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Order and potential resolvent families of kernels, by Aurel Cornea and Gabriela Licea, Lecture Notes in Mathematics, no. 494, Springer-Verlag, Berlin, Heidelberg, New York, 1975, 154 pp., $7.40.

The first title of this book is *Order and potential*. If the nonspecialist reader opens it at any page, just looking for familiar words, he can be sure to see some mention of *order*, and has reasonable chances to find *potentials*, but may wonder whether the use of the latter word has anything to do with newtonian *potential*, harmonic functions and similar things. After all, the word potential has different connotations in different contexts (the military potential of the United States, the industrial potential of Europe) and the recurrent mention of a mysterious "domination principle" might lead to further political misinterpretations. So let me tell first what the subject of the book really is.

We must come back to the early history of the subject. Between 1945 and 1950, H. Cartan proved some fundamental results in classical potential theory, which were rapidly digested, generalized and improved by the French school of potential theory around M. Brelot, G. Choquet and J. Deny. The axiomatic trend had always been felt in potential theory (the use of the old word "principle" to mean "axiom" may be good evidence for it), and anyhow the years 1950 were those of the big axiomatic boom in mathematics. Hence it is entirely natural that the interest shifted from potential *theory* to potential *theories* defined by suitable axioms. Among the interesting features of classical potential theory, the so called *complete maximum principle* came to play a leading role. It can be easily stated and understood, as follows. Let u and v be two newtonian potentials of positive measures λ and μ, and let a be a positive constant. Assume that

(1) $a + u \geqslant v$ on the closed support F on the measure μ corresponding to v.

Then the same inequality takes place everywhere. This is almost obvious. In the open set F^c complement of F, the function $a + u - v$ is superharmonic, so it dominates the infimum of its boundary values. Since the boundary is contained in F where $a + u - v$ is positive by assumption, $a + u - v$ must be positive in F^c. The point in the presence of a is the fact that a constant function is superharmonic, but is not a potential. The same statement with $a = 0$ is a substantially weaker axiom (domination principle), and the same is true if $u = 0$ (ordinary maximum principle). In the years 1950–1951 two papers were published, one by Cartan and Deny [**1**], one by Deny [**2**], which are relevant to us because of the following remark. Let μ be any positive measure on $\mathbf{R}^n$, of total mass $\leqslant 1$, and let α be the measure

$$\alpha = c\sum_0^\infty \mu^{*n} \tag{2}$$

where c is a positive constant, and μ^{*n} is the nth convolution power of μ. Assuming α is finite on compact sets, let f be a positive function[1], and call potential of f the convolution $\alpha * f = Uf$. Then U satisfies many "principles" of potential theory, including the complete maximum principle, in the form appropriate to functions instead of measures: if a is a positive constant, if f and g are positive functions, then $a + Uf \geqslant Ug$ on the set where $g > 0$ implies the same inequality everywhere. They also remarked that the measure $dx/|x|^{n-2}$ which defines the newtonian potentials, can be approximated by measures α_n which can be represented as (2). More generally, there seemed to be an intimate connection between the following two properties of the positive measure α: (1) The operator $Uf = \alpha * f$ satisfies the complete maximum principle, (2) α can be approximated by measures α_n as above. This was already the very heart of our subject!

Let us say that an operator U which takes positive functions f into positive functions Uf is a *kernel* if the value $Uf(x)$ of Uf at x is the integral of f with respect to some positive measure $U(x, dy)$. Let us say that a kernel U defines a *potential theory* if the complete maximum principle is satisfied. This is far more general than convolution kernels. The fundamental work of Doob [**3**, 1954], [**4**, 1955] and Hunt [**5**, 1957] in probability theory had led to the conclusion that if (P_t) is a *semigroup of kernels*, that is a family $(P_t)_{t\geqslant 0}$ such that

$$P_s(P_t f) = P_{s+t} f, \quad P_0 f = f \quad \text{for all } f \tag{3}$$

and also $P_t 1 = 1$ (this can be relaxed to $P_t 1 \leqslant 1$), then the kernel

$$Uf = \int_0^\infty P_t f\, dt \tag{4}$$

defines a potential theory. The original proof of this result was probabilistic, but more recent proofs are analytic and require practically no regularity assumptions. Even more striking was the converse: Hunt proved that any "reasonable" potential theory arises in this way. There is a relation between this result and that of Cartan and Deny: in (2), the powers of μ correspond to a *discrete* semigroup, instead of a continuous one. The gist of the passage from discrete to continuous semigroups is the replacement of an *approximation* of the potential kernel by an exact representation like (4).

