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# CELL-LIKE MAPPINGS AND THEIR GENERALIZATIONS[1]

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Cell-like maps are those whose point-inverses are cell-like spaces (as subspaces of the domain). A space is cell-like if it is homeomorphic to a cellular subset of some manifold. This definition was given in 1968, at which time I began to study proper, cell-like maps between euclidean neighborhood retracts (ENR's). At the time, I pointed out that such maps form a category which includes proper, surjective, contractible maps between polyhedra and proper, cellular maps from a manifold to an ENR. S. Smale had previously studied contractible maps between ANR's, proving a Vietoris-like theorem; M. Cohen had shown that PL contractible maps between finite polyhedra are simple homotopy equivalences; and, of course, proper, cellular maps on manifolds had been studied extensively ("point-like decompositions"). This unification seemed worthwhile at the time because of the equivalence of cell-like (for proper, surjective maps between ENR's) and a condition studied by D. Sullivan in connection with the Hauptvermutung: *the restriction to the inverse of any open set is a proper homotopy equivalence*. Using Sullivan's work it followed that a cell-like map between PL manifolds is often homotopic to a PL homeomorphism.

In 1971, L. C. Siebenmann identified the suspected red herring nature of "PL" in the above sentence: The set of cell-like maps $M \to N$ (where $M$ and $N$ are closed $n$-manifolds, $n \neq 4$) is precisely the closure of the set of homeomorphisms $M \to N$ in the space of maps $M \to N$. (The case $n = 3$ requires also that $M$ contain no fake cubes and was done earlier by S. Armentrout.)

The "cell-like" concept has since been studied, generalized, and analogized. I will attempt to recount some of this recent work.

Three main topics are identifiable:

A. Finiteness theorems and their global consequences;

B. Mapping cylinder neighborhoods; and

C. Desingularizations of spaces.

Recent major discoveries can be considered at least peripheral to the theme: The topological invariance of simple homotopy type (by T. Chapman), the finiteness of compact ANR's (by J. West), and the locally euclidean nature of the double suspension of certain homology spheres (by R.

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Edwards). Discussion of these results is included where appropriate.

**0. Notes to the reader.** The material presented here was selected. After the Introductory §1, there are four roughly defined collections of sections which could have been called "chapters": §§2, 3, 4 (Basics); §§5, 6, 7, 8 (Mappings between manifolds), §§9, 10 (Mapping cylinder neighborhoods); and §§11, 12, 13 (Desingularizing spaces). The section headings are as follows:



In addition, there are three appendices:



NOTATIONS AND CONVENTIONS. All spaces are assumed to be separable and metrizable; a map, or mapping, is a continuous function. A double arrow, $f: M \twoheadrightarrow N$, indicates a *surjective* mapping. A *compact* mapping is one $f: M \to N$ such that $f^{-1}(K)$ is compact for each compact set $K \subset N$. A *proper* mapping is a closed map with compact point-inverses. In our setting, a mapping is compact if and only if it is proper. (See Appendix II.)

An ANR is a space which is an absolute neighborhood retract (not necessarily compact) *for metric* spaces. An ENR is a "euclidean neighborhood retract", i.e., a retract of a neighborhood in euclidean space. Note that ENR is equivalent to "locally compact, finite-dimensional ANR".

We denote euclidean $n$-space by $\mathbf{R}^n$, its closed unit ball by $B^n$, and its unit sphere by $S^{n-1}$. Also, $\mathbf{R} = \mathbf{R}^1$ and $I = [0, 1]$. An *n-manifold* is a space which is locally homeomorphic to $\mathbf{R}^n$. An *n-manifold with boundary* (or a "$\partial$-manifold") is a space which is locally homeomorphic to $B^n$. The *boundary* $\partial N^n$ of a $\partial$-$n$-manifold is the set of points of $N^n$ which do not have neighborhoods homeomorphic to $\mathbf{R}^n$. Thus $\partial B^n = S^{n-1}$, and $\partial N^n = \varnothing$ is a possibility. We write $\mathring{N}^n = N^n - \partial N^n$. The superscript on a symbol for a manifold denotes its dimension.

The Hilbert cube, denoted by $Q$, is the countably infinite product of closed intervals $[-1, 1] = B^1 \subset \mathbf{R}^1$. A *Q-manifold* is a space which is locally homeomorphic to open subsets of $Q$.

$\mathbf{Z}$ denotes the ring of integers. "$\approx$" denotes homeomorphism.

A reference for notation and basic facts is Spanier [**1**].

REFERENCES. An asterisk in the bibliographical numbering indicates a paper containing proofs which are necessary for the logical development of the text. Some of the items listed in the Bibliography are not actually cited in the text; these are listed because they are nevertheless pertinent to the general subject matter.

**1. Introduction.**

A. *What is a finiteness theorem*? Suppose that $f$ is a mapping from some closed manifold $M$ onto another, say $N$. We shall be interested in placing local assumptions on the map $f$ and obtaining global conclusions. By a local property on $f$ we mean a property of point-inverses of $f$ (these can usually be interpreted as strictly local properties of the mapping cylinder; see §3.1 below). The type of global conclusions we have in mind may be geometric (e.g., concluding $M$ is homeomorphic to $N$ or the connected sum of $N$ and some other manifold), homotopical or homological (e.g., concluding $f$ has degree $\pm 1$). More particularly, we will be interested in "finiteness theorems" as *vehicles* for such conclusions. By a finiteness theorem we mean a result in which local assumptions are made on $f$ and, in turn, the set of points $y \in N$ for which $f^{-1}(y)$ fails to have some other property is proved to be a finite set. We illustrate this idea with a "geometric" finiteness theorem.

DEFINITION. A compact set $X$ in the interior of the $n$-manifold $M^n$ is said to have a *deleted annular neighborhood* in $M^n$ if there is a compact manifold neighborhood $W$ of $X$ such that $W - S \approx S^{n-1} \times [0, \infty)$.

If $f$: $M^n \to Y$ is a map on a manifold $M^n$, we define

$$C_f = \{y \in Y | f^{-1}(y) \text{ is not cellular in } M^n\}.$$

(See Appendix III for the basic facts on cellularity.)

THEOREM. *If $f$: $M^n \twoheadrightarrow Y$ is a map from a compact manifold onto a metric space, and if $f^{-1}(y)$ has a deleted annular neighborhood in $M^n$ for each $y \in Y$, then $C_f$ is finite.*

REMARK. Suppose $f$: $M^n \twoheadrightarrow N^n$ is a map between compact manifolds such that $C_f$ is finite. We can analyze the relationship between $M$ and $N$ as follows. Let $C_f = \{y_1, \ldots, y_t\}$, let $N_i$ be a neighborhood of $f^{-1}(y_i)$ such that $N_i - f^{-1}(y_i) \approx S^{n-1} \times [0, \infty)$, and let $\hat{N}_i$ be the manifold obtained by capping $\partial N_i$ with an $n$-cell. Denote by $N_0$ the space obtained from $M$ by shrinking each of the sets $f^{-1}(y_i)$ to a point. Clearly we have:

$N_0$ is a manifold, $M \approx N_0 \# \hat{N}_1 \# \cdots \# \hat{N}_t$, and
the induced map $f_0$: $N_0 \twoheadrightarrow N$ is cellular.

Thus, in case $n \neq 4$, $f_0$ is uniformly approximated by homeomorphisms $N_0 \twoheadrightarrow N$ by a result of Armentrout [3] ($n = 3$) and Siebenmann [8] ($n \geq 5$). For $n \neq 4$, we obtain: $M$ is homeomorphic to the connected sum $N \# \hat{N}_1 \# \cdots \# \hat{N}_t$. (See §§5, 7 below.)

PROOF. Define a set

$$F = \{y \in Y | f^{-1}(y) \text{ does not lie in a topological copy of } S^{n-1} \times \mathbf{R}\}.$$

Plainly $F$ is finite. We claim that $F = C_f$. One easily sees the irrelevent containment, so we proceed to the proof that $C_f \subset F$. Suppose $y \notin F$. Then $f^{-1}(y)$ has two neighborhoods $U$, $V$ with the properties

$$V \subset U, \quad U \approx S^{n-1} \times \mathbf{R}, \quad \text{and} \quad V - f^{-1}(y) \approx S^{n-1} \times \mathbf{R}.$$

Let $h$: $U \to S^n$ be an embedding. Then $hf^{-1}(y)$ has a neighborhood $h(V)$ in $S^n$ such that

$$h(V) - hf^{-1}(y) \approx S^{n-1} \times \mathbf{R}.$$

The generalized Schoenflies Theorem shows that $hf^{-1}(y)$ is cellular in $S^n$. It follows that $f^{-1}(y)$ is cellular in $U$ and hence in $M$. Therefore $y \not\in C_f$. $\square$

B. *Mapping cylinder neighborhoods of compacta* $X$ are compact $\partial$-manifold neighborhoods $N$ which admit the following structure: There exists a map $\phi$: $\partial N \to X$ such that $N$ and $Z_\phi$ are homeomorphic keeping $X$ fixed.

Three basic questions arise:

Does a locally flat submanifold have a mapping cylinder neighborhood?

Especially interesting in the light of Rourke's and Sanderson's examples [**2**] of locally flat submanifolds which admit no open tubular neighborhood, R. Edwards [**4**] has, in fact, answered this question affirmatively in most cases (see §10 below).

Must a submanifold with a mapping cylinder neighborhood be locally flat?

This question bigrades by dimension: If $M^m$ is an $m$-manifold topologically embedded in the $n$-manifold $N^n$, must $M^m$ be locally flat in $N^n$? The answer is "yes" whenever $n \leqslant 3$ and when $(m, n) = (1, 4)$ or (3.4), and is "no" in all other cases. The proofs in cases (1, 4) and (3, 4) depend on certain finiteness theorems for 3-manifolds; the case (1, 4) is especially interesting, depending on special 3-dimensional results as well as special high-dimensional techniques.

In a double suspension $H^{n-2} * S^1$, the copy of $S^1$ has a mapping cylinder neighborhood; assuming that $H^{n-2} * S^1$ is topologically $S^n$, it follows that $S^1$ is locally flat if and only if $H^{n-2}$ is simply connected ($n \geqslant 4$). Using Edwards' result [**2**] that $H^3 * S^2 \approx S^5$, where $H^3 = \partial$ Mazur manifold, one sees that there are 1-spheres with mapping cylinder neighborhoods in $S^5$ which are not locally flat. In fact, Edwards' results imply the existence of nonlocally flat $m$-spheres with mapping cylinder neighborhoods in $S^n$ whenever $n - m \geqslant 1, n \geqslant 5$.

Thus the "mapping cylinder implies locally flat" question is answered in all cases. Some subtle taming questions arise in this context.

What other kinds of spaces admit mapping cylinder neighborhoods?

Given a polyhedron, or a manifold, $X$, it has been known for some time that $X$ has a mapping cylinder neighborhood in *some* euclidean space. (Regular neighborhoods of polyhedra are mapping cylinder neighborhoods, as is a normal disk bundle neighborhood.)

On the other hand, if the compactum $X$ has a mapping cylinder neighborhood in some manifold then $X$ is the cell-like image of that neighborhood and consequently is an ENR. R. Miller [**1**] has shown recently that, if $X$ is a finite-dimensional compact ANR, then $X \times S^1$ has a mapping cylinder neighborhood in some euclidean space. Miller's result (as well as its infinite-dimensional analogue) is a powerful tool used in West's proof of the finiteness of compact ENR's (in fact, ANR's). The Edwards-Siebenmann version of West's proof yields as a corollary that any compact ENR has a mapping cylinder neighborhood in some euclidean space.

C. *Desingularization of spaces*. Certain geometric singularities in spaces can be smoothed, or removed, with a minimal geometric operation. Two methods of removal are particularly relevant. The first we refer to as *resolution* of singularities:

For which finite-dimensional compacta $X$ can one find a compact manifold $M^m$ and a cell-like mapping of $M^m$ onto $X$?

In paragraph B above, we pointed out that $X$ has an MCN in some $\mathbf{R}^n \Leftrightarrow X$ is the cell-like image of a compact $\partial$-manifold $\Leftrightarrow X$ is an ENR. Thus one needs somehow to capture the "manifold-likeness" of spaces intrinsically.

Bryant and Hollingworth have shown that a space which is a manifold except at a zero-dimensional set admits a resolution. Cohen and Sullivan have a general theory of resolutions (in the PL category).

Another way of desingularizing spaces is by *stabilization*:

For which spaces $X$ is it true that $X \times \mathbf{R}^k$ is a manifold for some $k$?

Beginning with pioneering works of R. H. Bing, a multitude of spaces of the type $X = \mathbf{R}^n/G$, where $G$ is a certain upper semicontinuous cell-like decomposition, have been discovered to have this stabilization property. Coming full circle, Edwards has begun with ideas of Bing and woven an ingenious argument to show that $\mathbf{R}^4/D \times \mathbf{R}^1 \approx \mathbf{R}^5$, where $D$ is the "linked" embedding of the dunce hat, thereby verifying that the double suspension of the $\partial$ of the Mazur manifold (Mazur [**1**] and Zeeman [**2**]) is $S^5$.

There is little evidence to suggest the falsity of what might be called the "main conjecture" in the theory of desingularization: that $X$ admits a resolution if and only if $X \times \mathbf{R}$ is a manifold.

**2.** ***UV*** **properties.** Throughout this section we assume that $\mathcal{F}$ is a homotopy invariant covariant functor from the category of topological spaces to the category of sets.

(2.1). DEFINITION. Suppose that $A$ is a closed set in the space $X$. We say that *the inclusion* $A \subset X$ *has property* $UV(\mathcal{F})$ provided the following condition is satisfied:

> For any open set $U$ of $X$ containing $A$, there exists an open set $V$ of $U$ containing $A$ and a point $* \in A$ such that $\mathcal{F}(j) = \mathcal{F}(c)$, where $j\colon V \to U$ is inclusion and $c\colon V \to U$ is constantly $*$.

Note that the above seems to depend on the embedding $A \subset X$. In a surprising number of cases, however, property $UV(\mathcal{F})$ depends only on the space $A$:

PROPOSITION. *The following two statements are equivalent for any compactum* $X$:

(a) *There exist an ANR* $M$ *and an embedding* $\alpha$ *of* $X$ *into* $M$ *such that* $\alpha(X) \subset M$ *has property* $UV(\mathcal{F})$.

(b) *For any ANR* $M$ *and any embedding* $\alpha$ *of* $X$ *into* $M$, *the inclusion* $\alpha(X) \subset M$ *has property* $UV(\mathcal{F})$.

PROOF. Suppose that $X$ is contained in an ANR $M$ and that $X \subset M$ has property $UV(\mathfrak{F})$. Suppose further that $f\colon X \to M_1$ is an embedding of $X$ into an ANR $M_1$. We wish to show that $f(X) \subset M_1$ has property $UV(\mathfrak{F})$.

Let $U_1$ be a neighborhood of $f(X)$ in $M_1$. Using the neighborhood extension property of ANR's, we can extend $f$ to a map $F$ of a neighborhood $U$ (of $X$ in $M$) into $U_1$. Since $X \subset M$ has property $UV(\mathfrak{F})$, there is a neighborhood $V$ of $X$ in $U$ such that the inclusion $V \subset U$ is $\mathfrak{F}$-trivial. Extend $f^{-1}$ to a map $G$ of $V'$ into $V$, where $V'$ is some neighborhood of $f(X)$ in $U_1$. We have the

$$\begin{array}{ccc} U & \xrightarrow{F} & U_1 \\ \cup & & \cup \\ X & \xrightarrow[\approx]{f} & X_1 \\ \cap & & \cap \\ V & \xleftarrow[G]{} & V' \end{array}$$

Since the inclusion $V \subset U$ is $\mathfrak{F}$-trivial, $\mathfrak{F}(F \circ G) = \mathfrak{F}(*)$ for some $* \in f(X) = X_1$.

Let $H$ be the map of $Y = f(X) \times I \cup V' \times 0 \cup V' \times 1$ into $U_1$ which is the identity on $f(X) \times I \cup V' \times 0$ and is $F \circ G$ on $V' \times 1$. $H$ extends to a map of $W$ into $U_1$, where $W$ is some neighborhood of $Y$ in $V' \times I$. Let $V_1$ be a neighborhood of $f(X)$ in $V'$ such that $V_1 \times I \subset W$. It follows that the inclusion $j$ of $V_1$ into $U_1$ is homotopic to the composition $F \circ G|_{V_1}\colon V_1 \to U_1$. Therefore

$$\begin{aligned} \mathfrak{F}(j) &= \mathfrak{F}(F \circ G|_{V_1}) = \mathfrak{F}(F \circ G \circ i) = \mathfrak{F}(F \circ G) \circ \mathfrak{F}(i) \\ &= \mathfrak{F}(c') \circ \mathfrak{F}(i) = \mathfrak{F}(c' \circ i) = \mathfrak{F}(c) \end{aligned}$$

(where $i\colon V_1 \to V'$ is inclusion and $c'\colon V' \to U_1$ and $c\colon V_1 \to U_1$ are constant maps) (cf. Lacher [5]). □

Now we can define the

PROPERTY $UV(\mathfrak{F})$ [INTRINSIC]. The compactum $X$ has property $UV(\mathfrak{F})$ if and only if (a) or (b) of the above Proposition is satisfied.

Table 1 gives the most often used $UV$ properties.

PROPOSITION. *Suppose $X$ is a compact ANR. Then $X$ has property $UV(\mathfrak{F})$ if and only if $\mathfrak{F}(\mathrm{id}_X) = \mathfrak{F}(c)$ for some constant map $c\colon X \to X$.*

PROOF. Suppose $X$ has property $UV(\mathfrak{F})$. Let $M$ be an ANR containing $X$, and let $U$ be a neighborhood of $X$ which retracts onto $X$ via $r_1\colon U \to X$.

TABLE 1. SOME $UV$ PROPERTIES

| $UV$ Property | requirement on the inclusion $V \subset U$ | functor $\mathcal{F}$ |
|---|---|---|
| $k - uv(R)$ | any singular $k$-cycle (over $R$) in $V$ is null-homologous in $U$ | $H_k(-;R)$ |
| $k - uv$ | $k - uv(\mathbf{Z})$ | |
| $uv^k(R)$ | any singular $j$-cycle (over $R$) in $V$ is null-homologous in $U$, $0 \leqslant j \leqslant k$ | $\bigoplus_{i=0}^{k} H_j(-;R)$. |
| $uv^k$ | $uv^k(\mathbf{Z})$ | |
| $uv^\infty(R)$ | any singular $j$-cycle (over $R$) in $V$ is null-homologous in $U$, $0 \leqslant j$ | $\prod_{j=0}^{\infty} H_j(-;R)$ |
| $uv^\infty$ | $uv^\infty(\mathbf{Z})$ | |
| $k - UV$ | any singular $k$-sphere in $V$ is null-homotopic in $U$ | $[S^k, -]$ |
| $UV^k$ | any singular $j$-sphere in $V$ is null-homotopic in $U$, $0 \leqslant j \leqslant k$ | $[K, -]$, $K = S^0 \vee \cdots \vee S^k$ |
| $UV^\infty$ | $V$ is null-homotopic in $U$ | $[K, -]$, $K$ = disjoint union of all ANR's. |

Let $V$ be a neighborhood of $X$ in $U$ such that $\mathcal{F}(j) = \mathcal{F}(c)$. Let $i_1$ be the inclusion of $X$ in $U$, $i$ = the inclusion of $X$ in $V$, and $r = r_1 | V$. We assume $V$ is small enough so that $i_1 \circ r \simeq j$. Then

$$\mathcal{F}(\mathrm{id}_X) = \mathcal{F}(r_1 \circ i_1 \circ r \circ i) = \mathcal{F}(r_1) \circ \mathcal{F}(i_1 \circ r) \circ \mathcal{F}(i) = \mathcal{F}(r_1) \circ \mathcal{F}(j) \circ \mathcal{F}(i)$$
$$= \mathcal{F}(r_1) \circ \mathcal{F}(c) \circ \mathcal{F}(i) = \mathcal{F}(r_1 \circ c \circ i) = \mathcal{F}(\bar{c})$$

where $\bar{c}$: $X \to X$ is constant.

