took the initiative to have it published in 1960 [**8**]. The independent development of linear programming in the West grew out of work carried out during World War II. In 1942 Koopmans, working on the British-American shipping board dealing with merchant shipping problems, wrote a memorandum "Exchange Ratios Between Cargoes on Various Routes". See [**19**] for references to this and related work of Koopmans. In this work, Koopmans explicitly introduced ideal prices to find the optimal shipping plan. In 1947 George Dantzig, working on similar problems for the Air Force, was led to formulate the general problem of linear programming as we now know it. In addition he presented his celebrated simplex method. This was first presented in [**10**] and the full theory developed in [**2**]. A typical problem is to find the maximum of a linear function defined on a bounded convex set of nonnegative vectors in R_n formed by the intersection of a finite number of half-planes. The maximum will occur on a boundary point and the simplex algorithm gives a method for starting at one corner and, if possible, moving to a neighboring corner where the function has a greater value. If no such move can be made the desired maximum has been found. A practical problem may require looking at hundreds of corners and the introduction of the high speed computer made his algorithm a powerful tool for solving these problems. Dantzig's work, unlike that of Kantorovich, was immediately appreciated. Von Neumann introduced and stressed the importance of duality in linear programming and conjectured the equivalence of games and linear programming problems. Koopmans was led to a new theory of economic production that he called activity analysis. In this theory he showed, in a very general setting, that the problem of efficient use of resources was intimately connected with a dual problem of assigning a theoretical value of price to the goods used in production. These results as well as a number of other fundamental results were presented by leading economists and mathematicians at a conference in 1947 organized by Koopmans. In particular, D. Gale, H. W. Kuhn and A. W. Tucker formalized the duality theory for linear programming and Dantzig verified von Neumann's conjecture. The proceedings of this conference were published in [**10**]. It was an exciting time and I can recall, as a student, hearing with great awe that new results were being transmitted by telephone between Princeton and California. I have dwelled upon this history because in this book and most others written as mathematics books, the treatment of the simplex method and duality comes out a bit dry and the dual prices, called shadow prices, are in fact presented as if the Shadow himself had invented them. One has to indulge in a little reading of economic theory to realize that production and prices play an equally fundamental role in economics, and this shows up in the mathematical theory in many subtle ways.

Many new problems in graph theory and combinatorics came out of studying special linear programming problems. Such problems are well presented in this book. A typical problem is the following. Given a road map of the country, find the shortest route from New York to San Francisco. More precisely, the problem is to find a good algorithm for finding such minimum path solutions. An algorithm is considered "good" if the number of operations required is bounded by a polynomial function of the number of inputs. It is possible to find such an algorithm for this problem. However, a long outstanding problem is to decide whether such an algorithm even exists for the slightly different problem of the traveling salesman. This is the problem of finding the shortest path starting at a particular city and returning to this city, having visited every city, at least once. Questions like this have led to a new branch of mathematics called computational complexity. Here one asks for example: for which mathematical problems is it possible to find a good algorithm? Recently Karp and others [**9**] have shown that for a fairly large class of problems either all problems in the class have a polynomial algorithm or none do. This class includes such problems as determining if a graph has a Hamiltonian path–a path which goes through all points exactly once. Another important problem in this area is the following. The simplex algorithm can require exponential time and hence is not good in the above sense. However, it is good in practice since it seems never to require a number of operations more than three times the number of equations in the problem. The question of the existence of algorithms which are good in some average or "typical" sense is apparently an even harder problem.

A second area of new research that has come out of economic theory is the work of H. Scarf [**15**] who has found an effective algorithm for determining a fixed point of a continuous transformation of a simplex into itself. Economists have made major progress in recent years in putting Walras' theory of the existence of an equilibrium price vector in a competitive market in a reasonable mathematical form. It is assumed that for any set of prices there are certain demands made on the set of all goods. Prices are changed as a function of the excess demands. This results in a continuous transformation of prices and an equilibrium price vector is a fixed point of this transformation. In any reasonable application there will be many commodities and so, as in the case of linear programming, without a way to find the fixed point it is hard to determine the validity of these equilibrium theories. The Scarf algorithm provides a major new tool for this theory comparable to that provided by the simplex algorithm for production theory. Scarf's algorithm bears some resemblance to the simplex algorithm. The proof is based on some very elegant combinatorial topology. The basic idea of the algorithm is the following. Let S be the simplex of nonnegative vectors in R_n with component sum 1. Let T be a continuous map of S into itself. Choose a set X of points on the simplex. Label the points of X as follows: if $x = (x_1, x_2, \ldots, x_n)$, label x by i if i is the first i such that $(Tx)_i \geqslant x_i$. A set of n points from X is called primitive if there is a subsimplex of S which contains only these n points from X one occuring on each edge. A primitive set is completely labeled if all the points have different labels. If the mesh determined by the points is small and x is any point in a completely labeled set, then for any i there is a point $\bar{x}$ near x such that $(T\bar{x})_i \geqslant \bar{x}_i$. By continuity and the fact that x and Tx have components with the same sum, x must be near Tx. Thus any point in a completely labeled set is nearly a fixed point. The Scarf algorithm provides a method for starting with a primitive set which is not completely labeled and, by exchanging one point at a time, arriving, after a finite number of steps, at a completely labeled set. While this gives a new proof of Brouwer's fixed point theorem the real interest is in its application to finding approximate fixed points.