[1] All "functions" here and below are assumed to be measurable with respect to appropriate σ-fields. We are not concerned with such details in a review.

To understand the book of Cornea and Licea, one step is still missing. Hunt's way from U to (P_t) is not direct. There is an intermediate construction, that of the operators

$$U_\lambda f = \int_0^\infty e^{-\lambda t} P_t f \, dt \quad \text{for } \lambda > 0 \tag{5}$$

which are connected to U by the "algebraic" relation

$$U = U_\lambda (I + \lambda U) \tag{6}$$

(meaning $Uf = U_\lambda f + \lambda U(U_\lambda f)$ if f is a positive function), and to each other by the so called *resolvent equation*

$$U_\mu = U_\lambda (I + (\lambda - \mu) U_\mu) \tag{7}$$

if $\mu < \lambda$. On the other hand, the inequality $P_t 1 \leqslant 1$ is reflected in the inequality $\lambda U_\lambda 1 \leqslant 1$ satisfied by the "resolvent" $(U_\lambda)_{\lambda \geqslant 0}$. Hunt's method consists in a very clever construction of the family (U_λ) from equation (6), depending in an essential way on the complete maximum principle (Hunt's proof was entirely freed from probabilistic "impurities" by Lion, [**6**, 1966]). So Hunt's way was

$$\text{Kernel} \xrightarrow[a]{} \text{Resolvent} \xrightarrow[b]{} \text{Semigroup} \left(\xrightarrow[c]{} \text{Markov process} \right)$$

Step (b) reduces to an application of the Hille-Yosida theorem in "good" cases, but in general situations may require some compactification of the space, Ray [**9**, 1959]. No matter: from the analytical point of view, the semigroup is a luxury. *Everything can be done using the resolvent only*. Besides that, the identity

$$\frac{1}{\lambda} I + U = \frac{1}{\lambda} \sum_0^\infty (\lambda U_\lambda)^n \tag{8}$$

is strikingly similar to the Cartan-Deny representation (2): one just adds to U a "small" multiple of the identity kernel I, and the situation reduces to the discrete one.

This is why, in the book, resolvents are defined in an axiomatic way, as the main object of potential theory. The authors even consider something more general than kernels (i.e. mappings of the cone of positive functions into itself), namely, mappings of some "σ-lattice cone" into itself, with the advantage that three forms of potential theory are taken under the same roof: the theory of superharmonic *functions*, that of superharmonic *measures* (in the classicial case, they turn out to be the same), and finally that of superharmonic *classes of functions* neglecting suitable sets of measure 0 (the "semiclassical potential theory" of Kac, see for instance Stroock [**10**, 1967]. The disadvantage is that of abstraction.

Now, does the book contain results which are general and deep? I would like to quote at least one of them, and this requires some definitions, though the reader of this review may note that the quite precise meaning of U, U_λ has not been given. A *potential* is a function[2] g which can be represented as

[2]We are returning to the case of functions, rather than σ-lattice cones.

Uf, $f \geqslant 0$; f isn't quite unique, but any two versions of it are equal a.e., that is, except on a *set of potential zero*. Potentials are particular cases of *excessive functions*, i.e. any potential g satisfies the relations

$$(9) \qquad g \geqslant 0, \quad \lambda U_\lambda g \leqslant g \quad \text{for all} \quad \lambda, \quad \lim_{\lambda\to\infty} \lambda U_\lambda g = g$$

if the third property is deleted, we have the slightly more general definition of a *supermedian* function. Another important definition is that of the *specific order* between functions: we say that $f \ll g$ if $g = f + h$, where h is a supermedian function.