Conversely, suppose $\mathcal{F}(\mathrm{id}_X) = \mathcal{F}(c)$. Let $U$ be a neighborhood of $X$ in $M$, and find a neighborhood $V$ of $X$ in $U$ and a retraction $r$: $V \to X$ such that $r$: $V \to U$ is homotopic to the inclusion $j$: $V \to U$. Then $F(j) = F(i \circ r) = F(i \circ \mathrm{id}_X \circ r) = F(i \circ c \circ r) = F(\bar{c})$. □

PROPOSITION. *For a compactum $X$, the following three conditions are equivalent*:

(a) *$X$ is connected*;

(b) *$X$ has $uv^0(R)$ for some $R \neq 0$; and*

(c) *$X$ has $UV^0$.*

Why study $UV$ properties? We need a property $P(\mathfrak{F})$ of compacta such that, for compact ANR's $M$, $M$ has $P(\mathfrak{F})$ if and only if $\mathfrak{F}(\mathrm{id}_M) = \mathfrak{F}(\text{constant})$, as in the conclusion of the second Proposition above. But there are two much more traditional ways of defining such a property $P$, each of which has the advantage of placing a natural algebraic structure on $\mathfrak{F}(X)$:

(1) simply say $X$ hat $P(\mathfrak{F})$ iff $\mathfrak{F}(\mathrm{id}_X) = \mathfrak{F}(\text{constant})$, or

(2) define $\check{\mathfrak{F}}(X) = \text{proj lim } \mathfrak{F}(U)$ where $U$ runs over all neighborhoods of $X$ in some ANR, and say $X$ has $P(\mathfrak{F})$ iff $\check{\mathfrak{F}}(\mathrm{id}_X) = \check{\mathfrak{F}}(\text{constant})$. Consider two examples:

EXAMPLE (1). Let $X$ be the graph of $y = \sin(1/x)$, $0 < x \leqslant 1$, together with the line $0 \times [-1, 1]$. Then $X$ has property $UV^\infty$ (and hence $UV(\mathfrak{F})$ for any $\mathfrak{F}$). Also, $S^2 - X \approx \mathbf{R}^2$, so $X$ should be "like a point". But $H_0(\mathrm{id}_X) \neq H_0(\text{constant})$. Thus $H_0$ would have us believe $X$ is complicated.

EXAMPLE (2). Let $X$ be the limit of the inverse sequence $S^1 \leftarrow S^1 \leftarrow \dots$ where each bonding map is $z \to z^2$. Then

$$\check{H}_*(X; \mathbf{Z}) = \check{H}_*(\text{point}) \quad \text{and} \quad \check{\pi}_*(X) = \check{\pi}_*(\text{point}),$$

having us believe $X$ is "like a point". But $S^3 - X$ is certainly not $\mathbf{R}^3$!

Thus neither method (1) nor (2) is an appropriate definition of $P(\mathfrak{F})$ from the point of view of decomposition spaces. That $UV(\mathfrak{F})$ *is* appropriate will become increasingly clear as this article progresses.[3]

(2.2) *The relationship between homological uv properties and Čech cohomology* is an exact generalization of the Universal Coefficient Theorem. (Lacher [**8**]). Let $R$ be a principal ideal domain.

PROPOSITION. *Let $X$ be a compactum, $k \geqslant 1$.*

(1) *If $X$ has properties $(k-1) - uv(R)$ and $k - uv(R)$, then*

$$\check{H}^k(X; R) = 0.$$

(2) *If $\check{H}^k(X; R) = \check{H}^{k+1}(X; R) = 0$, then $X$ has property $k - uv(R)$.*

COROLLARY. *A finite-dimensional compactum $X$ has property $uv^\infty(R)$ if and only if $\check{H}^*(X; R) = 0$.*

PROOF. Let $X$ be embedded in Hilbert space, and suppose that $U_1, U_2, \dots$ are neighborhoods of $X$ such that

$\bar{U}_{i+1} \subset U_i$ for each $i$,

Each $U_i$ is a finite union of open balls,

$X = \bigcap_{i=1}^\infty U_i$.

For an $R$-module $M$, let $TM = \{m \in M \mid rm = 0 \text{ for some } 0 \neq r \in R\}$, Hom $M = \mathrm{Hom}_R(M; R)$, and Ext $M = \mathrm{Ext}_R(M; R)$. Applying the Universal Coefficient Theorem relating homology and cohomology, we obtain commutative diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}\, H_{l-1}U_{i+1} & \longrightarrow & H^lU_{i+1} & \longrightarrow & \mathrm{Hom}\, H_lU_{i+1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Ext}\, H_{l-1}U_i & \longrightarrow & H^lU_i & \longrightarrow & \mathrm{Hom}\, H_lU_i & \longrightarrow & 0
\end{array}$$

[3] A compactum $X$ has property $UV(\mathfrak{F})$ if and only if pro-$\mathfrak{F}(X) = *$. Thus property $UV^n$ is the same (for compacta) as "approximatively $n$-connected". See Borsuk [**5**, p. 143].

in which the vertical arrows are induced by inclusion and the rows are exact. If $\phi_i: U_{i+1} \to U_i$ denotes inclusion, the vertical maps are Ext $H_{l-1}(\phi_i)$, $H^l(\phi_i)$, and Hom $H_l(\phi_i)$, respectively.

Suppose that $X$ has properties $(k-1) - uv$ and $k - uv$. Then we can choose the $U_i$ so that $H_{k-1}(\phi_i) = H_k(\phi_i) = 0$ for each $i$. It follows from the diagram that

$$\operatorname{im} H^k(\phi_i) \subset TH^kU_{i+1} \quad \text{and} \quad H^k(\phi_i)|TH^kU_i = 0$$

for each $i$. Therefore $H^k(\phi_{i+1}\phi_i) = 0$, and $\check{H}^kX = 0$.

Now suppose that $\check{H}^kX = \check{H}^{k+1}X = 0$. Then we can choose the $U_i$ so that $H^k(\phi_i) = H^{k+1}(\phi_i) = 0$ for each $i$. It follows from the diagram that

$$\text{Ext } H_k(\phi_i) = \text{Hom } H_k(\phi_i) = 0$$

for all $i$. Therefore $H_k(\phi_{i+1}\phi_i) = 0$, and $X$ has property $k - uv(R)$.

The last conclusion uses the fact that the functors $T$ and Ext $\circ$ Ext are naturally equivalent (see Appendix I). □

(2.3) *The relationship between homological and homotopical UV properties* is also as an exact generalization, this time of the Huréwicz Theorem (Lacher [8], compare Mardesič and Ungar).

PROPOSITION. *Let $X$ be a compactum.*

(1) *If $X$ has $UV^k$ then it has $uv^k$.*

(2) *If $X$ has $UV^{k-1}$ and $k - uv$ then it has $UV^k$, provided $k \geq 2$.*

COROLLARY. *The $1 - UV$ compactum $X$ has property $UV^k$ if and only if it has property $uv^k$.*

PROOF OF (1). For a given neighborhood $U$ of $X$ find open sets $U_0, \ldots, U_{k+1}$ with the properties:

$$\begin{aligned} &X \subset U_0 \subset \cdots \subset U_{k+1} \subset U, \text{ and} \\ &\text{any map } S^q \to U_q \text{ extends to a map} \\ &B^{q+1} \to U_{q+1}. \end{aligned}$$

Let $V = U_0$. If $K$ is a $k$-complex, then any map $K \to V$ can be extended to a map $vK \to U$ (where $vK$ = cone on $K$) using the inclusion $U_q \subset U_{q+1}$ to extend over the $(q+1)$-skeleton of $vK$. It follows immediately that any singular $q$-cycle in $V$ bounds in $U$ $(0 \leq q \leq k)$. Hence $X$ has $uv^k$.

PROOF OF (2). Given a neighborhood $U$ of $X$, let $V \subset U_0 \subset \cdots \subset U_k \subset U$ be path-connected neighborhoods of $X$, chosen so that every singular $k$-cycle in $V$ bounds in $U_0$ and every singular $q$-sphere in $U_q$ is null-homotopic in $U_{q+1}$, $0 \leq q \leq k-1$.

Let $\alpha: S^k \to V$ be a map. Then as a singular $k$-cycle $\alpha \sim 0$ in $H_kU_0$. Therefore there exists a subdivision $L$ of $S^k$ and an ordering of the simplexes of $L$ such that $\Sigma_i\alpha\tau_i = \partial c$ for some singular $(k+1)$-chain $c = \Sigma_jn_j\sigma_j$ in $U_0$. (Here $\tau_i$ is the simplicial mapping $\Delta^k \to S^k$ determined by $L$.) Let $K$ be a geometric realization of the abstract complex $\{\sigma_j\}$. Then $K$ is a complex containing $L$ as a subcomplex and, moreover, we can extend $\alpha$ to a mapping $\beta: |K| \to U_0$.

Let $K'$ be the union of $K$ and the cone on its $(k-1)$-skeleton. Using the inclusion $U_q \subset U_{q+1}$ we can extend $\beta$ to a map $\bar{\alpha}: K' \to U$. Note that $|K'|$ is $(k-1)$-connected and that $\Sigma_i\tau_i$ is null-homologous in $K$, hence in $K'$.

Therefore, by the classical Hurewicz theorem, $S^k$ is null-homotopic in $|K'|$. Therefore $\bar{\alpha}|S^k = \alpha$ is null-homotopic in $U$. □

**3. Local connectivity for maps and Vietoris-like theorems.** Let $\mathfrak{F}$ be a functor as in §2.

(3.1) *Local connectivity*. Suppose that $Y$ is a closed subset of a space $Z$. We say that $Z$ *is* $LC(\mathfrak{F})$ *mod* $Y$ at the point $z \in Z$ if for any neighborhood $U$ of $z$ in $Z$, there is a neighborhood $V$ of $z$ in $U$ such that $\mathfrak{F}(j) = \mathfrak{F}(c)$, where $j, c$: $(V - Y) \to (U - Y)$ are inclusion and constant, respectively. (See Table 2.)

PROPOSITION. *Suppose that* $f\colon X \twoheadrightarrow Y$ *is a proper mapping. Then the following are equivalent*:

(a) *The inclusion* $f^{-1}(y) \subset X$ *has property* $UV(\mathfrak{F})$ *for each* $y \in Y$.

(b) *The mapping cylinder* $Z_f$ *is* $LC(\mathfrak{F})$ mod $Y$ *at each point of* $Y$.

The proof is an exercise in the manipulation of definitions. (The notation for $Z_f$ is established in §9.1 below.)

*Notation*. Often we will use the phrase "$UV(\mathfrak{F})$-mapping" to mean a map satisfying (a) above. It is tempting to refer to maps satisfying (b) above as "$LC(\mathfrak{F})$-mappings". In view of the Proposition itself, and since the definition conflicts with Kozlowski's, we refrain.

TABLE 2. SOME $LC$ PROPERTIES

| $LC$ Property | Requirement on the inclusion $V - Y \subset U - Y$ |
|---|---|
| $k - lc(R)$ mod $Y$ | singular $R$-cycles over $R$ in $V - Y$ bound in $U - Y$ |
| $k - lc$ mod $Y$ | $k - lc(\mathbf{Z})$ mod $Y$ |
| $lc^k(R)$ mod $Y$ | $i - lc(R)$ mod $Y$ for $0 \leqslant i \leqslant k$ |
| $lc^k$ mod $Y$ | $lc^k(\mathbf{Z})$ mod $Y$ |
| $lc^\infty(R)$ mod $Y$ | singular $i$-cycles over $R$ in $V - Y$ bound in $U - Y$, $i = 0, 1, \ldots$ |
| $lc^\infty$ mod $Y$ | $lc^\infty(\mathbf{Z})$ mod $Y$ |
| $k - LC$ mod $Y$ | singular $k$-spheres in $V - Y$ are contractible in $U - Y$ |
| $LC^k$ mod $Y$ | $i - LC$ mod $Y$ for $0 \leqslant i \leqslant k$ |
| $LC^\infty$ mod $Y$ | $U - Y$ is contractible in $V - Y$ |

(3.2) *Limits*. The set of proper $UV(\mathfrak{F})$-mappings between two spaces is often a closed set (using the compact-open toplogy) in the mapping space. A similar proof shows that the set of cellular mappings $M^m \to Y$ is a closed set (see §§4 and 12). The argument below is a metric version of one in Kuratowski-Lacher [**1**]. (See also Finney [**1**].)

THEOREM. *Suppose that $f, f_n$: $X \to\to Y$ are maps between compact spaces and that $f$ is the uniform limit of the sequence $\{f_n\}_{n=1}^{\infty}$. If each $f_n$ is $UV(\mathfrak{F})$ then so is $f$.*

The proof is immediate from the following two lemmas (which assume only that $f$ is the limit of the $\{f_n\}$).

LEMMA A. *Let $V$ and $U$ be open sets in $Y$ such that $\bar{V} \subset U$. Then $\exists n_0$ such that $f_n^{-1}(V) \subset f^{-1}(U)$ for $n \geq n_0$.*

PROOF. Suppose not. Then $\exists x_n \in f_n^{-1}(V) - f^{-1}(U)$, and we may as well assume $\{x_n\}$ converges to $x_0$. Then $f(x_n)$ converges to $f(x_0)$, and since $f(x_n) \notin U$, we conclude $f(x_0) \notin U$. But $f_n(x_n)$ also converges to $f(x_0)$, and since $f_n(x_n) \in V$, we conclude $f(x_0) \in \bar{V} \subset U$, a contradiction. $\square$

LEMMA B. *Let $W$ be an open set in $Y$, $y \in W$. Then $\exists n_1$ such that $f^{-1}(y) \subset f_n^{-1}(W)$ for $n \geq n_1$.*

PROOF. Suppose not. Then $\exists x_n \in f^{-1}(y) - f_n^{-1}(W)$, and we assume $\{x_n\}$ converges to $x_0$. Since $f(x_n) = y$, we conclude $f(x_0) = y$. But $f_n(x_n) \notin W$ implies that $f(x_0) \notin W$, contradiction. $\square$

(3.3) *$UV^k$-mappings and a Vietoris-Smale theorem.* S. Smale [**1**] seems to have introduced the idea that a "Vietoris" theorem could be proved using homotopy, rather than homology. He used "$k$-connected" as a hypothesis on point-inverses (and consequently needed to assume point-inverses were $LC^k$), producing an exact analogy with the Vietoris-Begle theorem (Begle [**1**], [**2**]).

What follows is a similar result, obtained as an offshoot of the proof of Theorem 4.2 below (Lacher [**6**]). Several others have proved Vietoris-Smale theorems, including Kozlowski [**1**] and Armentrout-Price [**1**]. Dydak [**1**] has a Vietoris-Smale theorem in pro-homotopy. Special cases of many of these could be deduced from far more classical results using (3.1). (See Eilenberg-Wilder [**1**].)

THEOREM. *Suppose that $f$: $X \to\to Y$ is a proper $UV^{k-1}$-mapping between locally compact spaces.*[4] *Suppose further that $\exists$ subgroup $G$ of $\pi_k(X, *)$ such that, for each $y \in Y$ $\exists$ neighborhood $V$ of $f^{-1}(y)$ in $X$ such that every map $S^k \to V$ is homotopic (ignoring base points) to a representative of an element of $G$. Then the following conclusions hold*:

(1) *$f_\#$: $\pi_q(X, *) \to \pi_q(Y, f(*))$ is an isomorphism for $0 \leq q < k$;*

(2) *The kernel of $f_\#$: $\pi_k(X, *) \to \pi_k(Y, f(*))$ is contained in $G$; and*

(3) *If $Y$ is $LC^k$ then $f_\#$: $\pi_k(X, *) \to \pi_k(Y, f(*))$ is surjective.*

COROLLARY 1. *Suppose that $f$: $X \to\to Y$ is a proper $UV^{k-1}$-mapping and that $Y$ is a locally compact $ANR$.*

(1) *$f_\#$: $\pi_q(X, *) \to \pi_q(Y, f(*))$ is an isomorphism for $0 \leq q < k$ and an epimorphism for $q = k$.*

[4]Replacing epsilonics by open cover arguments, Armentrout and Price avoided the assumption of local compactness.

(2) *If for each* $y \in Y$ $\exists$ *neighborhood* $V$ *of* $f^{-1}(y)$ *in* $X$ *such that any map* $S^k \to V$ *is null-homotopic in* $X$, *then* $f_{\#}\colon \pi_k(X, *) \to \pi_k(Y, f(*))$ *is an isomorphism.*

COROLLARY 2. *If* $f\colon X \twoheadrightarrow Y$ *is a proper* $UV^k$*-mapping and* $Y$ *is locally compact and finite-dimensional with* $\dim Y < k$ *then* $Y$ *is an ENR.*

In particular, the *finite-dimensional* cell-like image of a compact ANR is an ENR. (See §§4 and 11.)

The subtlety involving the subgroup $G$ stems from an idea of McMillan [**6**]. See also §3.4 below.

The proof of the Theorem is based on two "lifting" lemmas. These are found in Lacher [**6**]. The first has been modified to include the possibility that $K$ be infinite-dimensional. The argument is much the same, except that we must give an inductive definition instead of an inductive proof.

LEMMA A. *Let the following be given*:

(0) *A positive integer* $k$, *or* $k = k - 1 = \infty$.

(i) *Locally compact spaces* $X$ *and* $Y$.

(ii) *A proper* $UV^{k-1}$*-mapping* $f\colon X \twoheadrightarrow Y$.

(iii) *A locally finite* $k$*-complex* $K$ *with a locally finite subcomplex* $L$.

(iv) *A proper map* $\phi\colon K \to Y$.

(v) *A proper map* $\psi\colon L \to X$ *such that* $f\psi = \phi|L$.

(vi) *A continuous function* $\varepsilon\colon Y \to (0, \infty)$.

(vii) *A metric* $d$ *on* $X$ *and* $Y$ *under which closed, bounded sets are compact. Then, there exists a proper map* $\bar{\phi}\colon K \to X$ *such that* $\bar{\phi}|L = \psi$ *and*

$$d\big(f\bar{\phi}, \phi\big) \leqslant \varepsilon\phi.$$

PROOF. For each $y \in Y$, let $U_y$ be a neighborhoof of $y$ with

(1) $\operatorname{diam} U_y \leqslant \min \varepsilon | U_y$, and

(2) $\operatorname{diam} f^{-1}(U_y) \leqslant \operatorname{diam} f^{-1}(y) + 1$.

Find neighborhoods

$$\cdots \subset V_{j,y} \subset \cdots \subset V_{-1,y} \subset V_{0,y} \subset U_y$$

of $y$ ($j = 0, -1, -2, \ldots$) satisfying

(3) any singular $p$-sphere in $f^{-1}(V_{j,y})$ is null-homotopic in $f^{-1}(V_{j+1,y})$, ($0 \leqslant p \leqslant k$ and $j \leqslant -1$), and

(4) the open cover $\mathcal{V}_j = \{V_{j,y}\}$ "star refines" $\mathcal{V}_{j+1}$, ($j \leqslant -1$).

I.e., if $v \in \mathcal{V}_j$ then $\exists$ $u \in \mathcal{V}_{j+1}$ such that ($w \in \mathcal{V}_j$ and $w \cap v \neq \varnothing$) implies $w \subset u$.

For each $\sigma \in K$, let

$$d(\sigma) = -2 \dim N(\sigma)$$

where $N(\sigma)$ is the simplicial neighborhood of $\sigma$ in $K$. Thus $d = d(\sigma)$ is a nonpositive integer, the "going down" number of $\sigma$. Since $K$ is locally finite, we may assume that $K$ is subdivided finely enough so that $\phi(\sigma)$ lies in some element of $\mathcal{V}_{d(\sigma)}$ for each $\sigma \in K$.