The final topic that we will discuss is that of control theory. A first problem in control theory came from Abraham Wald's formulation of statistical decision theory [**18**] in terms of game theory concepts. Faced with an unknown distribution the statistician is allowed to sample and make an intelligent decision based on the information obtained about the underlying distribution. It costs money to sample and wrong final decisions cost money. In the evaluation of optimal strategies Wald was led to a problem that is now called the optimal stopping problem. He assumed that at any stage of the sampling the statistician has certain information which we denote by v. If he stops at time n with information v he will suffer a loss of $u_n(v)$. Assume that the cost of another experiment is c. Then the value $w_n(v)$ of the statistician's position at time n with information v should satisfy the functional equation

$$w_n(v) = \min(u_n(v), E(w_{n+1}|v) + c).$$

He showed that this was the case, if the number of samples was bounded, using backward induction. He then showed that it was true by a limiting process for a wide class of situations. The optimal strategy for the statistician is to continue sampling until the first time that $w_n(v) = u_n(v)$ and then quit and make the best decision that can be made based on the information that he has at that time. This optimal stopping problem was put in a more general setting and shown to have very pretty connections with martingale theory and with potential theory for Markov processes. A discussion of the martingale approach may be found in Neveu [**13**] and the potential theory approach in Dynkin and Yushkevich [**3**]. Viewed as a control problem the statistician has at any stage to choose one of two controls, "continue" or "stop". Many natural generalizations of this control problem have been made. Bellman [1] gave a formal theory for both stochastic and deterministic control problems and made many contributions to the theory. A particularly well-developed theory is that of Markov decision processes. Here one assumes that at any stage one has to make one of a finite number of decisions. The decision determines an immediate reward and the probabilities for moving to a new state on the next step. It is desired to find a decision procedure which optimizes the total final reward. A theory of Markov decision processes may be found in the book of Howard [6] and a very general treatment with emphasis on economic problems may be found in the very recent book of Dynkin and Yushkevich [4]. The book under review considers only discrete time problems. The continuous time control problems is another very rich theory initiated by a Russian school led by Pontryagin [14]. An account of the continuous time case for both the deterministic and the stochastic case may be found in [5]. This theory has been shown to be closely connected with classical theory of the calculus of variations showing that good mathematics never dies or even fades away.

Malita and Zidăroiu present a quite condensed version of the mathematics of problems which grew out of linear programming and game theory. They also discuss a number of standard topics from applied probability and statistics such as queuing theory and analysis of variance. Probability and statistics also played an important role in early development of operations research. A group headed by Philip Morse at M.I.T. was formed in 1942 to work for the Navy on search techniques related to submarine warfare. An interesting account of the work of this group can be found in the book by Kimball and Morse [12].

The authors do a good job of showing the wealth of mathematical ideas that have come from quite modest applied problems.

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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 4, July 1977

Finite orthogonal series in the design of digital devices, by M. G. Karpovsky, John Wiley and Sons, New York, 1976, 251 pp., $35.00 with bibliography and subject index.

The subject of this book is an interesting mathematical approach to the design of digital logic, with the intended application of the results to be used in the design of digital computers. The central problem of the book concerns the methodology for designing networks that realize arbitrary boolean functions. In the most elementary case, the problem is to realize a single boolean valued function $f(x_1, x_2, \ldots, x_n)$ of the n boolean variables $x_1, x_2, \ldots, x_n$. The design objective is to obtain minimum cost designs where cost is measured in terms of the costs of the primitive functions used to construct the given function. More complex problems derived from the basic one include the design of networks that realize two or more boolean functions of the same arguments, the design of networks that realize sequential functions (functions of present and past values of arguments), and the design of networks for partially specified functions. In the latter case the design makes use of the freedom to complete the function specification arbitrarily, and picks a completion that achieves minimal cost. Yet another problem is the design of networks that exhibit error-correction properties in that failures of such a network result in a network that produces a different function from the desired one with low probability.

Traditional approaches taken by practitioners involve costly searches over many possible implementations to find the best one, or they rely on canonical realizations that are improved by hand on an *ad hoc* basis. Karpovsky exposes a very different mathematical viewpoint to the minimization process that has several interesting properties. His work is partially stimulated by work by Ninomiya and by Lechner on harmonic analysis of boolean functions.