One of the basic notions of potential theory is that of the *réduite*, due to Mokobodzki. Given any function f, possibly taking negative values, we denote by Rf the lower bound of all supermedian functions larger than f. It is not obvious that it exists as a measurable function, but it does. Even less obvious is the striking result that $f \ll g \Rightarrow Rf \ll Rg$. Mokobodzki has used the properties of the réduite as a tool to prove the following generalization of the Lebesgue derivation theorem on the line.

Consider on the line the semigroup (P_t) defined by $P_t f(x) = f(x + t)$. This semigroup has a resolvent (U_λ)–which we do not write–and the corresponding excessive functions are just the nonincreasing, right continuous functions. The potential operator is $Ug(x) = \int_x^\infty g(t)\,dt$. Thus an excessive function is a potential if and only if it is absolutely continuous, and tends to 0 at $+\infty$. Finally, set $D_h f(x) = (f(x) - f(x + h))/h$ for $h > 0$, and $D^*f(x) = \sup_h D_h f(x)$.

The Lebesgue theory of derivation tells us that:(1) for any excessive (= decreasing) f, $D_h f$ converges a.e. in the sense of Lebesgue measure.

(2) Denoting by Df this limit, the potential UDf is the absolutely continuous part of f.

(3) The potential $\int_x^\infty I_{\{D^*f \geqslant c\}}(t)\,dt$ is at most $f(x)/c$. This is the Hardy-Littlewood maximal lemma.

This result extends to resolvents as follows. For simplicity, assume there is a semigroup (P_t) associated to the resolvent, define $D_h f = (f - P_h f)/h$ (f excessive, $h > 0$) and $D^*f = \sup_h D_h f$. Then

(1) $D_h f$ converges a.e. (that is, except on a set of potential 0).

(2) Denoting by Df this limit, UDf is the largest potential dominated by f in the specific order sense.

(3) The potential $U(I_{\{D^*f \geqslant c\}})$ is at most $f(x)/c$.

You may find an abstract version of this remarkable theorem in Chapter III,§2. So we can say this "Lecture Notes" volume contains practically everything that is known to the present day about the potential theory of a resolvent. There is, however, a noteworthy exception, that of the "Martin representation" of excessive functions as integrals of extremal excessive functions, also due to Mokobodzki. In the case of a resolvent (U_λ) consisting of kernels, however, this theorem requires the absolute continuity of all measures $U(x, dy)$ with respect to a fixed measure, and requires techniques of a rather special kind. So the exception is quite understandable.[3]

As a conclusion, I would like to say this is a very valuable book, and certainly will long be used as a reference in the field of potential theory.

[3]For an abstract version, see Mokobodzki [**8**].

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Mathematics of organization, by Mircea Malita and Corneliu Zidăroiu, Abacus Press, Tunbridge Wells, Kent, England, 1974, 383 pp., $30.00.

The Nobel Prize in Economics for 1975 was awarded to Leonid V. Kantorovich and Tjalling C. Koopmans for their contributions to the theory of optimum allocations of resources. This event emphasized the fact that the mathematics of operations research has been developed in parallel with economic theory. Books on operations research, such as the one under review, emphasize optimization problems, especially linear programming, game theory and control theory. These topics have been developed in the past thirty years and a sketch of this development may help to put in perspective the mathematics, presented in this book in a rather terse style.

In 1928 John von Neumann [**16**] gave a mathematical formulation of games of strategy and proved the celebrated minimax theorem justifying his definition of the value of a noncooperative game. This work was not pursued further until the economist Oskar Morgenstern, having been forced to leave Vienna, came to Princeton University and, during the classical tea in Fine Hall, talked with von Neumann about games and economics. This conversation led to the collaboration between Morgenstern and von Neumann which resulted in the publication in 1944 [**17**] of their famous book *The theory of games and economic behavior*. A fascinating account of this collaboration may be found in [**11**].

In 1939 the Russian mathematician Leonid Kantorovich published a paper