We extend $\psi$ over the skeleta $K^i$ of $K$. The first step is to extend over $K^0 \cup L$ by choosing a point $\psi(\sigma) \in f^{-1}(V)$, where $V$ is some element of $\mathcal{V}_{d(\sigma)}$, for each $\sigma \in K^0 - L$.

Inductively, suppose $\psi$ has been extended over $K^p \cup L$ so as to take $p$-simplexes into pull-backs of elements of $\mathcal{V}_{d+2p}$. Then extend $\psi$ over the simplices $\sigma^{p+1} \in K^{p+1} - L$ so as to take $\sigma^{p+1}$ into the pull-back of an element of $\mathcal{V}_{d+2p+2}$. (4) above implies that $\psi(\partial\sigma^{p+1})$ lies in some element of $\mathcal{V}_{d+2p+1}$ and (3) shows how to extend $\psi$ over $\sigma^{p+1}$ as required.

Continuing the process indefinitely defines an extension $\phi: K \to X$ of $\psi$. (2) above implies that $\phi$ is proper and (1) shows that $d(f\phi, \phi) < \varepsilon$. □

LEMMA B. *Let the following be given*:

(i) *Locally compact spaces $X$ and $Y$.*

(ii) *A pair $(K, L)$ of finite simplicial complexes*, $\dim K \leqslant k$.

(iii) *A proper $UV^k$-mapping $f$ of $X$ onto $Y$.*

(iv) *A map $\phi: K \to Y$.*

(v) *A map $\psi: L \to X$ such that $f\psi = \phi|L$.*

*Then there exist maps $\Phi: K \times I \to Y$ and $\Psi: K \times (0, 1] \to X$ such that*

(1) $f\Psi_t = \Phi_t$ *for* $0 < t \leqslant 1$,

(2) $\Psi_t|L = \psi$ *for* $0 < t \leqslant 1$, *and*

(3) $\Phi_0 = \phi$.

PROOF. For each $y \in Y$ let $\{U_y^{(n)}\}$ be a sequence of neighborhoods of $y$ such that

$$\operatorname{diam} U_y^{(n)} \leqslant 1/n$$

and

$$\overline{U}_y^{(n+1)} \subset U_y^{(n)}.$$

Moreover, construct the $U_y^{(n)}$ so that any singular $q$-sphere in $f^{-1}(U_y^{(n+1)})$ bounds a singular $(q+1)$-disk in $f^{-1}(U_y^{(n)})$. Finally, construct $U_y^{(1)}$ to have compact closure.

Applying Lemma A carefully, we can find a sequence $\{\psi_n\}$ of maps of $K$ into $X$ such that

$$\psi_n|L = \psi,$$

and

$$f\psi_n(x) \in U_{\phi(x)}^{(n+1)}$$

for each $n$ and each $x \in K$. Being slightly more careful, we can find a descending sequence $K_1, K_2, \ldots$ of subdivisions of $K$ such that

$$f\psi_n(\sigma) \subset U_{\phi(x)}^{(n+1)} \qquad \text{for all } x \in \sigma$$

holds for each $n$ and each $\sigma \in K_n$.

SUBLEMMA. *For each $n$ there is a map $\Psi^{(n)}: K \times I \to X$ such that*

$$\Psi_0^{(n)} = \psi_{(k+1)(n+1)},$$

$$\Psi_1^{(n)} = \psi_{(k+1)n},$$

*and*

$$f\Psi^{(n)}(x \times I) \subset U_{\phi(x)}^{(k+1)n}.$$

PROOF. Extend $\Psi_0 \cup \Psi_1$ over the cells of $K_{(k+1)(n+1)} \times I$ as follows. Let $J = K_{(k+1)(n+1)}$. Use the triviality of the inclusion $f^{-1}(U_{\phi(x)}^{(k+1)(n+1)}) \subset$

$f^{-1}(U_{\phi(x)}^{(k+1)(n+1)-1})$ to extend over the cells $J \times I$, where $x \in J^0$. Then, using the triviality of the inclusions

$$f^{-1}\left(U_{\phi(x)}^{j+1}\right) \subset f^{-1}\left(U_{\phi(x)}^{j}\right)$$

for each $j$ and each $x \in \sigma \in J$, extend over the cells of $J^p \times I$ for each $p \leqslant K$.

Now let $\Psi$: $K \times (0, 1] \to X$ be the composition of the $\Psi^{(n)}$, where $\Psi^{(n)}$ is to be copied on the interval $[1/(n+1), 1/n]$. Let $\Phi$: $K \times I \to Y$ be defined by

$$\Phi(x, t) = \begin{cases} f\Psi(x, t) & \text{if } 0 < t \leqslant 1, \\ \phi(x) & \text{if } t = 0. \end{cases}$$

Clearly $\Phi$ is a well-defined function. $\Phi$ is continuous, since $\Phi_t$ converges uniformly to $\phi$ as $t \to 0$. □

PROOF OF THE THEOREM. Conclusion (1) follows immediately from Lemmas A and B (with B supplying surjectivity and A injectivity). Conclusion (3) follows from the fact that "sufficiently close" maps of a $k$-complex into an $LC^k$ space are homotopic, along with Lemma A.

Conclusion (2), a refinement due to McMillan **[6]**, does not follow by direct application of Lemma A. One could modify the proof of Lemma A easily enough. Armentrout and Price **[1]** have a version which applies directly. □

Coram and Duvall **[1]** have defined "approximate fibrations" using the idea that homotopies can be "almost" lifted in a sense similar to the conclusions of Lemmas A and B. The concept generalizes that of a fibration and seems to warrant further study. (See also Husch **[1]**.)

(3.4) *$uv^k$-mappings and a Vietoris-Begle theorem.* The classical Vietoris theorem, as proved by Begle **[1]**, **[2]**, is probably better placed in a cohomology setting where there is a complete analysis via the Leray spectral sequence (see Bredon **[1]**). A "$uv$" version, however, is very useful in a homology setting. The following, due to R. Soloway **[1]**, is analogous to the Theorem of (3.3).

THEOREM. *Suppose that $f$: $X \twoheadrightarrow Y$ is a proper $uv^{k-1}(R)$-mapping, where $R$ is a principal ideal domain. Suppose further that $\exists$ subgroup $G$ of $H_k(X; R)$ such that, for each $y \in Y$ $\exists$ neighborhood $V$ of $f^{-1}(y)$ in $X$ for which the image of*

$$H_k(V; R) \to H_k(X; R)$$

*is contained in $G$. Then the following hold:*

(1) *$f_*$: $H_q(X; R) \to H_q(Y; R)$ is a monomorphism for $0 \leqslant q < k$.*

(2) *The kernel of $f_*$: $H_k(X; R) \to H_k(Y; R)$ is contained in $G$.*

(3) *If in addition $Y$ is $lc^k(R)$ then $f_*$: $H_q(X; R) \to H_q(Y; R)$ is an epimorphism for $0 \leqslant q \leqslant k$.*

A proof of this theorem can be constructed by analogy with §3.3, replacing "sphere" with "cycle", "complex" with "chain", and "homotopy" with "homology" throughout. The details of such a program have been worked out in detail by Soloway **[1]**.

## 4. Cell-like spaces and mappings.

(4.1) *Spaces.* Originally, a space was called *cell-like* iff $\exists$ manifold $N^n$ and an embedding $\phi: X \to N^n$ such that $\phi(X)$ is cellular in $N^n$. (Cf. Lacher [2].)[5] Since that time, the proposition below has been used to define *cell-like* for infinite-dimensional compacta. (See Mardesič [1].)

PROPOSITION. *For a finite-dimensional compactum $X$, the following statements are equivalent*:

(a) *$X$ admits an embedding as a cellular subset of some manifold*;

(b) *$X$ has property $UV^\infty$; and*

(c) *$X$ has the shape of a point* (*cf. Borsuk* [3],[4]).

We sketch a proof according to the plan (a) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

(a) $\Rightarrow$ (c). We may assume that $X$ is a cellular subset of $\mathbf{R}^n$. Thus

$$X = \bigcap_{i=1}^{\infty} Q_i$$

where $\mathring{Q}_{i+1} \subset Q_i \approx B^n$ for each $i$. We assume that $Q_i = f^{-1}(q_i)$, where $q_i$ is the ball of radius $1/i$ centered at the origin in $\mathbf{R}^n$ and $f: \mathbf{R}^n \twoheadrightarrow \mathbf{R}^n$ is a map whose only nondegenerate point-inverse is $\phi^{-1}(0) = X$. Also, we assume that $0 \in X$.

Let $f_i$ be a homeomorphism of $\mathbf{R}^n$ which agrees with $f$ on the complement of $Q_i$, and let $g_i = f_i^{-1}$. Then

$$\mathbf{f} = \{f_i x \text{ id}\}: X \to \{0\}$$

and

$$\mathbf{g} = \{g_i x \text{ id}\}: \{0\} \to X$$

are shape morphisms such that $\mathbf{g} \circ \mathbf{f} = \{\mathrm{id}_X\}$ and $\mathbf{f} \circ \mathbf{g} = \{\mathrm{id}_{\{0\}}\}$. $\square$

(c) $\Rightarrow$ (b). Assume that $X \subset \mathbf{R}^n$ and that $X$ has the shape of a point $\{x_0\}$. I.e., there exist shape morphisms

$$\mathbf{f} = \{f_i\}: X \rightleftarrows \{x_0\}: \{g_i\} = \mathbf{g}$$

such that $\mathbf{g} \circ \mathbf{f} = \{\mathrm{id}_X\}$ and $\mathbf{f} \circ \mathbf{g} = \{\mathrm{id}_{\{x_0\}}\}$. Since $\{x_0\}$ is an ANR, we may assume that each $f_i$ is the constant map $Q \to \{x_0\}$.

Let $U$ be an open set in $Q$ containing $X$. Determine an open set $V$, $X \subset V \subset U$, such that $g_i \circ f_i|_V \simeq \mathrm{id}_V$ in $U$ for almost all $i$. Since $f_i$ is constant, $g_i \circ f_i|_V$ is constant, and $\mathrm{id}_V \simeq$ constant in $U$. I.e., $X \subset Q$ has property $UV^\infty$. $\square$

(b) $\Rightarrow$ (a). Assume $X \subset \mathbf{R}^n$ has property $UV^\infty$, where $n \geqslant 2$, and consider $X \times 0 \subset \mathbf{R}^n \times 0 \subset \mathbf{R}^n \times \mathbf{R}^3 = \mathbf{R}^{n+3}$. Then $X \times 0 \subset \mathbf{R}^{n+3}$ has $UV^\infty$, and an easy general position argument shows that loops in $(V \times rB^3 - X \times 0)$ are homotopic to loops in $V \times rS^2$ (modulo $X \times 0$). Since $V$ may be chosen null-homotopic in $U$, it follows that loops in $(V \times rB^3 - X \times 0)$ are null-ho-

[5] I have recently learned of Sher and Alford's "like a point", apparently essentially the same concept.

motopic in $(U \times rB^3 - X \times 0)$; i.e., $X \times 0$ satisfies condition CC in $\mathbf{R}^{n+3}$. (See Appendix III.) $\square$

(4.2) *Mappings.* A mapping $f: X \longrightarrow Y$ is *cell-like* if $f^{-1}(y)$ is a cell-like space for each $y \in Y$. The main characterization of such maps is given in the next theorem. Applications of this result range from such basics as the verification that cell-like maps between compact ANR's form a category to geometric facts such as verifying that cell-like maps between manifolds are cellular. The proof is virtually the same as the one given in Lacher **[6]** for the finite-dimensional case.[6] Refined versions may be found in Kozlowski **[2]** and Haver **[3]**.

THEOREM. *If $f: X \twoheadrightarrow Y$ is a proper map between locally compact ANR's, then the following are equivalent:*

(a) *$f$ is cell-like;*

(b) *For every open set $V \subset Y$, the restriction $f|f^{-1}(V)$ is a proper homotopy equivalence: $f^{-1}(V) \rightarrow V$.*

Since $Y$ is locally contractible, it is obvious that (b) $\Rightarrow$ (a). For the converse, it suffices to assume $f: X \twoheadrightarrow Y$ is a proper map between locally compact ANR's and deduce that $f$ is a proper homotopy equivalence.

We set things up to apply §3.3, Lemma A, using Hanner's **[1]** characterization of ANR's: each of $X$ and $Y$ is a "proper homotopy retract" of a locally finite complex; i.e., $\exists$ locally finite complexes $P$ and $Q$ and proper maps

$$i: X \rightleftarrows P: r$$
$$j: Y \rightleftarrows Q: s$$

such that $r \circ i$ is properly homotopic to $\mathrm{id}_X$ and $s \circ j$ is properly homotopic to $\mathrm{id}_Y$. Let $\varepsilon: Y \rightarrow (0, \infty)$ be continuous, and apply §3.3, Lemma A, with $K = Q$, $L = \varnothing$, and $\phi = s$. We obtain a proper map $v: Q \rightarrow X$ such that $d(f \circ v, s) \leqslant \varepsilon \circ s$. Let $g = v \circ j$. Then $g: Y \rightarrow X$ is a proper map such that

$$d(f \circ g, s \circ j) \leqslant \varepsilon \circ s \circ j.$$

Clearly $\varepsilon$ can be chosen small enough so that $f \circ g$ is properly homotopic to $s \circ j$ (and, hence, to $\mathrm{id}_Y$). Let $h: Y \times I \rightarrow Y$ be a proper homotopy with the properties

$$h_0 = \text{identity on } Y,$$
$$h_1 = f \circ g,$$
$$d(y, h_t(y)) \leqslant \delta(y) \qquad \text{for } 0 \leqslant t \leqslant 1 \text{ and } y \in Y,$$

where $\delta: Y \rightarrow (0, \infty)$ is proper.

Now, define $H: P \times I \rightarrow Y$ by

$$H(x, t) = h(f \circ r(x), t), \quad x \in P, \quad 0 \leqslant t \leqslant 1.$$

$H$ is a proper map. Define $\bar{h}: P \times \{0, 1\} \rightarrow X$ by

$$\bar{h}_0 = r, \quad \bar{h}_1 = g \circ f \circ r.$$

Then $f \circ \bar{h}_0 = f \circ r = h_0 \circ f \circ r = H_0$, and $f \circ \bar{h}_1 = f \circ g \circ f \circ r = h_1 \circ f \circ r = H_1$; i.e.,

---

[6] R. B. Sher **[1]** has shown that a cell-like map between finite-dimensional compacta induces a Shape equivalence. His assumption of finite-dimensionality is essential in the Shape category. (See Taylor **[1]**)

$$f \circ \bar{h} = H|P \times \{0, 1\}.$$

Apply §3.3.A again, with $K = P \times I$, $L = P \times \{0, 1\}$, $\phi = H$, $\psi = \bar{h}$, and $\varepsilon = \delta$. We get an extension $\bar{H}$ of $\bar{h}$ over $P \times I$,

$$\bar{H}: P \times I \to X,$$

such that $d(f \circ \bar{H}, H) \leqslant \delta \circ H$.

$\bar{H}$ is a proper homotopy between $r$ and $g \circ f \circ r$, so $\bar{H}_t \circ i$ $(0 \leqslant t \leqslant 1)$ is a proper homotopy between $r \circ i$ and $g \circ f \circ r \circ i$. Since $r \circ i$ is properly homotopic to $\mathrm{id}_X$, so is $g \circ f$. □

COROLLARY 1. *If $f: X \to Y$ is a proper cell-like map between locally compact ANR's and $B$ is a cell-like subset of $Y$ then $f^{-1}(B)$ is cell-like.*

COROLLARY 2. *If $X$, $Y$, and $Z$ are locally compact ANR's and $f: X \to Y$ and $g: Y \to Z$ are proper cell-like maps then $g \circ f$ is cell-like.*

The last corollary is *false* when $X$ is not assumed an ANR (Taylor **[1]**).

The results of §4.2 hold equally well for closed cell-like maps between separable ANR's. The proofs use covers instead of epsilons (cf. Haver **[2]**).

(4.3) T. Chapman **[3]** gave the first proof that simple homotopy type is a topological invariant, and later extended his result **[4]** to show that a cell-like map between $Q$-manifolds is a simple homotopy equivalence. J. West's **[1]** result on ANR's, together with Chapman's work, implies the following much more delicate version of the Theorem of §4.2.

THEOREM. *If $f: X \to Y$ is a cell-like map between compact ANR's then $f$ is a simple homotopy equivalence.*

Independently, R. Edwards **[3]** and Chapman **[5]** gave proofs for the case in which $X$ and $Y$ are finite polyhedra. The general result follows from this special case and the result of Edwards listed as (11.3) below.

Chapman's proof is quite similar in nature to Siebenmann's proof of (5.3) below (a proof which is omitted from this article). We chose to include the one here because it is, in its simplicity, perhaps the best known illustration of the "power of the torus" and, as such, makes an excellent introduction to Siebenmann's paper **[8]**. The organization is that of Chapman **[5]** which is by design that of Siebenmann **[8]**. Polyhedra are assumed locally finite and finite dimensional.

MAIN LEMMA. *If $X$ is a polyhedron and $f: X \to \mathbf{R}^n$ is a proper cell-like map, then there exists a polyhedron $Y$ and a proper cell-like map $g: Y \to \mathbf{R}^n$ such that $g$ is a PL homeomorphism over a neighborhood of $\infty$ and $g = f$ over a neighborhood of* 0.

PROOF. Consider the following Kirby **[3]** diagram.

$$\begin{tikzcd}
\mathbf{R}^n \times D^k \arrow[rr, "h_2"] & & \mathbf{R}^n \\
r_1\mathring{D}^n \times D^k \arrow[u, "\cup"] \arrow[rr, "h_1"] \arrow[d, "\gamma \times \mathrm{id}"'] & & r_1\mathring{D}^n \arrow[u, "\cup"'] \arrow[d, "\gamma"] \\
\mathbf{R}^n \times D^k \arrow[rr, "\tilde{h}"] \arrow[d, "e^n \times \mathrm{id}"'] & & \mathbf{R}^n \arrow[d, "e^n"] \\
T^n \times D^k \arrow[rr, "h"] & & T^n \\
T_0^n \times D^k \arrow[u, "\cup"] \arrow[r, "p"] & X_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"'] & T_0^n \arrow[u, "\cup"'] \arrow[d, "\alpha"] \\
 & X \arrow[r, "f"] & \mathbf{R}^n
\end{tikzcd}$$

$e^n$: $\mathbf{R}^n \to T^n$ is $e(x) = \exp(\pi i x/4)$ in each factor, where $T^n = S^1 \times \cdots \times S^1$ is the $n$-torus.

$\alpha$: $T_0^n \to \mathbf{R}^n$ is a $PL$ immersion of the punctured $n$-torus $T_0^n = T^n$-point into $\mathbf{R}^n$. $\alpha$ is chosen so that $\alpha \circ e^n|3D^n$ is the inclusion of $3D^n$ into $\mathbf{R}^n$. ($rD^n = [-r, r]^n$ for $r > 0$.)

$f_1$: $X_1 \to T_0^n$ is the projection on the second factor of the pull-back

$$X_1 = \{(x,y) \in X \times T_0^n | f(x) = \alpha(y)\}.$$

Note that $f_1$ is cell-like: $f_1^{-1}(y) \approx f^{-1}(\alpha(y))$.

$\alpha_1$: $X_1 \to X$ is projection on the first factor.

Note that $X_1$ is a polyhedron and that $\alpha_1$ is a PL immersion. Also note that $\alpha_1|f_1^{-1}e^n(3D^n)$ is $1-1$.

$p$: $T_0^n \times D^k \to X_1$. By §(4.2), $f_1$ is a proper homotopy equivalence. Since all fundamental groups involved are free, $f_1$ is a simple homotopy equivalence (Seibenmann [7]). Therefore (ibid) $X$ and $T_0^n$ have closed regular neighborhoods (in some euclidean space) which are PL homeomorphic. It follows that $X_1$ has a regular neighborhood which is PL homeomorphic to $T_0^n \times D^k$ for some $k$. $p$ is the composition of first the inverse of this PL homeomorphism and then the PL mapping cylinder retraction onto $X_1$. Thus $p$ is a proper PL cell-like map. We may choose $p$ so that $f_1 \circ p$ is proper homotopy equivalent to the projection map $T_0^n \times D^k \to T_0^n$.

$h$: $T^n \times D^k \to T^n$. This map extends $p \circ f_1$ by sending $* \times D^k$ to $*$, where $T_0^n = T^n - \{*\}$. $h$ is cell-like by §4.2, and $h \simeq$ projection.

$\tilde{h}$: $\mathbf{R}^n \times D^k \to \mathbf{R}^n$. This map is a covering of $h$. Since $h^{-1}(y)$ has property $UV^1$ for each $y$, point-inverses of $\tilde{h}$ are homeomorphic to point-inverses of $h$, so $\tilde{h}$ is cell-like. Moreover, there is a $\delta > 0$ such that

$$d(\tilde{h}(x,y), x) < \delta$$

for all $x \in \mathbf{R}^n$, $y \in D^k$. Finally: $e^n \times \mathrm{id}$ maps $\tilde{h}^{-1}(3D^n)$ homeomorphically onto $h^{-1}(3D^n)$.

$\gamma$: $r_1\mathring{D}^n \to \mathbf{R}^n$. Find $r_1 > r \geqslant 3$ so that $rD^n \times D^n$ contains $\tilde{h}^{-1}(3D^n)$. Let $\gamma$ be a PL homeomorphism which maps rays onto rays and is the identity on $rD^n$.

$h_1$: $r_1\mathring{D}^n \times D^k \to r_1\mathring{D}^n$ is the composition $\gamma^{-1} \circ \tilde{h} \circ (\gamma \times \mathrm{id})$.

$h_2$: $\mathbf{R}^n \times D^k \to \mathbf{R}^n$. We can extend $h_1$ to $h_2$ by defining $h_2$ to be the projection

$$\mathbf{R}^n \times D^k \to \mathbf{R}^n$$

where $h_1$ is not already defined. Since $\tilde{h}$ is uniformly close to the projection, $h_2$ is continuous. In fact, $h_2$ is a proper cell-like map.

$g$: $Y \to \mathbf{R}^n$. This map is a "quotient" of the map $h_2$; we define $q_1$ and then $q$ in the diagram

$$\begin{tikzcd}
\mathbf{R}^n \times D^k \arrow[d, "q_1"'] \arrow[ddr, "h_2"] & \\
Y_1 \arrow[d, "q"'] \arrow[dr, "g_1"] & \\
Y \arrow[r, "g"] & \mathbf{R}^n.
\end{tikzcd}$$

The point-inverses of $q_1$ are the sets of the form

$$x \times D^k, \quad x \in \mathbf{R}^n - r_1\mathring{D}^n,$$

together with singleton sets.

Note that $\alpha \circ e^n|3D^n = $ identity, and hence that $h_2 = f \circ \alpha_1 \circ p \circ (e^n \times \mathrm{id})$ over $3\mathring{D}^n$. Since $g_1 = h_2$ over $3\mathring{D}^n$, and since $\beta = \alpha_1 \circ p \circ (e^n \times \mathrm{id})$: $h_2^{-1}(3\mathring{D}^n) \to f^{-1}(3\mathring{D}^n)$ is a PL collapse, we can let the point-inverses of $q$ be sets of the form

$$\beta^{-1}(y), \quad y \in f^{-1}(2D^n),$$

together with singletons. $\square$

MAIN THEOREM. *If $X$ is a polyhedron and $f$: $X \to \mathbf{R}^n$ is a proper cell-like map, then there exists a polyhedron $Y$ and a proper cell-like map $g$: $Y \to \mathbf{R}^n$ such that $g$ is a PL homeomorphism over a neighborhood of* 0 *and $g = f$ over a neighborhood of $\infty$.*

PROOF. The proof is the "Siebenmann inversion trick" [**8**].

Let $g_1$: $Y \to \mathbf{R}^n$ be the $(g, Y)$ supplied by the Main Lemma, where $g_1$ is a PL homeomorphism over a neighborhood of $\infty$ and $g_1 = f$ over $2D^n$. Let $\hat{Y}_1$ be the one-point compactification of $Y_1$, and extend $g_1$ to

$$\hat{g}_1\colon \hat{Y}_1 \to S^n,$$

where $S^n = \mathbf{R}^n \cup \{\infty\}$. Let $Y_2 = \hat{Y}_1 - g_1^{-1}(0)$, and let

$$g_2\colon Y_2 \to S^n - \{0\}$$

be the restriction of $\hat{g}_1$ to $Y_2$. Apply the lemma again, obtaining a polyhedron $Y_3$ and a proper cell-like map

$$g_3\colon Y_3 \to S^n - \{0\}$$

such that $g_3$ is a PL homeomorphism over $\mathring{D}^n - \{0\}$ and $g_3 = g_2$ over $S^n - 2\mathring{D}^n$. (The numbers 1 and 2 in the above sentence can be obtained by applying the proof of the lemma.) Note that $g_3 = f$ over $\partial(2D^n)$. Thus the desired $g\colon Y \to \mathbf{R}^n$ is obtained by setting

$$Y = \{0\} \cup g_3^{-1}(2D^n - 0) \cup f^{-1}(\mathbf{R}^n - 2D^n)$$

and letting $g$ be the obvious map. □

Proof of the theorem (assuming $X$ and $Y$ are finite polyhedra). The proof is by induction on dim $Y$, beginning trivially when dim $Y = 0$.

For the inductive step, assume dim $Y = n$, and let $Y^{n-1}$ be the $(n-1)$-skeleton of $Y$. Then $Y - Y^{n-1}$ is the disjoint union of finitely many open sets homeomorphic to $\mathbf{R}^n$. Applying the Main Theorem to each of these open $n$-cells, we obtain a polyhedron $Z$ and a cell-like map

$$g\colon Z \to Y$$

such that $g = f$ over a regular neighborhood $N_0$ of $Y^{n-1}$ and $g$ is a PL homeomorphism over the complement of a regular neighborhood $N^n$ of $Y^{n-1}$.

By our induction hypothesis (since $N^n$ collapses to $Y^{n-1}$) $g|g^{-1}(N^n)$: $g^{-1}(N^n) \to N^n$ is simple. It follows from the Sum Theorem (cf. Siebenmann [7]) that $g$ is simple. We can find a map (using §4.2)

$$h\colon Z \to X$$

such that $h|g^{-1}(N_0) =$ identity and $h|Z - g^{-1}(\mathring{N}_0)$: $(Z - g^{-1}(\mathring{N}_0)) \to (X - f^{-1}(\mathring{N}_0))$ is a homotopy equivalence. Since $f \circ h \simeq g$ and $g$ is simple, we need only check that $h$ is simple. But

$$h|g^{-1}(N_0) \text{ and } h|\big(Z - g^{-1}(\mathring{N}_0)\big)$$

are each simple, so $h$ is simple by the Sum Theorem. □

**5. Cellular maps between manifolds.** A mapping $f\colon M^m \to Y$, where $M^m$ is a manifold, is called *cellular* iff $f^{-1}(y)$ is a cellular subset of $M$ for each $y \in Y$. Note that cellularity is a property of an inclusion $X \subset M^m$, certainly not an intrinsic property of $X$. For example, an arc may fail to be cellular in $\mathbf{R}^n$, $n \geqslant 3$ (see Blankenship), and polyhedral copies of the dunce hat may fail to be cellular in $\mathbf{R}^4$ (see Zeeman [2]). It follows from the Generalized Schoenflies Theorem, however, that $\mathbf{R}^n/X$ is a manifold if and only if $X$ is cellular in $\mathbf{R}^n$.

On the other hand, factoring such sets as $S^1 \vee S^1$ out of $S^1 \times S^1$, yielding $S^2$ makes it clear that this last statement doesn't generalize far without careful study. There is a placement problem as well as an intrinsic one. Cellularity is studied in more detail in Appendix III. For mappings, we have three clarifying results

(5.1) THEOREM. *If $f\colon M^n \to\to N^n$ is a (proper) $UV^k$-mapping between manifolds, $2k + 1 \geqslant n$, then $f$ is cell-like.*

PROOF. If $n = 1$, $f$ is monotone, hence cell-like (2.1). Suppose $n \geqslant 2$. Then $k \geqslant 1$, so each $f^{-1}(y)$ has $UV^1$. Combining (2.2) and (2.3), we need only show that $\check{H}^i(f^{-1}(y); \mathbf{Z}) = 0$ for $i > k$ and $y \in N^n$.

We may assume, by restricting $f$ to a simply connected inverse open set, that $M^n$ and $N^n$ are orientable manifolds; therefore

$$\check{H}^i\big(f^{-1}(y)\big) \simeq H_{n-i}\big(M, M - f^{-1}(y)\big)$$

(cf. §6 below). By combining (2.2) and (3.4), or by (3.3) and the Hurewicz Theorem,

$$H_{n-i}\big(M, M - f^{-1}(y)\big) \simeq H_{n-i}\big(N, N - f^{-1}(y)\big) \simeq 0$$

whenever $n - i \leqslant k$ and $y \in N$. $\square$

(5.2) THEOREM. *Let $f\colon M^n \to N^n$ be a (proper) cell-like map between manifolds. If $n \geqslant 5$ then $f$ is cellular.*

Thus the embeddings $f^{-1}(y) \subset M^n$ are in a sense tame.

PROOF. Let $V$ be an open $n$-cell in $N^n$. Then, by §4.2, $f^{-1}(V)$ is an open manifold with the proper homotopy type of $\mathbf{R}^n$. Since $n \geqslant 5$, a result of Siebenmann [**2**] implies that $f^{-1}(V) \approx \mathbf{R}^n$. (See Appendix III.) It follows easily that $f^{-1}(y)$ is cellular for each $y \in N^n$. $\square$

COROLLARY. *If $f\colon M^n \to N^n$ and $g\colon N^n \to Q^n$ are (proper) cellular maps between manifolds then so is the composition $g \circ f$ ($n \geqslant 5$).*

(5.3) THEOREM (ARMENTROUT-SIEBENMANN). *Suppose $f\colon M^n \to N^n$ is a (proper) cellular map between manifolds, $n \neq 4$. Then $\exists$ proper homotopy $h$: $M^n \times I \to N^n$ such that $h_0 = f$ and $h_t$ is a homeomorphism for each $0 < t \leqslant 1$.*

COROLLARY. *Let $M^n$ and $N^n$ be manifolds, and denote by Map$(M^n, N^n)$ the space of proper mappings of $M^n$ and $N^n$ with the compact-open topology. Among the subsets*

$$\mathrm{CL}(M^n, N^n) = \textit{cell-like mappings } M^n \to N^n,$$

$$\mathrm{Cell}(M^n, N^n) = \textit{cellular mappings } M^n \to N^n,$$

$$\mathrm{Hom}(M^n, N^n) = \textit{homeomorphisms } M^n \to N^n,$$

*we have the following relationships*:

$$\mathrm{CL}(M^n, N^n) = \mathrm{Cell}(M^n, N^n) = \overline{\mathrm{Hom}(M^n, N^n)}, \quad n \not\in \{3, 4\},$$

$$\mathrm{Cell}(M^3, N^3) = \overline{\mathrm{Hom}(M^3, N^3)}, \qquad n = 3.$$

We defer to Siebenmann's paper [**8**] for the proof of (5.3), with McMillan [**8**] has supplementary reference when $n = 3$. (Armentrout [**3**] and Moore [**1**] are the original papers when $n = 3$ and 2. See also Roberts and Steenrod [**1**].)

(5.4) REMARKS. The (proper) assumption made on maps in this §5 is deducible from the other hypotheses in each case. See Appendix II.[7]

[7] T. Chapman has proved a theorem analogous to (5.3) for $Q$-manifolds. The (proper) assumption cannot be deduced in the infinite-dimensional case.

The 3-dimensional analogue of Theorem 5.2 is equivalent to the Poincaré conjecture. The best possible result obtainable without solving the Poincaré conjecture is that all but finitely many point-inverses of $f$ are cellular. This result, due to McMillan [3], is proved in §8.

When $n \geqslant 5$, D. Sullivan has given a complete set of invariants which determines the topological type of a simply connected $n$-manifold (characteristic submanifolds, $KO$-orientations, etc.). From his work, the simply connected $n \geqslant 5$ case of (5.3) follows as a corollary; this was announced by Sullivan at the Georgia Topology Conference, 1969.

**6. Duality; a finiteness theorem for partially acyclic mappings.** Let $R$ be a PID, and take $G$ to be a finitely generated $R$-module.

DUALITY. Suppose $N$ is an $R$-oriented $n$-manifold, $X$ a closed set in $N$. Then there exist isomorphisms

$$H_c^i(N; G) \xrightarrow{D} H_{n-i}(N; G)$$

and

$$\check{H}_c^i(X; G) \xrightarrow{D_X} H_{n-i}(N, N - X; G)$$

which are constructed in such a way that

$$\begin{tikzcd} H_c^i(N; G) \arrow[r] \arrow[d, "D"] & \check{H}^i(X; G) \arrow[d, "D_X"] \\ H_{n-i}(N; G) \arrow[r] & H_{n-i}(N, N-X; G) \end{tikzcd}$$

commutes up to sign.

Suppose further that $N$ is closed. Then, setting $X = N$ in Appendix II determines a generator $[N] \in H_n(N; R)$ which, by the universal coefficient theorem, determines a generator $[N]$ of $H_n(N; G)$ (as an $R$-module). In this case, the isomorphism $D$ is given by the formula

$$D(u) = u \cap [N], \qquad u \in H^i(N; G).$$

DEFINITION. If $f$: $M \to N$ is a map with compact point-inverses, define two sets as follows:

$$A_i(f; G) = \{y \in N | f^{-1}(y) \text{ does not have property } i - uv(G)\},$$

$$A^i(f; G) = \left\{y \in N | \tilde{\check{H}}^i(f^{-1}(y); G) \neq 0\right\}.$$

The following results may be found in slightly more general form in Lacher-McMillan [1].

(6.1) THEOREM. *Suppose $f$: $M^n \to\!\!\to N^n$ is a proper map between $R$-orientable $n$-manifolds, $k \in \mathbf{Z}$. If $A_i(f; G) = \phi$ for $i < k$ then $A^i(f; G) = \phi$ for $i > n - k$ and $A^{n-k}(f; G)$ is a locally finite set in $N$.*

For the proof, we shall need some lemmas.

(6.2) LEMMA. *With $f$ as in the hypothesis of* (6.1), *suppose $A_i(f; G) = \phi$ for $i < k$. If $Y$ is a compact set in $N$ such that $\check{H}^{n-k}(Y; G) = 0$, then*

$$H^{n-k}(M; G) \to \check{H}^{n-k}(f^{-1}(Y); G)$$

*is epic.*

PROOF. Consider the diagram

$$\begin{tikzcd} H_c^{n-k}(M; G) \arrow[r, "\psi"] \arrow[d] & \check{H}^{n-k}(X; G) \arrow[d] & & \\ H_k(M; G) \arrow[r, "\phi"] & H_k(M, M - X; G) \arrow[r, "\partial_M"] & H_{k-1}(M - X; G) \arrow[r, "\beta"] \arrow[d] & H_{k-1}(M; G) \arrow[d] \\ & H_k(N, N - Y; G) \arrow[r, "\partial_N"] & H_{k-1}(N - Y; G) \arrow[r, "\alpha"] & H_{k-1}(N; G) \end{tikzcd}$$

where $X = f^{-1}(Y)$. The rows of the diagram are exact (being portions of homology sequences); the upper vertical arrows are duality isomorphisms; the lower vertical arrows are induced by $f$; and the diagram commutes up to sign. Since $f$ is $uv^{k-1}(G)$, the lower vertical arrows are isomorphisms. (See §3 above.)

Now, $H_k(N, N - Y; G) \simeq \check{H}^{n-k}(Y; G) = 0$, so $\partial_N$ is the zero map. But $\partial_N = 0 \Leftrightarrow \alpha$ monic $\Leftrightarrow \beta$ monic $\Leftrightarrow \partial_M = 0 \Leftrightarrow \phi$ epic $\Leftrightarrow \psi$ epic. Now consider the diagram

$$\begin{tikzcd} H_c^{n-k}(M; G) \arrow[r, "\psi"] & \check{H}^{n-k}(X; G) \\ H^{n-k}(\hat{M}; G) \arrow[r] \arrow[u, "\zeta"] & H^{n-k}(M; G) \arrow[u, "\eta"'] \end{tikzcd}$$

Since $\psi$ is epic and $\zeta$ is an isomorphism (see Spanier [1]) we conclude that $\eta$ is epic. □

(6.3) LEMMA. *Let $X$ be a compact set in $M$, where $M$ is an ANR. Then there exist a finite polyhedron $P$ and maps $X \to^{\alpha} P \to^{\beta} M$ such that $\beta\alpha$ is homotopic to the inclusion $X \subset M$.*

PROOF. Let $W$ be a complex for which $\exists$ compact maps $r$: $W \rightleftarrows M$: $i$ where $r \circ i \simeq$ identity. Let $T$ be a locally finite triangulation of $W$, and let $P$ be the union of all simplices of $T$ which intersect $i(X)$. Clearly $P$ is a finite polyhedron containing $i(X)$. Letting $\alpha$ be the map $i$: $X \to P$ and $\beta = r|P$, we have the maps desired. □

(6.4) LEMMA. *Suppose that $M$ and $N$ are $R$-orientable $n$-manifolds and that $f$: $M \twoheadrightarrow N$ is a proper map with $A_i(f; G) = \phi$ for $i < k$, where $k < n$. If $y \in N$, then there exists a saturated compact neighborhood $X$ of $f^{-1}(y)$ in $M$ such that $\check{H}^{m-k}(X; G)$ is finitely generated (over $R$).*

PROOF. Let $Y$ be a closed $n$-cell neighborhood of $y$ in $N$, $X = f^{-1}(Y)$. Then $X$ is a compact set in $M$, so there exist $P$, $\alpha$ and $\beta$ as given by (6.3). Thus, the image of $H^*(M; G) \to \check{H}^*(X; G)$ is finitely generated, since the homomorphism factors through $H^*(P; G)$. But $\check{H}^{n-k}(Y; G) = 0$, so (6.2) implies that $H^{n-k}(M; G) \to \check{H}^{n-k}(X; G)$ is epic, and the proof is complete. □

(6.5) REMARK. Suppose $f$ satisfies the hypothesis of (6.1). Then considering (6.2) and (6.4) we see that $\check{H}^{n-k}(f^{-1}(y); G)$ is finitely generated for each $y \in N^n$.

PROOF OF (6.1). That $A^i(f; G) = \phi$ for $i > n - k$ follows immediately from duality. Now we show $A^{n-k}(f; G)$ is locally finite.

Suppose to the contrary that $A^{n-k}(f; G)$ has a limit point in $N$, say $y_0$. Let $X$ be a compact neighborhood of $f^{-1}(y_0)$ such that $\check{H}^{n-k}(X; G)$ is finitely generated. Let $\{y_j\}_{j=1}^{\infty}$ be an infinite sequence of points of $A^{n-k}(f; G)$ which converges to $y_0$ and lies in $f(X)$; and define $Y$ as the set $\{y_j\}_{j=0}^{\infty}$. $Y$ is a compact, zero-dimensional set in $N$. In the commutative diagram

$$\begin{tikzcd} H^{n-k}(M; G) \arrow[rr] \arrow[dr] & & \check{H}^{n-k}(f^{-1}(Y); G) \\ & \check{H}^{n-k}(X; G) \arrow[ur] & \end{tikzcd}$$

Lemma 6.2 implies that the upper arrow is epic, hence $\check{H}^{n-k}(X; G) \to \check{H}^{n-k}(f^{-1}(Y); G)$ is epic. But, under the assumption that $y_j \in A^{n-k}(f; G)$ for $j \geqslant 1$, it is clear that $\check{H}^{n-k}(f^{-1}(Y); G)$ is not a finitely generated $R$-module, contradicting the choice of $X$. □

Dualizations, generalizations, removal of orientability hypothesis, and applications may be found in Lacher-McMillan [**1**].

**7. Some geometric finiteness theorems.** We study the critical dimensions, the ones just omitted from consideration in (5.1). I.e., we consider $UV^{k-1}$-maps between $n$-manifolds when $2k = n$ and $2k + 1 = n$. The conclusion of (5.1) does not hold, in general, under the weakened dimensional restrictions as examples below show. However, we are able to analyze the situation fairly completely, obtaining appropriate global conclusions, using finiteness theorems. First, some examples.

BING MAPS. We consider the $n$-sphere $S^n$ as the join $S^k * S^l$, where $k + l + 1 = n$. Let

$$\phi: S^k * S^l \to \sum (S^k \times S^l)$$

be a map with only two nondegenerate point-inverses, the two spheres $(S^k, -1)$ and $(S^l, +1)$ at the lower and upper ends of the join. Thus $\Sigma(S^k \times S^l)$ is the suspension of, say, the 0-level of the join. Let

$$\psi: S^k \times S^l \to S^{k+l}$$

be a map with precisely one nondegenerate point-inverse, a wedge $S^k \vee S^l$. Then we have a "Bing map" defined as the composition

$$\begin{tikzcd} S^n \approx S^k * S^l \arrow[r, "\phi"] \arrow[rr, bend right, "b"'] & \sum(S^k \times S^l) \arrow[r, "\Sigma(\psi)"] & \sum(S^{k+l}) \approx S^n. \end{tikzcd}$$

(Cf. Bing [**6**].) Assuming that $2k + 1 \leqslant n$, we see that $b$ is a $UV^{k-1}$-map: There is an arc $A \subset S^n$ with endpoints $p, q$ such that

$$b^{-1}(y) \approx \begin{cases} \text{point} & \text{if } y \in S^n - A, \\ S^k \vee S^l & \text{if } y \in \mathring{A}, \\ S^k & \text{if } y = p, \\ S^l & \text{if } y = q. \end{cases}$$

Thus further analysis is necessary if global conclusions will be forthcoming when $2k + 1 = n$.

SPINE MAPS. These are our prototype for "simplest maps other than homeomorphisms". Let $N^n$, $K_1$, $K_2, \ldots,$ be $n$-manifolds, and let $M^n$ denote the locally finite connected sum

$$M^n = N^n \# K_1 \# \cdots \# K_t \# \ldots .$$

$K_i$ are compact. $N^n$ must be noncompact if $\{K_i\}$ is infinite. For $i = 1, 2, \ldots$ let $X_i$ be a spine of $K_i$-{point}, chosen so that $X_i \subset M^n$. (I.e., $X_i$ is a compact subset of $K_i$ such that $K_i - X_i \approx \mathbf{R}^n$.) There is a proper map

$$s\colon M^n \to N^n$$

whose nondegenerate point-inverses are precisely the sets $X_i$. Assuming each $K_i$ is $(k-1)$-connected, $s$ is a $UV^{k-1}$-map.

In the case in which $2k = n$ and each $K_i$ is $(k-1)$-connected, our $s$ is a $UV^{k-1}$-map between $2k$-manifolds. We show below that these are essentially the only such maps.

THE SINGULAR SET $C_f$. Recall from §1 the set of noncellular values of a map $f\colon M^n \to Y$ (where $M^n$ is a manifold) is denoted by

$$C_f = \{ y \in Y \mid f^{-1}(y) \text{ is not cellular in } M^n \}.$$

Our conclusions in this section will often be that the set $C_f$ is locally finite.

(7.1) LEMMA. *Suppose that $f\colon M^n \twoheadrightarrow N^n$ is a proper map between n-manifolds, $n \neq 4$, and that $C_f$ is locally finite. Then there exists a commutative diagram*

$$\begin{tikzcd} M^n \arrow[rr, "f"] \arrow[dr, "s"'] & & N^n \\ & N^n \arrow[ur, "f_1"'] & \end{tikzcd}$$

*where $s$ is a spine map and $f_1$ is cellular.*

PROOF. We apply the Armentrout-Siebenmann theorem (5.3) several times. First, let $y \in N^n$; then $f|f^{-1}(U - y)$ is cellular for some open $n$-cell neighborhood $U$ of $y$; therefore, by (5.3), $f^{-1}(y)$ has a neighborhood $V = f^{-1}(U)$ such that $V - f^{-1}(y) \approx S^{n-1} \times \mathbf{R}$.

Let $C_f = \{y_i\}_{i=1}^{\infty}$ and $X_i = f^{-1}(y_i)$. Define

$$s : M^n \to N_1$$

to be the quotient map which shrinks the sets $X_i$ to distinct points ($i = 1, 2, \ldots$). The existence of the neighborhoods $V$ implies that $N_1$ is an $n$-manifold and that $\bar{s}$ is a spine map.

A mapping

$$\bar{f}_1 : N_1 \to N^n$$

is induced by $f$ on $N_1$. Clearly $\bar{f}_1$ is cellular. By (5.3), there is a homeomorphism

$$h: N_1 \to N^n.$$

Letting $s = h \circ \bar{s}$ and $f_1 = \bar{f}_1 \circ h^{-1}$ completes the proof. $\square$

(7.2) THEOREM. *If $f$: $M^{2k} \twoheadrightarrow N^{2k}$ is a proper $UV^{k-1}$-map between topological manifolds then*

$$\tilde{C}_f = \{y \in N \,|\, f^{-1}(y) \text{ is not cell-like}\}$$

*is a locally finite set in $N^{2k}$.*

(7.3) COROLLARY. *If $f$: $M^{2k} \twoheadrightarrow N^{2k}$ is a proper $UV^{k-1}$-map between manifolds, $k \neq 2$, then $C_f$ is locally finite in $N^{2k}$.*

The corollary follows from (7.2) and (5.2). To prove (7.2), we consider the cases k = 1, k > 1 separately.

*Case* 1 ($k = 1$). By the theorem of §1A, we need only show that $f^{-1}(y)$ has a "deleted annular neighborhood" for each $y \in N^2$. Let $y \in N^2$ be fixed, and let $W$ be a compact $\partial$-manifold neighborhood of $f^{-1}(y)$ in $M^2$. Since $f$ is monotone, it maps $\pi_0$ bijectively, so $W - f^{-1}(y)$ is "connected at infinity". An easy argument shows that the answer to the question following this proof is "yes" when $k = 1$, so $V = W - f^{-1}(y)$ is a connected sum

$$V = M_0 \# M_1 \# M_2 \# \ \ldots$$

of compact $\partial$-manifolds where $\partial M_0 = \partial V$ and $\partial M_i = \varnothing$ for $i \geqslant 1$. But $V \subset W$ and $W$ is compact, so we conclude that all but finitely many of the $M_i$ are 2-spheres. Therefore $V$ contains a compact set $K$ such that

$$V - K \approx S^1 \times \mathbf{R}.$$

$W - K$ is then a deleted annular neighborhood of $f^{-1}(y)$.

*Case* 2 ($k > 1$). The map $f$ is $UV^1$, so $f^{-1}$ (simply connected open set) is simply connected. (See §3.3.) Thus, by restricting $f$ when necessary, we may assume without loss of generality that $M^{2k}$ and $N^{2k}$ are orientable manifolds. Therefore, Theorem 6.1 applies, and we see that $A^k(f; \mathbf{Z})$ is locally finite while $A^i(f; \mathbf{Z}) = \varnothing$ for $i \neq k$ (see §2.2). It follows from §2.3 that

$$\tilde{C}_f = A^k(f; \mathbf{Z}),$$

proving the theorem. $\square$

A more geometric proof of Case 2 may be found in Lacher [**8**]. A proof of Case 2 along the lines of that of Case 1 could be constructed if the answer to the following question were "yes":

QUESTION 1. *Suppose that $W^{2k}$ is a noncompact manifold (a $\partial$-manifold with $\partial W^{2k}$ compact and null-cobordant). Suppose further that $W^{2k}$ is $(k-1)$-connected at infinity. Must $W^{2k}$ be an infinite connected sum*

$$W = M_0 \# M_1 \# M_2 \# \ \ldots$$

*where $M_i$ is compact for each $i \geqslant 0$ and $\partial M_i = \varnothing$ for each $i \geqslant 1$?*

Thus, in the even-dimensional case, the "critical" dimensions admit a complete analysis via (7.3), (7.1), and (5.3). The results are as good as could be hoped (except when $2k = 4$).

The odd-dimensional case is a bit more complicated. The $UV^{k-1}$ Bing map $b\colon S^{2k+1} \to S^{2k+1}$ shows that one cannot conclude $C_f$ is finite in this case. Examining the situation more carefully yields a *local* reason for the difficulty.

(7.4) *The deficiency of a $UV^{k-1}$-map between $(2k+1)$-manifolds.* Assume that

$$f\colon M^{2k+1} \twoheadrightarrow N^{2k+1}$$

is a proper $UV^{k-1}$-map. The results of §6 imply that

$$\check{H}^i(f^{-1}(y); \mathbf{Z}) = 0 \quad \text{for } i \notin \{k, k+1\} \quad \text{and}$$
$$\check{H}^{k+1}(f^{-1}(y); \mathbf{Z}) \text{ is finitely generated for each } y \in N^{2k+1}.$$

We define the *(local) deficiency*

$$d(f,y) = \operatorname{rank} \check{H}^k(f^{-1}(y); \mathbf{Z}_2) - \operatorname{rank} \check{H}^{k+1}(f^{-1}(y); \mathbf{Z}_2)$$

for each $y$. Note that

$$d(f,y) = (-1)^k\left[\chi(f^{-1}(y); \mathbf{Z}_2) - 1\right]$$

where $\chi(X; \mathbf{Z}_2) = \Sigma_i(-1)^i \operatorname{rank} \check{H}^i(X; \mathbf{Z}_2)$ is the "Čech mod 2" Euler characteristic. The *(total) deficiency* is given by

$$d(f) = \sup_{y \in N} d(f,y).$$

Thus $d(f) \in \mathbf{Z}$ or $d(f) = +\infty$.

Notice that if $s\colon M^{2k+1} \to N^{2k+1}$ is a $UV^{k-1}$ spine map and $b\colon S^{2k+1} \to S^{2k+1}$ is a $UV^{k-1}$ Bing map then

$$d(b) = 2 \text{ while } d(s) = 0,$$

the latter by duality in $K_i$.

(7.5) THEOREM. *Let $f\colon M^{2k+1} \twoheadrightarrow N^{2k+1}$ be a proper $UV^{k-1}$-map between topological manifolds. If $d(f) \leqslant 0$ then $C_f$ is locally finite in $N^{2k+1}$.*

PROOF. Suppose $k > 1$. Then, just as in the proof of (7.2), we may assume $M$ and $N$ are orientable. Therfore, by (6.1), the set $A^{k+1}(f; \mathbf{Z})$ is locally finite.

Suppose $y \in N - A^{k+1}(f; \mathbf{Z})$. Then $f^{-1}(y)$ has property $UV^1$ and $i$ - $uv(\mathbf{Z})$ for $i \neq k$. (See §2.2.) Also, $\check{H}^{k+1}(f^{-1}(y); \mathbf{Z}_2) = 0$ by the Universal Coefficient Theorem, so $\check{H}^k(f^{-1}(y); \mathbf{Z}_2) = 0$ (using the assumption that $d(f) \leqslant 0$). It follows that $\check{H}^k(f^{-1}(y); \mathbf{Z}) = 0$, hence $f^{-1}(y)$ has $uv^\infty(\mathbf{Z})$, hence $UV^\infty$. (See §2.)

Applying (5.2) to $f|(M - f^{-1}(A))$, we see that $C_f = A^{k+1}(f; \mathbf{Z})$, completing the proof in case $k > 1$.

Now suppose $k = 1$. We cannot assume in this case that $M^3$ is orientable, and the Poincaré conjecture would follow from the conclusion that $C_f = \tilde{C}_f$. Examples show that $C_f \neq A^2(f; \mathbf{Z}_2)$. We are forced to work with $\mathbf{Z}_2$ coefficients and to do some special 3-$D$ geometry. The latter is postponed until §8.

Assuming $k = 1$, it follows immediately that $A^2(f; \mathbf{Z}_2)$ is locally finite and hence (because $d(f) \leqslant 0$) that $A(f; \mathbf{Z}_2)$ is locally finite, where

$$A(f; \mathbf{Z}_2) = \bigcup_i A^i(f; \mathbf{Z}_2).$$

We will take up the matter further in §8. □

(7.6) COROLLARY. *If $f$: $M^{2k+1} \twoheadrightarrow N^{2k+1}$ is a proper $UV^{k-1}$-map between manifolds then $d(f) \geqslant 0$.*

Note that if $t \geqslant 2$ is an integer, by judiciously composing Bing maps one can construct a $UV^{k-1}$-map

$$b_t: S^{2k+1} \to S^{2k+1}$$

such that $d(b_t) = t$. Thus the values

$$0, 2, 3, \ldots$$

are possible for $d(f)$. What about $d(f) = 1$?

This last question is answered by Bryant-Lacher [3]. Therein, it is shown that the value $d(f) = 1$ is possible if and only if $k \in \{1, 2, 4, 8\}$:

(7.7) THEOREM. *Let $f$: $M^{2k+1} \twoheadrightarrow N^{2k+1}$ be a proper $UV^{k-1}$-map. If $d(f) = 1$ then $k \in \{1, 2, 4, 8\}$ and $C_f = A \cup B$ where*

(1) *$A$ is locally finite in $N$;*

(2) *$B$ is the closure in $N$ of a union of nondegenerate continua; and*

(3) *For each $y \in B$, the kth Stiefel-Whitney class of $M$ is nonzero when restricted to $f^{-1}(y)$.*

Examples of maps with $d = 1$ occur in exactly the situations not ruled out by (7.7).

HOPF MAPS. Let $k \in \{1, 2, 4, 8\}$ and

$$\phi: S^{2k-1} \to S^k$$

be the Hopf fibration [1]. The mapping cylinder $Z_\phi$ is a $2k$-manifold with $\partial = S^{2k-1}$, so $Z_\phi \cup B^{2k} = P^{2k}$ is a closed manifold (the projective plane over the division algebra $\mathbf{R}^k$). There is a spine map

$$s: P^{2k} \to S^{2k}$$

whose only nondegenerate point-inverse is $S^k$. The map

$$h = s \times \mathrm{id}: P^{2k} \times S^1 \to S^{2k} \times S^1$$

is a $UV^{k-1}$-map with $d(h) = 1$. A spherical modification of $h$ produces a similar example whose range is $S^{2k+1}$.

The proof of (7.7) hinges on symmetry of a (mod 2) linking invariant defined for disjoint $k$-cycles in the total space of a $(k + 1)$-plane bundle $\xi^{k+1}$ over a compact $k$-manifold. This linking obeys the equation

$$\lambda(x,y) + \lambda(y,x) = w_k(\xi^{k+1})$$

for disjoint $k$-cycles $x$, $y$, where $w_k$ is the $k$th Stiefel-Whitney class. In the proof of (7.7), $\xi^{k+1}$ is a bundle whose total space is a neighborhood of a point-inverse of $f$ and whose base is a $k$-sphere. In such a situation, $w_k(\xi)$ is the Hopf invariant associated with the Thom space of $\xi$ (see Thom [1]) and hence $\lambda$ is symmetric (Adams [1]) whenever $k \notin \{1, 2, 4, 8\}$. Symmetry of $\lambda$ allows an argument by contradiction that $d(f) = 1$ is impossible. The proof seems sufficiently deep to make one wonder:

QUESTION 2. *Can one give an alternativ proof of the "nonexistence of maps of deficiency one" and thereby obtain a geometric proof of the theorem of Adams [1]?*

**8. Some special finiteness theorems for 3-manifolds.** The case $k = 1$ of (7.5) requires special treatment, both because of the unsolved nature of the Poincaré conjecture and because the hypothesis of (7.5) is especially weak in this situation. The methods developed here for this case of (7.5) also apply to the proof of a Finiteness Theorem for maps $W^3 \twoheadrightarrow \mathbf{R}$, an essential step in the proof of the tameness of 1-manifolds with mapping cylinder neighborhoods in 4-space. Setting $k = 1$ in (7.5) yields the following equivalent statement (see §2.1).

(8.1) THEOREM. *Suppose that $M^3$ and $N^3$ are 3-manifolds and that $f$: $M^3 \twoheadrightarrow N^3$ is a proper map with connected point-inverses. If*

$$\chi(f^{-1}(y); \mathbf{Z}_2) \geqslant 1$$

*for each $y \in N^3$ then $C_f$ is a locally finite set in $N^3$.*

(See §7.4 for the definition of $\chi(\ ; \mathbf{Z}_2)$.)

Notice that (7.1) applies and that, conversely, any spine mapping (where $K_i^3$ are connected) between 3-manifolds satisfies the hypothesis.

COROLLARY (A. WRIGHT [2]). *Suppose $f$: $M^3 \twoheadrightarrow N^3$ is a proper $uv^\infty(\mathbf{Z}_2)$ map. Then $C_f$ is locally finite.*

COROLLARY (D. R. MCMILLAN, JR. [3]). *Suppose $f$: $M^3 \to N^3$ is a proper cell-like map. Then $C_f$ is locally finite.*

The above three facts are listed in inverse chronological order of discovery. In fact, the above-mentioned result of McMillan seems to be the first published example of a "finiteness theorem" in the sense used in this article. Using ideas from the same McMillan paper, one deduces

(8.2) COROLLARY. *Suppose that $M^3$ and $N^3$ are compact 3-manifolds and that $f$: $M^3 \twoheadrightarrow N^3$ and $g$: $N^3 \twoheadrightarrow M^3$ are maps with connected point-inverses. If*

$$\chi(f^{-1}(y); \mathbf{Z}_2) \geqslant 1 \leqslant \chi(g^{-1}(x); \mathbf{Z}_2)$$

*for each $y \in N^3$ (resp. $x \in M^3$) then $M^3$ and $N^3$ are homeomorphic.*

PROOF OF (8.2). By (8.1) and (7.1), $\exists$ spine mappings $M^3 \to N^3$ and $N^3 \to M^3$. Therefore $\exists$ compact 3-manifolds $H^3$ and $K^3$ such that

$$M^3 \approx N^3 \# H^3 \text{ and } N^3 \approx M^3 \# K^3;$$

hence $M^3 \approx M^3 \# H^3 \# K^3$. It follows from Kneser [1] that $H^3 \# K^3 \approx S^3$ and hence that $H^3 \approx K^3 \approx S^3$. $\square$

REMARK. In (8.1) and (8.2), the inequality $\chi \geqslant 1$ could be replaced by $\chi \geqslant 0$ under the assumption that $M^3$ is orientable. (See §7.7 and recall that $w_1$ is the obstruction to orientability.) This result, in dimension 3, should be credited to Tom Knoblauch [1], [3].

The proof of (8.1) depends heavily on "finiteness" of a different nature, that found in Kneser's Theorem [1] (which gives an upper bound on the number of nontrivial bounding 2-spheres in a compact 3-manifold) and in the later result of W. Haken [1] (giving an upper bound on the number of mutually nonparallel incompressible surfaces one can find in a given compact 3-manifold). We will use the latest such result, due to Knoblauch [2]:

(8.3) THEOREM (KNOBLAUCH). *Suppose $M^3$ is a compact orientable 3-manifold with $\partial$. Then $\exists$ integer $k$ such that, whenever $X_1, \ldots, X_{k+1}$ are pairwise disjoint compact sets in $M^3$ then at least one $X_i$ has a neighborhood which embeds in $\mathbf{R}^3$.*

We defer to the original paper for the proof of (8.3).

Another result we shall need, but not prove, is the following very special case of a theorem of McMillan [6, Theorem 2]. (See Appendix III.) A "punctured ball with handles" is a compact $\partial$-manifold $W^3$ which is obtained from $S^3$ by removing the interiors of finitely many pairwise disjoint 3-cells and attaching finitely may solid 1-handles to the boundary of the resulting punctured cell. (It is assumed that $\partial W^3 \neq \varnothing$.)

(8.4) THEOREM (MCMILLAN). *Suppose $X$ is a compact set in $\mathbf{R}^3$ and that $X$ has property $1 - uv(\mathbf{Z}_2)$. Then $X$ has arbitrarily small neighborhoods in $\mathbf{R}^3$ whose components are punctured balls with handles.*

PROOF OF (8.1). Suppose $f\colon M^3 \twoheadrightarrow N^3$ satisfies the hypothesis of (8.1). As we saw at the end of §7, $A = A(f; \mathbf{Z}_2)$ is a locally finite set in $N^3$. Moreover, the set

$$K = \{y \in N^3 | f^{-1}(y) \text{ has no neighborhood which embeds in } \mathbf{R}^3\}$$

is locally finite by Knoblauch's theorem (8.3), because $y \in K - A$ implies $f^{-1}(y)$ has an orientable neighborhood. We claim that

$$C_f = K \cup A.$$

Containment in one direction is obvious. The difficult part is to take $y \in N^3 - K - A$ and prove that $f^{-1}(y)$ is cellular in $M^3$. It suffices to show that for such $y$, $f^{-1}(y)$ has property $UV^\infty$ (see Appendix III). Finally, it suffices to show that for such $y, f^{-1}(y)$ has property $1 - UV$ (see §2.3).

So assume $y \in N^3 - K - A$, and let $U$ be a neighborhood of $f^{-1}(y)$ in $M^3$. By (8.4), we may as well assume that $U$ is the interior of a punctured ball with handles; the crucial fact is that

$$\pi_1(U) \text{ is free}$$

on finitely many generators. Now let $W$ be an open 3-cell neighborhood of $y$ in $N^3$ such that $f^{-1}(W) \subset U$; set $V = f^{-1}(W)$. By §3.2,

$$H_1(V; \mathbf{Z}_2) = 0.$$

The technique of A. Wright [2], explained in the next paragraph, proves that any loop in $V$ is contractible in $U$.

For any group $G$, let $\mathbf{S}G$ be the subgroup of $G$ generated by all squares of elements of $G$. $\mathbf{S}G$ is normal and $G/\mathbf{S}G$ is abelian. Define

$$\mathbf{S}_0 G = G \text{ and } \mathbf{S}_n G = \mathbf{S}(\mathbf{S}_{n-1} G)$$

for $n = 1, 2, \ldots$, and finally

$$\mathbf{S}_\omega G = \bigcap_{n=0}^{\infty} \mathbf{S}_n G.$$

Note that any homomorphism $G \to H$ of groups must carry $\mathbf{S}_\omega G$ into $\mathbf{S}_\omega H$. Also, for any space $X$, $\mathbf{S}\pi_1(X, *)$ is the kernel of the "mod 2" Hurewicz homomorphism

$$\pi_1(X, *) \to H_1(X; \mathbf{Z}) \to H_1(X; \mathbf{Z}_2).$$

Therefore $\mathbf{S}_\omega\pi_1(V) = \pi_1(V)$. But $\mathbf{S}_\omega\pi_1(U) = \{1\}$, so every homomorphism $\pi_1(V) \to \pi_1(U)$ is trivial. (Cf. Stallings [4].) $\square$

Here is a 3-$D$ Finiteness Theorem needed in the next section.

(8.5) THEOREM. *Let $W^3$ be a 3-manifold and suppose that $f\colon W^3 \twoheadrightarrow \mathbf{R}$ is a proper map. Suppose further that $f^{-1}(t)$ has property $uv^1(\mathbf{Z}_2)$ for each $t \in \mathbf{R}$. Then there exists a locally finite set $F$ in $\mathbf{R}$ such that if $J$ is an open interval of $\mathbf{R}$ which contains no point of $F$ then $f^{-1}(J) \approx S^2 \times \mathbf{R}$.*

For the proof we need

(8.6) THEOREM. *Let $f$ be a proper map of the 3-manifold $W^3$ onto $\mathbf{R}$ such that $f^{-1}(t)$ has property $UV^1$ and has a neighborhood in $W^3$ which contains no fake 3-cells for each $t \in \mathbf{R}$. Then $W^3 \approx S^2 \times \mathbf{R}$.*

PROOF OF (8.6). Since $f$ is $UV^1$, $W^3$ has two simply connected ends. Similarly, $J^* = f^{-1}(J)$ has two simply connected ends for each open interval $J \subset \mathbf{R}$.

If $J$ is chosen small enough so that $J^*$ contains no fake 3-cells, the result of Husch and Price [1] concerning ends of 3-manifolds (see Appendix III) implies that $\exists$ compact $\partial$-manifold $B \subset J^*$ such that $J^* - \mathring{B}$ is the disjoint union of two copies of $S^2 \times [0, +\infty)$. Now $\partial B$ is the union of two 2-spheres and $B$ is simply connected. Since $B$ contains no fake 3-cells, $B$ is homeomorphic to $S^2 \times [0, 1]$. It follows that $J^* \approx S^2 \times \mathbf{R}$.

Now $\mathbf{R}$ can be covered by a locally finite collection $\{J_i\}_{i \in \mathbf{Z}}$ of open intervals such that $J_i^*$ contains no fake 3-cells for each $i$. Using the $S^2 \times \mathbf{R}$ structure of $J_i^*$ to pull $J_0^*$ along, we see that any compact set can be engulfed by $J_0^*$, so $W^3$ contains no fake 3-cells. Therefore $W^3 \approx S^2 \times \mathbf{R}$. $\square$

PROOF OF (8.5). A repeat of the argument in the latter part of the proof of (8.1) shows that

$$F' = \left\{t \in R \,|\, f^{-1}(y) \text{ does not have } 1 - UV\right\}$$

and

$$F'' = \left\{t \in R \,|\, f^{-1}(y) \text{ has no neighborhood which embeds in } \mathbf{R}^3\right\}$$

are locally finite. Set $F = F' \cup F''$. If $J$ is a given interval containing no points of $F$ then $f|\colon f^{-1}(J) \to J$ satisfies the hypothesis of (8.6). $\square$

**9. Mapping cylinder neighborhoods: Taming.**

(9.1) *Notation and definitions*. For a mapping $f\colon X \to Y$, the *mapping cylinder* of $f$, $Z_f$, is the quotient space

$$\left[X \times [0, 1] \cup Y \times 2\right]/\sim$$

where $(x, 1) \sim (f(x), 2)$ for each $x \in X$. The domain $X$ is naturally included in $Z_f$ under the map $x \to [(x, 0)]$. The range $Y$ is naturally included under the map $y \to [(y, 2)]$. The *open mapping cylinder* $\mathring{Z}_f$ is $Z_f - X$.

Now let $M$ be a subset of the $n$-manifold $N^n$. A *(global) mapping cylinder neighborhood of $M$ in $N$* is a neighborhood $W$ of $M$ in $N^n$ such that the pair $(W, M)$ is homeomorphic to a pair $(Z_\phi, M)$ where $\phi$ is a proper mapping of a manifold $U^{n-1}$ onto $M$. A *(local) mapping cylinder neighborhood of $M$ in $N^n$ at*

$x \in M$ is an open neighborhood $W$ of $x$ in $N^n$ such that the pair $(W, W \cap M)$ is homeomorphic to a pair $(\mathring{Z}_\phi, W \cap M)$ where $\phi$ is a proper mapping of a manifold $U^{n-1}$ onto $W \cap M$.

REMARK. The requirement that $U^{n-1}$ be a manifold implies that $Z_\phi$ is a $\partial$-manifold with $\partial Z_\phi = U^{n-1}$. When $n = 3$, one need make no assumptions on $U^2$: That $U^2$ is a manifold follows from homology considerations (cf. Wilder [**1**]). For $n > 3$, it is plausible that $U$ (unrestricted) might be *replaced* by a manifold; we avoid this question here. (See §§11, 12.) In any case, the *interior* of a mapping cylinder neighborhood is unique (cf. Kwun and Raymond [**2**]). In the rest of this section, we will be concerned with local properties of the embedding $M^m \subset N^n$, where $M^m$ is a $\partial$-manifold with local mapping cylinder neighborhoods in the manifold $N^n$.

(9.2) THEOREM. *Suppose that $M^m$ is a topologically embedded $\partial$-manifold in the manifold $N^n$ and that $M^m$ has mapping cylinder neighborhoods in $N^n$ at each point. If $n \leqslant 3$, or if $(n, m) = (4, 3)$ or $(4, 1)$, then $M^m$ is locally flat in $N^n$.*

We will give a proof of the most difficult case $(n, m) = (4, 1)$, proved in Bryant-Lacher [**1**]. The other cases follow from similar arguments (or the original ones; see Nicholson [**1**], Lacher-Wright [**1**]). It is convenient to isolate a special case:

SPECIAL CASE. Suppose that $\phi$: $S^2 \times \mathbf{R} \to\!\!\!\to \mathbf{R}$ is a proper map such that $\phi^{-1}(J) \approx S^2 \times \mathbf{R}$ for any open interval $J \subset \mathbf{R}$. Suppose further that $\psi$: $Z_\phi \to \mathbf{R}^4$ is an embedding of $Z_\phi$ onto a neighborhood of $\psi(\mathbf{R})$ in $\mathbf{R}^4$. Then $\psi(\mathbf{R})$ is locally flat in $\mathbf{R}^4$.

PROOF OF SPECIAL CASE. It follows from the hypothesis that $Z_\phi$ is a $\partial$-manifold with $\partial = S^2 \times \mathbf{R}$, and it suffices to prove that $\mathbf{R}$ is locally flat in $Z_\phi$.

We want to construct a 3-cell $B$ in $Z_\phi$. Let $\alpha$: $S^2 \times (-1, 1) \to \phi^{-1}(-1, 1)$ be a homeomorphism. Let

$$S_t = \alpha(S^2 \times t) \times t, \quad \text{copied in } Z_\phi \qquad (0 \leqslant t < 1).$$

Clearly

$$B = \Big(\bigcup_{0 \leqslant t < 1} S_t\Big) \cup \{1\}$$

is a 3-cell in $Z_\phi$ which meets $\partial Z_\phi$ nicely and is locally flat in $Z_\phi$ except possibly at the point 1. A result of Kirby [**1**] (and, independently, Černavskiĭ [**1**]) implies that $B$ is locally flat in $Z_\phi$.

Let $s_1 < s_2 < \cdots < s_k$ be points of $\mathbf{R}$. Using the idea of the preceding paragraph, we can construct 3-cells $B_i$ such that

$$B_i \subset Z_{\phi|\phi^{-1}(s_{i-1}, s_{i+1})},$$

$B_i$ is locally flat in $Z_\phi$, and $B_i \cap B_j = \varnothing$ for $i \neq j$. Let $A_i$ be the annulus in $\partial Z_\phi$ bounded by $\partial B_i$ and $\partial B_{i+1}$. Then $B_i \cup A_i \cup B_{i+1}$ is a 3-sphere in $Z_\phi$ which bounds a 4-cell $C_i$ in $Z_\phi$ (cf. Brown [**2**], [**3**]). Using $B_i$ and $C_i$ one can easily construct an embedding

$$h: B^3 \times [1, k] \to Z_\phi$$

which takes $B^3 \times [i, i + 1]$ onto $C_i$.

It follows from the above construction that **R** can be isotoped off any 2-complex $K$ in $Z_\Phi$ with arbitrarily small isotopies: just move $h(0 \times [1, k])$ and $K$ into general position and then squeeze towards $h(0 \times [1, k])$ in $h(B^3 \times [1, k])$. Choosing the $s_i$ close together and working in the image $Z_t$ of $S^2 \times \mathbf{R} \times [t, 1]$ in $Z_\Phi$ instead of $Z_\Phi$ guarantees that the 4-cells $C_i$ have small diameter, thus restricting the size of the squeeze. A standard engulfing-taming argument now completes the proof (cf. Bryant-Sumners **[1]**, Bryant **[4]**, **[6]**). $\square$

PROOF OF (9.2). We are concentrating on the case $(n, m) = (4, 1)$.

For each $x \in M^1$ let $V_x$ be an open arc in $M^1$ containing $x$ such that there exist an open 3-manifold $U_x$, a proper map $\phi_x$ of $U_x$ onto $V_x$, and an embedding $\psi_x$ of $Z_{\phi_x}$ into $N^4$ such that $\psi_x(V_x) = V_x$. A local duality argument shows that $N^4$ is $lc^1(\mathbf{Z})$ mod $V_x$. It follows from §3.1 that $\phi_x$ is a $uv^1(\mathbf{Z})$-map.

Applying §8.5, we see that there is a locally finite set $F_x$ in $V_x$ such that

$$\phi_x^{-1}(J) \approx S^2 \times \mathbf{R}$$

for any open interval $J$ in $V_x - F_x$. Shrinking the $V_x$ somewhat, we may assume that $F_x$ is actually finite.

Let $J_{x_1}, \ldots, J_{x_i}, \ldots$ be a locally finite cover of $M^1$, and let

$$F = \bigcup_i F_{x_i}.$$

It follows from the Special Case that $M^1$ is locally flat in $N^4$ at each point of $M^1 - F$. A result of Cantrell **[2]** implies that $M^1$ is locally flat. $\square$

(9.3) THEOREM. *If $n \geqslant 4$ and $n - m = 2$, or if $n \geqslant 5$ and $0 < m < n$, then there exists a nonlocally flat m-sphere with a mapping cylinder neighborhood in $S^n$.*

PROOF. First suppose $n - m = 2$ and $n \geqslant 4$. Let $3 \leqslant k < n$, and let $K$ be a PL locally flat $(k-2)$-sphere in $S^k$ such that $(S^k, K)$ is not homeomorphic to $(S^k, S^{k-2})$. Consider

$$M^{n-2} = K * S^{n-k-1} \subset S^k * S^{n-k-1} \approx S^n,$$

where " $*$ " denotes "join". Since $M^{n-2}$ is a PL subset of $S^n$, it certainly has a mapping cylinder neighborhood in $S^n$. But $M^{n-2}$ fails to be locally flat precisely at the points of $S^{n-k-1}$.

Suppose $n \geqslant 5$, $0 \leqslant l < m < n$, and $3 \leqslant n - m + l - 1$. Then, by R. Edwards recent work **[6]**, there is a nonsimply connected homology sphere $H^{n-m+l-1}$ whose double suspension is a topological sphere. Let $S^{l-1}$ be a locally flat sphere in $H^{n-m+l-1}$, and consider

$$M^m = S^{l-1} * S^{m-l} \subset H^{n-m+l-1} * S^{m-l} \approx S^n.$$

One easily verifies that $M^m$ has a mapping cylinder neighborhood in $S^n$ and that $M^m$ fails to be locally flat precisely along the set $S^{m-l}$. $\square$

(9.4) *Taming problems.*

DEFINITIONS. Suppose $M^m$ is a $\partial$-submanifold of the manifold $N^n$.

(1) $M^m$ is said to be $M\,C\,N$ in $N^n$ iff $M^m$ has mapping cylinder neighborhoods in $N^n$ at each point.

(2) The singular set is

$$LF = \{x \in M^m | M^m \text{ is not locally flat in } N^n \text{ at } x\}.$$

The behavior of $M^m$ in $N^n$ at points of $LF$, assuming $M^m$ is $M\ C\ N$ in $N^n$, should be an interesting field of study. The singularities are subtle–the local fundamental groups of the complement are finitely generated. The problems are truly high-dimensional in nature, as evidenced by the Galewski-Hollingsworth-McMillan result on fundamental groups of arc-complements [1]. Consideration of the examples of (9.3) leads one to ask:

QUESTION 3. *If $M^m$ is $M\ C\ N$ in $N^n$ must* dim $LF \leqslant n - 4$?

QUESTION 4. *If $M^m$ is $M\ C\ N$ in $N^n$ and $n - m \neq 2$, must* dim $LF \neq 0$?

The answer to the latter question is "yes" if one assumes $LF$ is tame in $M^m$. The answer in general is related to the following (the answer is "yes" when $m = 3$ or $\overline{C}_f$ is tame and 0-dimensional; see Lambert [1]).

QUESTION 5. *If $f$: $M^m \twoheadrightarrow N^n$ is a map between compact manifolds and if* dim $C_f \leqslant 0$, *must $C_f$ be finite*? *What if* dim $\overline{C}_f \leqslant 0$?

One can ask for conditions under which $M\ C\ N$ submanifolds are locally flat. The freeness condition of Gillman [1], modified slightly and localized, is as follows.

DEFINITION. Let $M^m$ be a $\partial$-manifold in $N^n$. We say $M^m$ is *free in $N^n$ at the interior point $x \in \mathring{M}^m$* provided there exists a neighborhood $V$ of $x$ in $M^m$ such that: for each sufficiently small $\delta > 0$ there exists a compact $(n - m - 1)$-manifold $L^{n-m-1}$ and a map $\lambda$: $V \times L \to (N - M)$ satisfying

(i) $L^{n-m-1}$ is simply connected if $n - m \geqslant 3$ and is $S^{n-m-1}$ if $n - m \leqslant 2$;

(ii) $\lambda(v \times L^{n-m-1}) \subset$ ($\delta$-neighborhood of $v$) for each $v \in V$; and

(iii) the linking number of $M^m$ and $\lambda(x \times L^{n-m-1})$ is $\pm 1$.

The "sufficiently small" condition on $\delta$ is to guarantee that linking is well defined up to sign. For boundary points $x \in \partial M$ the definition is similar, with $\partial(V \times K^{n-m})$ playing the role of $V \times L^{n-m-1}$ for some simply connected $\partial$-manifold $K^{n-m}$.

THEOREM. *If $M^m$ is $M\ C\ N$ in $N^n$ and is free in $N^n$ at each point then $M^m$ is locally flat in $N^n$ ($n \geqslant 5$). (If $n - m = 2$, we need to assume $M^{n-2}$ is locally flat in $N^n$ at some point of each of its components.)*

The proof is given in Bryant-Lacher [2], along with a proof that "strongly free" implies locally flat.

QUESTION 6. *If $M^m$ is $M\ C\ N$ in $N^n$ and the maps $\phi$: $U \to V$ can be chosen so that $H_{n-m-1}(\phi^{-1}(W))$ is generated by spherical elements for any open $m$-cell $W \subset V$, must $M^m$ be locally flat*?

In codimension two there are free embeddings which are not locally flat (Bing [2]).

QUESTION 7. *Suppose $M^m$ is free in $N^n$ at each point and that $n - m \neq 2$. Must $M^m$ be locally flat*?

**10. Mapping cylinder neighborhoods: Existence.** Rourke and Sanderson [2] have given examples of (PL) locally flat submanifolds which have no open tubular neighborhood. On the other hand, regular neighborhoods of subpolyhedra have the structure of a mapping cylinder neighborhood (as do closed disk-bundle neighborhoods). In fact, Block Bundles (cf. Rourke-Sanderson [1], Cohen [2]) have the structures of a mapping cylinder neighborhood where the map has point-inverses homotopy equivalent to a sphere.

Edwards [4] has introduced the idea of a TOP regular neighborhood and proved what appears to be the strongest existence theorem possible (within the dimensional restrictions). A corollary to his methods (as he points out) is a finite-dimensional proof of the topological invariance of simple homotopy type for finite polyhedra (see §11).

THEOREM (EDWARDS). *Suppose $M^m$ is a locally flat submanifold of the manifold $N^n$, with $n \geqslant 6$. Then $M^m$ has a mapping cylinder neighborhood in $N^n$. Moreover, the mapping $\phi$: $U^{n-1} \to M^m$ can be chosen so that $\phi^{-1}(x)$ has the shape of an $(n - m - 1$ phere for each $x \in M^m$.*

Edwards proves a relative version (for ∂-manifolds, $n \geqslant 7$) and a strong uniqueness theorem.

**11. Characterizations of ANR's.** Using mapping cylinder techniques, cell-like mappings between $Q$-manifolds, and a result of R. Miller [1], J. West [1] has solved the conjecture of Borsuk [1] that a compact ANR has finite homotopy type. The result of Miller is that if $X$ is a compact ENR (resp. ANR) then $X \times S^1$ has a mapping cylinder neighborhood in some euclidean space (resp. in $Q$). Edwards-Siebenmann's modification of West's proof (Edwards, [5]) yields the following

(11.1) THEOREM (MILLER-WEST-EDWARDS-SIEBENMANN). *Suppose $X$ is a compact ENR in the topological manifold $N^n$, where* $2 \dim X + 3 \leqslant n$. *If $N^n$ is $LC^1$* mod $X$ *(equivalently, if $N^n - X$ is $1 - LC$ at each point of $X$) then $X$ has a mapping cylinder neighborhood in $N^n$.*

Any compact ENR admits an embedding into some $\mathbf{R}^n = N^n$ satisfying the hypothesis of (11.1). Also, the retraction of a mapping cylinder neighborhood onto the range of the map is a cell-like map. Thus an application of (11.1) is

(11.2) COROLLARY. *For finite-dimensional compact metric spaces $X$ the following statements are equivalent*:

(a) *$X$ is an ENR*;

(b) *$X$ admits a mapping cylinder neighborhood in some euclidean space*;

(c) *$X$ is the cell-like image of a compact ∂-manifold.*

The implication (c) ⇒ (a) was noted in §3.3. It is precisely this implication which at present seems to require the finite-dimensionality of $X$:

QUESTION 8. *If $X$ is the cell-like image of a compact ENR, must $X$ be finite-dimensional?*

If the answer is "yes" then $X$ is an ENR. If the answer is "no", one would hope to know whether $X$ is at least an ANR.

For possibly infinite-dimensional $X$, R. Edwards characterization [7] of ANR's is beautiful:

(11.3) THEOREM (EDWARDS). *Let $X$ be a compact metric space. $X$ is an ANR if and only if $X \times Q$ is a $Q$-manifold.*

In particular, the infinite-dimensional analogues of the implications (a)⇔(b) ⇒ (c) in (11.2) follow. (Note, that Miller's mapping cylinder result is needed for Edwards' proof.) However, the statement analogues to (c) ⇒ (a) is false: J. Taylor has constructed a cell-like map of $Q$ onto a non-ANR [1].

Combining Edwards' Theorem (11.3) with the Chapman (-West) theory of $Q$-manifolds [**8**] provides the neatest (though not the first) way of assigning a well-defined simple homotopy type to a compact ANR.

**12. Shrinking decompositions.** In a paper published in 1959, R. H. Bing [**4**] introduced a technique for proving that $M^m/G$ is homeomorphic to $M^m$, where $G$ is an upper semicontinuous decomposition of $M^m$ into compact sets. (See Appendix II.) Now known as the Bing Shrinking Criterion, the idea is to uniformly shrink the elements of $G$ to arbitrarily small size with an isotopy of $M^m$. The limit of such isotopies produces a (necessarily cellular, see §3.2) map

$$h_1: M^m \twoheadrightarrow M^m$$

such that

$$G = \{h_1^{-1}(y) | y \in M^m\}.$$

Then $h_1 \circ q^{-1}$: $M^m/G \to M^m$ is a homeomorphism. (See Chapman [**8**], Marin and Visetti for other versions of the Bing criterion.)

Bing used his ideas to prove that $\mathbf{R}^3/D \times \mathbf{R} \approx \mathbf{R}^4$, where $D$ is the "dog-bone" decomposition of $\mathbf{R}^3$. The shrinking criterion is shown to hold for the decomposition $G = \{d \times t | d \in D \text{ and } t \in \mathbf{R}\}$ of $\mathbf{R}^4$.

Thus, the first nonmanifold factor of euclidean space was discovered.

There have been many results following this pioneering work of Bing. Without an attempt to recount the history, here are some of the latest results on euclidean factors. Each of these results, as well as (11.3), uses the Bing Shrinking Criterion.

(12.1) THEOREM (EATON-PIXLEY, EDWARDS-MILLER). *Let $M^3$ be a 3-manifold and let G be a cell-like upper semicontinuous decomposition of $M^3$. Assume that each element of G has an irreducible neighborhoods in $M^3$ and that the closure in $M^3/G$ of the image of the nondegenerate elements of G is 0-dimensional. Then $M^3/G \times \mathbf{R} \approx M^3 \times \mathbf{R}$.*

(12.2) THEOREM (EDWARDS). *If X is a cell-like set in $\mathbf{R}^n$ then $\mathbf{R}^n/X \times \mathbf{R} \approx \mathbf{R}^{n+1}$.*

Theorem (12.1) represents the ultimate generalization of Bing's original. (12.2) is the heart of Edwards' proof [**6**] that if $M^{n-1}$ is a manifold which bounds a contractible $\partial$-manifold in $\mathbf{R}^n$ then the double suspension of $M^{n-1}$ is $S^{n+1}$. The result represents the ultimate in a sequence of results beginning with Andrews-Curtis [**1**]. (See Bryant [**3**], [**9**] and Glaser [**4**].) The logarithmic shrinking process of Bing [**1**] also plays a role. The $n$-dimensional analogue of (12.1) follows, leaving open

QUESTION 9. *If G is a cell-like (or even cellular) upper semicontinuous decomposition of $\mathbf{R}^n$ and the dimension of the image in $\mathbf{R}^n/G$ of the set of nondegenerate elements is zero, is $\mathbf{R}^n/G \times \mathbf{R} \approx \mathbf{R}^{n+1}$?*

**13. Desingularizing spaces.**

(13.1) *Resolutions.* Certain geometric singularities in spaces can be "removed", or "smoothed over", by stabilization and/or resolution. For example, the results of §11 show how to "smooth" any ENR by either infinite stabilization or resolution to a $\partial$-manifold.

The results of §12 deal with a much more subtle geometric singularity, a type which can be removed by *finite* stabilization.

Another way of desingularizing, perhaps inspired by results in algebraic geometry (cf. Hironaka [**1**]), was introduced by M. Cohen and D. Sullivan in the PL category [**1**], [**2**] (see also Sato [**1**]), and by J. L. Bryant and J. Hollingsworth [**1**] in the present context.

THEOREM (BRYANT-HOLLINGSWORTH). *Suppose that $X$ is a space such that $X \times \mathbf{R}^k$ is an $(m+k)$-manifold ($m \geqslant 5$, $k \geqslant 1$) and that $X$ is a manifold except possibly at the points of a 0-dimensional set. Then there exists an $m$-manifold $M^m$ and a proper cell-like map of $M^m$ onto $X$.*

DEFINITION. An *$m$-resolution* of a space $X$ is a pair $(M^m, f)$ where $M^m$ is an $m$-manifold and $f\colon M^m \to X$ is a proper, cell-like map.

MAIN CONJECTURE (QUESTION 10). *A space $X$ admits a resolution if and only if $X \times \mathbf{R}$ is a manifold.*

In order to disentangle the main conjecture from a previous question (Question 8 in §11), one might want to assume that $X$ is finite-dimensional.

This conjecture is audacious, certainly; but there is little evidence for its falsity. Theorems on existence of resolutions are rare; and all known manifold factors have been constructed as cell-like decompositions of manifolds.

(13.2) *Stable resolutions.* In order to promote their study, we introduce some classes of spaces:

$$\mathfrak{M}_k^m = \{X \mid X \times \mathbf{R}^k \text{ is the image of a proper, cell-like map on an } (m+k)\text{-manifold}\},$$

$$\mathfrak{M}^m = \bigcup_{k \geqslant 0} \mathfrak{M}_k^m.$$

Thus $\mathfrak{M}_0^m$ is the class of spaces which admit $m$-resolutions and $\mathfrak{M}^m$ is the class of spaces which admit stable $m$-resolutions. Obviously we have

$$\mathfrak{M}_0^m \subset \mathfrak{M}_1^m \subset \cdots \subset \mathfrak{M}^m.$$

One wonders about equalities. Two particularly interesting ones are mentioned in the next

QUESTION 11. *Is $\mathfrak{M}_0^m = \mathfrak{M}_1^m$, $\mathfrak{M}_1^m = \mathfrak{M}^m$, or both?*

A more careful look at the proof given by Bryant-Hollingsworth (using Siebenmann's Theorem 5.3) yields:

THEOREM. *If $X \in \mathfrak{M}^m$ is locally euclidean except at the points of a 0-dimensional set then $X \in \mathfrak{M}_0^m$ ($m \geqslant 5$).*

A related result, due to J. Cannon [**3**]:

THEOREM (CANNON). *Let $X$ be a compact homology $m$-manifold which is a retract of a neighborhood in $\mathbf{R}^{m+1}$. Then $X \in \mathfrak{M}_2^m$.*

Cannon's theorem provides evidence for another wild conjecture:

CONJECTURE (QUESTION 12). *$\mathfrak{M}^m$ is the set of ENR homology $m$-manifolds.*

A final question is related to uniqueness of resolutions (cf. Kato [**1**], problem 7.4):

QUESTION 13. *If $M_i^m$ is a compact manifold and $f_i\colon M_i^m \to X$ is a cell-like map, $i = 1, 2$, must $M_1^m$ and $M_2^m$ be homeomorphic?*

## APPENDIX I. $(\mathrm{Ext})^2 =$ TORSION

Let $R$ be a principal ideal domain, $\mathfrak{M}$ the category of finitely generated $R$-modules. We consider four functors. If $G$ is an $R$-module, let

$$\mathbf{Ext}\, G = \mathbf{Ext}_R(G; R), \quad \mathbf{Hom}\, G = \mathbf{Hom}_R(G; R)$$

$$\mathbf{T}G = \{ g \in G | rg = 0 \text{ for some } 0 \neq r \in R \}, \mathbf{F}G = G/\mathbf{T}G.$$

If $\psi\colon G \to H$ is an $R$-homomorphism, let

$$\begin{aligned} &\mathbf{Ext}\,\psi\colon \mathbf{Ext}\, H \to \mathbf{Ext}\, G \\ &\mathbf{Hom}\,\psi\colon \mathbf{Hom}\, H \to \mathbf{Hom}\, G \\ &\mathbf{T}\psi\colon \mathbf{T}G \to \mathbf{T}H \\ &\mathbf{F}\psi\colon \mathbf{F}G \to \mathbf{F}H \end{aligned}$$

be the induced homomorphisms. Because of the natural equivalence of $\mathbf{Hom} \circ \mathbf{Hom} \simeq \mathbf{F}$ as functors $\mathfrak{M} \to \mathfrak{M}$, it is generally accepted that **Hom** is a kind of "free part dual space" functor. In the same way, one can think of **Ext** as a "torsion part dual space" functor:

THEOREM. *There is a natural equivalence*

$$\mathbf{Ext} \circ \mathbf{Ext} \simeq \mathbf{T}$$

*as functors* $\mathfrak{M} \to \mathfrak{M}$.

COROLLARY. *Let* $F \to^{\phi} G \to^{\psi} H$ *be R-homomorphisms between finitely generated R-modules. If* $\mathbf{Hom}\,\phi = \mathbf{Ext}\,\psi = 0$, *then* $\psi \circ \phi = 0$.

PROOF. Let $G$ be a torsion $R$-module, and let $C_1$ and $C_0$ be finitely generated free $R$-modules such that

$$0 \to C_1 \to C_0 \to G \to 0$$

is exact. Then **Ext** $G$ makes the "**Hom**" sequence

$$0 \leftarrow \mathbf{Ext}\, G \leftarrow \mathbf{Hom}\, C_1 \leftarrow \mathbf{Hom}\, C_0 \leftarrow 0$$

exact. ($\mathbf{Hom}\, G = 0$, since $G$ is a torsion module.)

Since rank $\mathbf{Hom}\, C_1 =$ rank $\mathbf{Hom}\, C_0$, $\mathbf{Ext}\, G$ is a torsion module. (In fact, $\mathbf{Ext}\, G \simeq G$.) Thus $\mathbf{Ext}\,\mathbf{Ext}\, G$ makes the sequence

$$0 \to \mathbf{Hom}\,\mathbf{Hom}\, C_1 \to \mathbf{Hom}\,\mathbf{Hom}\, C_0 \to \mathbf{Ext}\,\mathbf{Ext}\, G \to 0$$

exact. It is well known that there are natural isomorphisms $h_i\colon C_i \simeq \mathbf{Hom}\,\mathbf{Hom}\, C_i$ ($i = 0, 1$), since the $C_i$ are free. Thus a unique homomorphism $h\colon G \to \mathbf{Ext}\,\mathbf{Ext}\, G$ is induced, and this is the desired isomorphism. The naturality of $h$ follows from that of the $h_i$.

It follows from the above that $\mathbf{Ext} \circ \mathbf{Ext} = \mathbf{Identity}$ on the category of finitely generated torsion $R$-modules. The general result follows from the special case. □

## APPENDIX II

### THE WHYBURN CONJECTURE AND SOLOWAY'S COMPACTNESS CRITERION

In 1959, G. T. Whyburn formalized the following question [2]: If $f$ is a mapping of $\mathbf{R}^3$ onto itself such that $f^{-1}(y)$ is compact and connected for each $y \in \mathbf{R}^3$, must $f$ be compact? Replacing $\mathbf{R}^3$ by $\mathbf{R}^2$, Whyburn showed that the answer was indeed "yes" [2]. The question has proved difficult to answer, finally being solved by R. H. Bing [9] (the answer is "no") after L. C. Glaser

[3] gave negative answers to the "generalized" Whyburn conjectures (for maps on $\mathbf{R}^n$, $n \geqslant 4$). More recently D. C. Wilson [1] has given examples of noncompact $UV^1$ mappings of $\mathbf{R}^n$ into itself ($n \geqslant 5$).

Thus, there is a need for criteria under which such monotone maps are compact. In fact, several such criteria were derived before it was known that the conjectures were false. (See, for example, Connell [1], Väsäillä [1], [2] Connor-Jones [1].) Recently, R. Soloway [1] has generalized most known criteria using local degree of a map. We present here an exposition of Soloway's results as well as some related facts about partially acyclic maps between manifolds.

Several problems are listed in the text.

**1. Some preliminary results and definitions.** We must begin by giving definitions.

A. Definitions. Suppose, for the definitions, that $f\colon X \twoheadrightarrow Y$ is a map.

Closed. $f$ is *closed* if $f(A)$ is closed in $Y$ whenever $A$ is closed in $X$.

Compact. $f$ is *compact* if $f^{-1}(B)$ is compact whenever $B$ is a compact subset of $Y$.

Proper. $f$ is *proper* if it is closed and $f^{-1}(y)$ is compact for all $y \in Y$.

Monotone. $f$ is *monotone* if $f^{-1}(y)$ is compact and connected for all $y \in Y$.

Reflexive compact. $f$ is *reflexive compact* if $f^{-1}f(A)$ is compact for all compact sets $A \subset X$.

While working with maps between locally compact spaces, the following terminology will be convenient: a subset $A$ of a space $X$ is said to be *bounded* (in $X$) if $\bar{A}$, the closure of $A$, is compact.

B. Some preliminary results. There are some implications among the above properties of maps. In general, "proper" is equivalent to "compact". For maps between locally compact (separable, metric) spaces, we have further implications as indicated by the arrows below:

$$\begin{tikzcd}
 & \text{proper} \arrow[r, leftrightarrow] \arrow[dl] & \text{compact} \arrow[d] \\
\text{closed} & & \text{reflexive compact} \\
 & & \text{monotone} \arrow[u]
\end{tikzcd}$$

and these are the only ones which hold in that generality.

First we give some examples. Let $X$ be the half-open interval $[0, 1)$; and let $Y$ be the compact space homeomorphic to the letter $P$. Clearly there is a one-one map $f\colon X \twoheadrightarrow Y$. Thus, $f$ is monotone but not compact and not closed. To see a closed map which is not proper, project $\mathbf{R}^n$ to $\mathbf{R}^{n-1}$.

For the remainder of this §1B, assume that $f\colon X \twoheadrightarrow Y$ is a map between separable metric spaces.

(1.1) Proposition. *A surjection is proper if and only if it is compact.*[8]

Proof. Assume proper. Suppose $B$ is compact in $Y$. Let $\{x_n\} \subset f^{-1}(B)$ be a sequence of distinct points. If $f^{-1}(B)$ is not compact, we may choose $\{x_n\}$

[8] See also Siebenmann [9, p. 152, footnote].

to be a closed set with no limit point. Now, $\{f(x_n)\}$ has a convergent subsequence, so we assume $\{f(x_n)\}$ converges to $f(x)$. Since $f^{-1}f(x)$ is compact, $f^{-1}f(x) \cap \{x_n\}$ is a finite set. Therefore, we can find an integer $N$ such that $x_n \not\in f^{-1}f(x)$ for $n \geqslant N$. The set $S = \{x_n\}_{n \geqslant N}$ is then a clos in $X$ and $f(S) = \{f(x_n)\}_{n \geqslant N}$ is a nonclosed subset of $Y$, a contradict the assumption that $f$ is closed.

Now assume compact. We need only show $f$ is closed. Let $A$ be a clo set in $X$. Suppose we can find $y_n \in f(A)$, $n = 1, 2, \ldots$, such that $\{y_n\}$ converges to $y$. Let $B = \{y_n\}_n \cup \{y\}$, $B$ is a compact set in $Y$, so $f^{-1}(B) \cap A$ is compact. Choose $x_n \in f^{-1}(y_n) \cap A$, $n \geqslant 1$, and let $\{x_{n_i}\}$ be a convergent subsequence of $\{x_n\}$, say converging to $x \in A$. Then $\{f(x_{n_i})\}$ converges to $f(x)$ and $y$, so $f(x) = y$. Thus $f(A)$ is closed. $\square$

(1.2) PROPOSITION. *If $X$ and $Y$ are locally compact and $f$ is monotone then $f$ is reflexive compact.*

PROOF. Let $A$ be a compact set in $X$. Let $\{x_n\}_n$ be a sequence in $f^{-1}f(A)$. We need to show that $\{x_n\}$ has a convergent subsequence.

Let $a_n \in A \cap f^{-1}f(x_n)$ for each $n$. Since $A$ is compact, we may assume that $\{a_n\}$ converges to some $a \in A$. Let $U$ be a neighborhood of $f^{-1}f(a)$ such that $\bar{U}$ is compact. We claim that there exists an integer $N$ such that $x_n \in U$ for $n \geqslant N$. This claim completes the proof, since then $\{x_n\}_{n \geqslant N}$ has a convergent subsequence by compactness of $\bar{U}$.

Suppose $x_n \not\in U$ for infinitely many $n$. Since $a_n \in U$ for all but finitely many $n$, there exists infinitely many $n$ for which $a_n \in U$ and $x_n \not\in U$. Using connectedness of $f^{-1}f(x_n)$, we see that there is a subsequence $\{x_{n_i}\}_i$ of $\{x_n\}$ such that $b_i \in f^{-1}f(x_{n_i}) \cap \dot{U}$, where $\dot{U} = \bar{U} - U$. But $\dot{U}$ is compact, so we may assume $\{b_i\}$ converges to some $b \in \dot{U}$. This is a contradiction, since $\{f(b_i)\}$ converges to $f(b)$ and to $f(a)$, forcing $b$ to belong to $f^{-1}f(a) \subset U$. $\square$

(1.3) PROPOSITION. *If $f$ is reflexive compact, and $X$, $Y$ are locally compact there exists a nonvoid open set $V$ of $Y$ such that $f^{-1}(V)$ is bounded (and, hence, $f|f^{-1}(V)$ is compact).*

PROOF. Write $X$ as the union of countably many compact sets $A_1, A_2, \ldots$. Then, $Y$ is the union of $f(A_1), f(A_2), \ldots$. By the Baire theorem, some $f(A_i)$ has nonvoid interior, say $V$. Since $f^{-1}f(A_i)$ is compact and contains $f^{-1}(V)$, it is clear that $f^{-1}(V)$ is bounded. $\square$

(1.4) PROPOSITION. *If each $y \in Y$ has a neighborhood $V_y$ such that $f^{-1}(V_y)$ is bounded, then $f$ is compact.*

PROOF. Any compact set in $Y$ can be covered by finitely many of the $V_y$. $\square$

C. UPPER SEMICONTINUOUS DECOMPOSITIONS. Upper semicontinuous decompositions have played an important role in point-set topology. (See, e.g., Moore [2].) We will display here the relationship between upper semicontinuous decompositions and proper maps.

DEFINITIONS. Let $X$ be a space. A *decomposition* of $X$ is defined to be a collection $G$ of closed subsets of $X$ such that no two elements of $G$ have a point in common and every point of $X$ belongs to some element of $G$. A decomposition is called *monotone* if each element is compact and connected. A decomposition $G$ of $X$ is called *upper semicontinuous* (abbreviated u.s.c.) if the following condition holds:

For any open set $U$ of $X$, the set
$U^* = \{x | x \in g \subset U \text{ for some } g \in G\}$
is an open set in $X$.

If $G$ is a decomposition of a space $X$, the *quotient space* $X/G$ is the space whose points are the elements of $G$ and whose open sets are those of the quotient topology; i.e., if $q$: $X \to X/G$ is the quotient function, then $V$ is open in $X/G$ if and only if $q^{-1}(V)$ is open in $X$.

(1.5) PROPOSITION. *Suppose $G$ is a decomposition of the metric space $X$. Then $G$ is u.s.c. if and only if the quotient map $q$: $X \to X/G$ is closed.*

(1.6) PROPOSITION. *Suppose $f$: $X \twoheadrightarrow Y$ is a closed map between metric spaces. Let $G = \{f^{-1}(y) | y \in Y\}$. Then $G$ is u.s.c. and the induced function $X/G \to Y$ is a homeomorphism.*

Proposition 1.5 follows from the identity

$$U^* = X - f^{-1}f(X - U)$$

(using the notation of (1.6)). The same identity shows that if $f$ is closed then $G$ is u.s.c. The induced function $X/G \to Y$ is always continuous, and has a continuous inverse whenever $f$ is closed.

**2. Local degree of a reflexive compact map; Soloway's criterion.** Throughout this section, suppose $R$ is a (nonzero) principal ideal domain. All homology is understood to have coefficients in $R$.

ORIENTATIONS. Suppose $M^n$ is an $n$-manifold without boundary. If $x \in M^n$, then

$$H_n(M^n, M^n - x) \simeq R$$

by excision. An *$R$-orientation* of $M^n$ is a selection of elements

$$g_X \in H_n(M^n, M^n - X),$$

where $X$ ranges over all compact subsets of $M$ (with $X \neq \varnothing$), such that the following hold:

(i) If $X \supset Y$ then $g_X \to g_Y$ under $H_n(M^n, M^n - X) \to H_n(M^n, M^n - Y)$; and

(ii) If $x \in M^n$ then $g_x$ is a generator of $H_n(M^n, M^n - x)$.

We say that $M^n$ is *$R$-orientable* if it admits an $R$-orientation. A manifold equipped with an $R$-orientation is called an $R$-oriented manifold. (These notions of orientability are essentially the same as the standard ones. See §6.3 of Spanier.)

LOCAL DEGREE. Suppose $M^n$ and $N^n$ are $R$-oriented $n$-manifolds ($n > 0$) and that $f$: $M^n \to N^n$ is a map such that $f^{-1}(y)$ is compact for some $y \in N^n$. Then we have

$$f_*: H_n\big(M^n, M^n - f^{-1}(y)\big) \to H_n(N^n, N^n - y)$$

where $H_n(N^n, N^n - y)$ is generated by $g_y$. Thus

$$f_*(g_{f^{-1}(\dot{y})}) = rg_y$$

for some $r \in R$. We define

$$r = \deg(f, y),$$

the *degree of f at y*.

The following theorem is due in its present form to R. Soloway [1]. Similar (but not as general) ideas were exploited by E. H. Connell [1], J. Väisälä [1], [2] and by A. C. Connor and S. L. Jones [1].

THEOREM (SOLOWAY). *Suppose $f\colon M^n \twoheadrightarrow N^n$ is a reflexive compact map between open R-oriented n-manifolds. If there exists a point p of $N^n$ such that* $\deg(f, p) \neq 0$, *then f is compact.*

Soloway uses the essentiality of $f$ near $p$ to drag the $V$ of (1.3) around (homologically) in $N^n$, supplying the $V_y$ of (1.4).

**3. Monotone maps between 2-manifolds.** The results in this section may be obtained as corollaries to the results of §5. (Recall that "monotone" is equivalent to "$UV^0$".) It is therefore logically unnecessary to cover the material in §3. However, in view of the historical interest, it seems worthwhile to make this special case more accessible.

If $f\colon M \twoheadrightarrow Y$ is a map, where $M$ is a manifold, we define

$$C_f = \{y \in Y \mid f^{-1}(y) \text{ is not cellular in } M\}.$$

(See Appendix III for the definition of cellular sets, as well as basic facts on cellularity.) The following was proved in §7.2 of the text assuming compactness of $f$. The proof given is special for 2-manifolds and is elementary.

(3.1) THEOREM. *If $f\colon M^2 \twoheadrightarrow N^2$ is a monotone map between 2-manifolds, then $C_f$ is a locally finite subset of N.*

The following is Soloway's generalization [1] of Whyburn's theorem [2]. The two results are proved simultaneously.

(3.2) THEOREM. *If $f\colon M^2 \twoheadrightarrow N^2$ is a monotone map between 2-manifolds then f is compact.*

PROOFS. Assume the hypothesis of (3.2). By (1.3), there is a nonvoid open set $V \subset N$ such that $f|f^{-1}(V)$ is compact. By the special case of (3.1) proved above $C_f|f^{-1}(V)$ is a locally finite subset of $V$. Therefore, there exists an open 2-cell $D \subset V$ such that $f^{-1}(y)$ is acyclic over $\mathbf{Z}_2$ (Čech homology or cohomology) for each $y \in D$. Let $p$ be an interior point of $D$. We have the following commutative diagram, in which the horizontal maps are excision isomorphisms:

$$\begin{array}{ccc} H^2(M, M - f^{-1}(p)) & \longleftarrow & H^2(f^{-1}(D), f^{-1}(D) - f^{-1}(p)) \\ \uparrow f^* & & \uparrow (f|)^* \\ H^2(N, N - p) & \longleftarrow & H^2(D, D - p). \end{array}$$

By the Vietoris mapping theorem (§3.4 of text, or §3.3 also applies), $(f|)^*$ is an isomorphism. Thus $f^*$ (and, hence, $f_*$) is an isomorphism, and

$\deg(f, p) = 1$ over $R = \mathbf{Z}_2$. It follows from §2 that $f$ is compact. □

(3.3) COROLLARY. *If $f$: $M \twoheadrightarrow N$ is a monotone map between R-orientable 2-manifolds* ($R = \mathbf{Z}$ *or* $\mathbf{Z}_2$) *then* $\deg f = \pm 1$.

(3.4) COROLLARY. *If $f$: $M \twoheadrightarrow N$ is a reflexive compact map between 2-manifolds such that $f|f^{-1}(U)$ is monotone for some nonvoid open set $U \subset N$, then $f$ is compact.*

R. Sher has pointed out that the "somewhere monotone" hypothesis in (3.4) is necessary (although it is not when $M = N = \mathbf{R}^2$ according to Duda [**1**]): Let $M = S^1 \times \mathbf{R}$, $N = S^1 \times S^1$, and define $f$ by sending $(x, t)$ to $(x, g(|t|))$, where $g$: $[0, \infty) \to S^1$ is continuous and bijective. $f$ is reflexive compact but not compact.

**4. Exotic monotone maps in higher dimensions.**

(4.1). We will describe some examples based on two constructions, the first due to R. H. Bing [**9**].

THEOREM (BING). *If $n \geqslant 3$, there exists a monotone map $\beta_0$: $S^n \twoheadrightarrow B^n$ such that $\beta_0^{-1}(p)$ = point for some $p \in \partial B^n$.*

Actually, Bing only states the theorem for $n = 3$, but suspension of his map produces the others.

D. C. Wilson [**1**] has a higher-dimensional version, as follows:

THEOREM (WILSON). *If $n \geqslant 5$, there exists a $UV^1$-map $\beta_1$: $S^n \twoheadrightarrow B^n$ such that $\beta_1^{-1}(p)$ = point for some $p \in \partial B^n$.*

(4.2). Another important step in constructing exotic maps is the following result of L. C. Glaser [**3**] (see also [**1**]).

THEOREM (GLASER). *If $n \geqslant 3$, there exists a bijective map*

$$g\colon \mathbf{R}^{n-1} \times [0, \infty) \twoheadrightarrow \mathbf{R}^n.$$

(4.3). *Applications.*

(1) If $n \geqslant 3$, $\exists$ noncompact monotone map of $\mathbf{R}^n$ onto itself. If $n \geqslant 5$, $\exists$ noncompact $UV^1$ map of $\mathbf{R}^n$ onto itself.

PROOF. The composition

$$\mathbf{R}^n \approx S^n - \beta^{-1}(p) \xrightarrow{\beta|} B^n - \{p\} \approx \mathbf{R}^{-1} \times [0, \infty) \xrightarrow{g} \mathbf{R}^n$$

is certainly monotone (resp. $UV^1$). Since $g$ cannot be compact (otherwise it would be a homeomorphism) the composition is not compact. □

Thus, Whyburn's conjecture is false. For $n \geqslant 4$, this was discovered by Glaser [**3**]. The case $n = 3$ is due to Bing [**9**].

Perhaps equally surprising is the following corollary. (The question of whether monotone maps must have degree ±one was first raised, to my knowledge, by A. C. Connor while he was a student at the University of Georgia in 1964.)

(2) If $n \geqslant 3$ (resp. $n \geqslant 5$), $\exists$ monotone (resp. $UV^1$) map of $S^n$ onto itself which is null-homotopic.[9]

[9] A PL monotone map must have degree plus or minus one. D. Schoenfeld [**2**] has shown that, for $n = 3$, degree one monotone maps are approximable by PL monotone maps. See also Walsh [**1**] for a result on approximating mappings which are surjections on $\pi_1$ by monotone maps.

PROOF. Map $B^n$ onto $S^n$ by shrinking $\partial B^n$ to a point. Then

$$S^n \xrightarrow{\beta} B^n \to S^n$$

is an example as required. $\square$

In answer to a question of Lacher-McMillan [1] we have

(3) If $n \geqslant 4$ (resp. $n \geqslant 6$) $\exists$ monotone (resp. $UV^1$) map of $S^n$ onto a manifold which is not a homotopy sphere.

PROOF. We get a $UV^{k-2}$-map of $B^n$ onto $S^k \times S^{n-k}$ by taking the composition

$$B^n \approx B^k \times B^{n-k} \to S^k \times B^{n-k} \to S^k \times S^{n-k}$$

(assuming $2k \leqslant n$). Hence

$$S^n \xrightarrow{\beta_j} B^n \to S^{j+2} \times S^{n-j-2}$$

is monotone ($j = 0$) (resp. $UV^1$ ($j = 1$)) provided $2j + 4 \leqslant n$. $\square$

QUESTION 14. *Let $2k + 3 \leqslant n$. Is there a $UV^k$-map $\beta_k$ of $S^n$ onto $B^n$ such that $\beta_k^{-1}(p) =$ point for some $p \in \partial B^n$?*

If the answer were "yes", one could generalize all of the above applications, and probably also answer the following:

QUESTION 15. *Is there a $UV^k$-map of $S^n$ to itself with degree greater than one* ($2k + 3 \leqslant n$)?

Wilson [1] has answered Question 15 in the case $k \leqslant 1$.

QUESTION 16. *Is there a $UV^k$-map of $S^n$ onto a manifold which is not a homotopy sphere* ($2k + 4 \leqslant n$)?

**5. Compactness and finiteness theorems for monotone maps.** Suppose $f$: $M^n \twoheadrightarrow N^n$ is a map, with compact point-inverses, between manifolds. By §2.2 of the text,

$$\bigcup_{i \geqslant 0} A_i(f; R) = \bigcup_{i \geqslant 0} A^i(f; R),$$

and we define $A(f; R)$ to be this set. (See §6 of the text.)

HYPOTHESIS. Throughout this section we assume that $f$ is as above and that $A_i(f; R) = \phi$ for $i < k$.

Note that when $k \geqslant 0$, $f$ is monotone and hence reflexive-compact.

(5.1) THEOREM. (1) *If $2k > n$ then $A(f; R) = \phi$.*

(2) *If $2k = n$ then $A(f; R)$ is a locally finite subset of $N^n$.*

(5.2) THEOREM. *If $2k \geqslant n$ then $f$ is compact.*

Note that (5.1) is proved in §6 of the text, assuming $f$ is compact. Therefore (5.1) and (5.2) can be proved simultaneously just as (3.1) and (3.2) were proved above.

(5.3) COROLLARY. *If $2k \geqslant n$ and $M^n$ and $N^n$ are orientable then* $\deg f = \pm 1$.

(5.4) THEOREM. *If* $\dim A(f; R) \leqslant k$ *then $f$ is compact.*

PROOF. By a result of E. G. Skljarenko [1], $f|f^{-1}(U)$ has degree $\neq 0$ at some $y \in V$, where $V$ is an open set for which $f|f^{-1}(V)$ is compact. See Lacher-McMillan [1] for detail. $\square$

(5.5) COROLLARY. *If $M^n$ and $N^n$ are orientable, and if* $\dim A(f; \mathbf{Z}) \leqslant k$, *then* $\deg f = \pm 1$.

QUESTION 17. *Are the codimensional restrictions on k in* (5.2) *and* (5.3) *best possible*? (*Compare with Question* 14 *in* §4 *above.*)

QUESTION 18. *Are the dimensional restrictions on* $A(f; R)$ *in* (5.4) *and* (5.5) *best possible*?

If one looks carefully at the constructions of Bing and Wilson, one sees that the answers to Questions 17 and 18 are "yes" when $n \leqslant 5$.

If one assumed in the above that $2k + 1 = n$ and $d(f) \leqslant 0$ (see §7 of the text) then $A(f; R)$ could be proved locally finite and $f$ proved compact. Moreover, if one looks at Bing's map, one sees that $d(\beta_0) = 3$. An affirmation of the following would show that all results of this section are "best possible":

CONJECTURE (QUESTION 19). *There exists a $UV^k$-map*

$$\beta_k\colon S^{2k+3} \twoheadrightarrow B^{2k+3},$$

*such that* (1) $\beta_k^{-1}(p) =$ *point for some* $p \in \partial B^{2k+3}$

(2) $\beta_k^{-1}(y)$ *has the shape of either* $S^{k+1} \vee S^{k+1}$, $S^{k+1}$, *or a point for each* $y \in B^{2k+3}$.

(3) $\dim \{y \mid \beta_k^{-1}(y)$ *does not have* $UV^\infty\} = k + 2$.

## APPENDIX III. CELLULARITY.

DEFINITION. A compact set $X$ in the $\partial$-manifold $N^n$ is *cellular* in $N^n$ if $X = \bigcap_{j=1}^{\infty} Q_j$ where $Q_j \approx B^n$ and $Q_{j+1} \subset \mathring{Q}_j$ for each $j$.

This definition was given by Morton Brown and used by him in his proof of the Generalized Schoenflies Theorem [**2**]. This idea has germed a great deal of mathematics, as is evidenced in part by this article.

### 1. Characterizations.

THEOREM (BROWN [**2**]). *If $X$ is cellular in $N^n$ and $U$ is a neighborhood of $X$ in $N^n$ then $\exists$ map $f\colon N^n \twoheadrightarrow N^n$ such that $f|N^n - U =$ identity and $X$ is the only nondegenerate point-inverse of $f$.*

It is clear from the definition that a cellular set has property $UV^\infty$. On the other hand, it is clear from the Theorem that cellularity depends on the embedding $X \subset N^n$, not just on $X$. A criterion for cellularity was given by McMillan (see also Hollingsworth and Sher [**1**]).

THEOREM (MCMILLAN[**1**]). *Suppose $X$ is a $UV^\infty$-compact set in the interior of the $\partial$-manifold $N^n$, $n \geqslant 5$. $X$ is cellular in $N^n$ if and only if the following "condition* CC*" is satisfied*:

> CC: *For any neighborhood $U$ of $X$ in $N^n$, $\exists$ neighborhood $V$ of $X$ in $U$ such that any loop in $V - X$ is null-homotopic in $U - X$.*

Again by Brown's Theorem, it is clear that condition CC is a necessary condition for cellularity of $X \subset N^n$ ($n \geqslant 3$). (For $n \leqslant 2$, the Phragmén-Brouwer Theorem shows that $X \subset N^n$ is cellular iff $X$ has property $uv^\infty$. See Whyburn [**1**].)

Note that $X$ has $UV^\infty$ iff $X$ has $uv^\infty$ and $1 - UV$ (see §2.3 of the text). Since property $uv^\infty$ (equivalently, vanishing of Čech cohomology) is relatively easy to recognize, high-dimensional recognition of cellularity is reduced to two "fundamental group" problems.

In contrast, almost nothing is known about cellularity in 4-manifolds, and the 3-dimensional analogue of McMillan's Theorem is equivalent to the Poincaré conjecture.

**2. The** 3-$D$ **case.** McMillan's Theorem does in fact admit a 3-$D$ version. The idea is to use Kneser's Theorem to "push the trouble to $X$".

THEOREM (MCMILLAN [3]). *Suppose $X$ is a compact set in the interior of the $\partial$-manifold $N^3$. If $X$ separates none of its neighborhoods and satisfies condition* CC, *then $X$ has a neighborhood $W$ in $N^3$ such that $W^3 - X \approx S^2 \times \mathbf{R}$.*

COROLLARY. *If the $UV^\infty$ compactum $X \subset N^3$ satisfies* CC *and if some neighborhood of $X$ contains no fake* 3-*cells, then $X$ is cellular in $N^3$.*

McMillan's 3-$D$ Theorem can be deduced rapidly from Kneser's Theorem [**1**] and the (more recent) result of Husch-Price [**1**].

**3. Detection of** $1 - UV$; **point-like sets.** It is known that property $1 - UV$ can be *deduced* from condition CC under certain circumstances. E.g., suppose $X \subset N^n$ satisfies CC and is acyclic; then $X$ has $UV^\infty$ if either $\dim X \leqslant n - 2$ or $N^n$ is simply connected (Lacher [**13**]; Lacher-McMillan [**1**]).

A set $X \subset N^n$ is *point-like* in $N^n$ iff $N^n - X \approx N^n$-point. By Brown's Theorem, cellular sets are point-like. The converse is an interesting question, especially in dimensions 3 and 4. (Point-like compacta clearly must be $UV^\infty$ and satisfy CC.)[10] Christenson and Osborne [**1**] show the converse is true in dimension 3. The Schoenflies Theorem (Brown [**2**]) shows that point-like sets in $S^4$ are cellular, but a similar statement replacing $S^4$ by $S^2 \times S^2$ is conjectural.

**4. Characterization of** $\mathbf{R}^n$, $n \neq 4$. Let $U^n$ be a noncompact $n$-manifold, $n \neq 4$, and assume the conditions below.

(i) *$U^n$ is contractible.*

(ii) *$U^n$ is simply connected at infinity*

(*i.e., for any compact set $K \subset U^n$ $\exists$ compact set $L \subset U^n$ such that $U^n - L$ is connected and any loop in $U^n - L$ is null-homotopic in $U^n - K$*).

(iii) *If $n = 3$ then $U^3$ contains no fake* 3-*cells.*

THEOREM. *Under the above hypotheses, $U^n \approx \mathbf{R}^n$.*

The result follows from Husch-Price [**1**] when $n = 3$ and is the main theorem of Siebenmann [**2**] when $n \geqslant 5$. It has antecedents as far back as C. H. Edwards, Jr. [**1**] and Stallings [**2**].

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